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Osaka J. Math.
4 (1967), 313-326

ON THE EXTREME VALUES OF GAUSSIAN PROCESSES

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(Received July 14, 1967)

1. Introduction

Let us consider a separable and measurable Gaussian process[1] $\boldsymbol{X}=\{X(t), t\geqq 0\}$ with mean zero and with the covariance function $\rho(t,s)=EX(t)X(s)$. We assume that $\rho(t,t)$ is independent of t, say $v(>0)$. The asymptotic behaviours of $\sup_{t\in[0,T]} X(t)$ as $T\to\infty$ have been studied by various authors [2] [3] [4] [8] [9]. For example, Pickands [8] proved that

$$(*)\qquad \frac{\sup_{t\in[0,T]} X(t)}{\sqrt{2v\log T}} \xrightarrow[T\uparrow\infty]{} 1 \qquad \text{a. s.},$$

under the following conditions (for stationary Gaussian process),

$$\limsup_{t\to 0} t^{-\alpha}(v-\rho(t,0))<\infty\,, \qquad \text{for some} \quad \alpha>0\,,$$

and $\lim_{t\to\infty}\rho(t,0)=0$.

In this note we shall prove (*) under certain conditions (condition A and B in Section 2) weaker than Pickands' conditions. As an application of (*), we can prove the Hölder continuity as well as the uniform Hölder continuity for *stationary* Gaussian processes;

$$\limsup_{t\downarrow 0} \frac{|X(t)-X(0)|}{\sqrt{4(v-\rho(0,t))\log\log\frac{1}{t}}} = 1\,, \qquad \text{a.s.}\,,$$

$$\limsup_{h\downarrow 0} \frac{\sup_{t,s\in[0,1],|t-s|=h} |X(t)-X(s)|}{\sqrt{4(v-\rho(0,h))\log\frac{1}{h}}} = 1\,, \qquad \text{a.s.}$$

There are many references on this subject (see, for example, [10]). Our method, different from the usual Borel-Cantelli method, consists in making use of some transformations of path functions to reduce the behaviour of path functions near $t=0$ to that near $t=\infty$.

(1) We mean a real valued process.

2. Results

Let $X=\{X(t),\ t\geqq 0\}$ be a separable and measurable Gaussian process with $EX(t)=0$, $\rho(t,s)=EX(t)X(s)$ and $EX^2(t)=v(>0)$. We shall introduce the following two conditions:

CONDITION A. For any t and s,

$$2(v-\rho(t,s))(=E(X(t)-X(s))^2)<\psi^2(|t-s|) \tag{1}$$

where ψ is a non-decreasing and continuous function on $[0,\infty)$ such that

$$\int_0^\infty \psi(e^{-x^2})\,dx<\infty\,. \tag{2}$$

CONDITION B.

$$\lim_{T\uparrow\infty}\sup_{|t-s|>T}\rho(t,s)\leq 0\,. \tag{3}$$

(This condition A implies the continuity of almost all sample paths, by a theorem due to X. Fernique [6]).

In Section 3, we shall prove the following theorems,

Theorem 1. *Under condition A, we have*

$$\limsup_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}|X(t)|}{\sqrt{2v\log T}}\leq 1$$

with probability 1.

Theorem 2. *Under condition B, we have*

$$\limsup_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}X(t)}{\sqrt{2v\log T}}\geq 1$$

with probability 1.

Therefore, if condition A as well as condition B are satisfied, we can conclude that

$$\lim_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}|X(t)|}{\sqrt{2v\log T}}=\lim_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}X(t)}{\sqrt{2v\log T}}=1$$

holds with probability 1.

Suppose that X is stationary and stochastically continuous. So, the covariance function $\gamma(t-s)=\rho(t,s)$ is expressible in the form

$$\gamma(\tau)=\int_{-\infty}^{\infty}e^{i\tau\lambda}\,dF(\lambda)$$

with a bounded measure dF, symmetric with respect to 0. Moreover, the meausre dF can be split into the continuous part dF_c and the discontinuous part dF_d; $dF=dF_c+dF_d$.

Corollary. *Let* $v_c=F_c(R^1)$. *If* v_c *is positive and if condition A is satisfied, then we have*

$$\limsup_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}|X(t)|}{\sqrt{2v_c\log T}}\le 1$$

with probability 1. *Moreover, if condition B is also satisfied, then*

$$\lim_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}|X(t)|}{\sqrt{2v_c\log T}}=\lim_{T\uparrow\infty}\frac{\sup_{t\in[0,T]}X(t)}{\sqrt{2v_c\log T}}=1$$

with probability 1.

In Section 4, we shall show the following theorems for the stationary and stochastically continuous process $\boldsymbol{X}$, using Theorems 1 and 2.

Theorem 3. *Let* $\sigma(t)=\sqrt{E|X(t)-X(0)|^2}>0$.
Suppose that there exist two positive constants β *and* L *such that*

$$(4)\qquad \frac{\sigma(ts)}{\sigma(t)}\le Ls^\beta,\qquad \text{for}\quad t,s\in(0,1],$$

and that

$$(5)\qquad \sigma^2(t)-\sigma^2(t-h)\le L\sigma^2(h),\qquad \text{for}\quad \text{small } t \text{ and } h.$$

Then we have

$$\limsup_{t\downarrow 0}\frac{|X(t)-X(0)|}{\sigma(t)\sqrt{2\log\log\frac{1}{t}}}=1$$

with probability 1.

Theorem 4. *If the assumption* (4) *of Theorem* 3 *is valid and if* $\sigma^2(t)$ *is concave in a small interval* $(0,\delta)$, *then we have*

$$\limsup_{h\downarrow 0}\frac{\sup_{t,s\in[0,1],|t-s|=h}|X(t)-X(s)|}{\sigma(h)\sqrt{2\log\frac{1}{h}}}=1$$

with probability 1.

3. Proof of Theorem 1 and 2

Without loss of generality, we may assume that $v=1$. Since $\boldsymbol{X}$ has continuous paths under condition A, Theorem 1 follows immediately from the statement

A. For any $\varepsilon>0$ and for almost all ω, we can find a finite $T_0(\varepsilon, \omega)$ such that, for all values of T greater than $T_0(\varepsilon, \omega)$, the inequality

$$\frac{\sup_{t\in[T_0, T]} |X(t, \omega)|}{\sqrt{2 \log T}} < 1+\varepsilon$$

holds.

Let $a(n)=[n^{\varepsilon}]$(2) and define ξ by

$$\xi(n, k) = X\left(n+\frac{k}{1+a(n)}\right), \qquad k=0, 1, \cdots, a(n), \quad n=0, 1, \cdots$$

Using the following well-known inequality;

$$(6) \qquad \frac{1}{\sqrt{2\pi}}\int_{|x|\geq c} e^{-x^2/2}dx < \frac{2}{c}e^{-c^2/2}$$

we can get

$$(7) \qquad \sum_{n=0}^{\infty}\sum_{k=1}^{a(n)} P(|\xi(n, k)| \geq \sqrt{2 \log n}(1+\varepsilon)) < \infty .$$

Therefore, using Borel-Cantelli's Lemma, we see that, for almost all ω,

$$\max_{k=0,\cdots,a(n)} |\xi(n, k)| < \sqrt{2 \log n}(1+\varepsilon), \qquad \text{for large } n.$$

Define η by

$$\eta(n, k, j) = X\left(n+\frac{1}{1+a(n)}+\frac{j}{b(n)}\right)-X\left(n+\frac{k}{1+a(n)}\right),$$

$$j=1, 2, \cdots, \frac{b(n)}{1+a(n)}, \quad k=0, 1, \cdots, a(n), \quad n=0, 1, \cdots$$

where $b(n)=(1+a(n))\left[\exp \frac{\varepsilon^3 (\log n)^2}{K}\right]$ and $K=2\int_0^{\infty}\psi(e^{-x^2})dx$.

By virtue of condition A, we have

$$\begin{aligned} &P(|\eta(n, k, j)| \geq \varepsilon\sqrt{2 \log n}) \\ &\leq P\left\{\frac{|\eta(n, k, j)|}{D(\eta(n, k, j))} \geq \varepsilon\frac{\sqrt{2 \log n}}{D(\eta(n, k, j))}\right\} \qquad \text{for } n=1, 2, \cdots, \\ &\leq P\left\{\frac{|\eta(n, k, j)|}{D(\eta(n, k, j))} \geq \varepsilon\frac{\sqrt{2 \log n}}{\psi(n^{-\varepsilon})}\right\} , \end{aligned}$$

where $D(\cdot)$ stands for the standard deviation of a random variable. On the other hand, (2) implies

$$(8) \qquad \psi(e^{-c^2}) \leq \frac{\sqrt{K}}{\sqrt{2}\, c}, \qquad c<0 .$$

(2) $[c]$ is the integer part of c.

Hence, appealing to the inequality (6), we see

$$\sum_{n=1}^{\infty}\sum_{k=0}^{a(n)}\sum_{j=1}^{b^*(n)} P(|\eta(n,k,j)|\geq \varepsilon\sqrt{2\log n})$$
$$\leq \frac{1}{\varepsilon}\sum_{n=1}^{\infty}\frac{2\psi(n^{-\varepsilon})}{\sqrt{2\log n}}\exp\left(\frac{\varepsilon^3(\log n)^2}{K}+\varepsilon(\log n)-\frac{\varepsilon^2\log n}{\psi^2(n^{-\varepsilon})}\right)$$

where $b^*(n)=\dfrac{b(n)}{1+a(n)}$. Combining this inequality with (8), we have

(9) $$\sum_{n=1}^{\infty}\sum_{k=0}^{a(n)}\sum_{j=1}^{b^*(n)} P(|\eta(n,k,j)|\geq \varepsilon\sqrt{2\log n})<\infty\,.$$

We set $c(p)=2^{2^p}$ and define ζ by

$$\zeta(n,i,p,q,r)$$
$$=X\left(n+\frac{i}{b(n)}+\frac{q}{b(n)c(p)}+\frac{r}{b(n)c(p+1)}\right)-X\left(n+\frac{i}{b(n)}+\frac{q}{b(n)c(p)}\right),$$
$$i=0,1,\cdots,b(n)-1,\quad q=0,1,\cdots,c(p)-1,\quad r=1,\cdots,c(p),$$
$$p=1,2,\cdots,\quad n=0,1,\cdots.$$

Let $Y(n,p)=\max\limits_{i,q,r}|\zeta(n,i,p,q,r)|$ and $Z(l,p)=\max\limits_{c(l)\leq n\leq c(l+1)} Y(n,p)$. Then we have, for any $h>0$,

$$EZ(l,p)\leq h+\sum_{n=c(l)}^{c(l+1)}\sum_{r=1}^{c(p)}\sum_{q=0}^{c(p)-1}\sum_{i=0}^{b(n)-1}\int_h^{\infty}|x|\,d\mu_{\zeta(n,i,p,q,r)}(x)$$

where μ_ζ is the probability law of ζ, ([5], Proposition 2). Hence

$$EZ(l,p)\leq h+\sqrt{\frac{2}{\pi}}\sum_{n=c(l)}^{c(l+1)}\sum_{r,q,i}D(\zeta(n,i,p,q,r))\exp\left(\frac{-h^2}{2D^2(\zeta(n,i,p,q,r))}\right)$$
$$\leq h+b(c(l+1))c(p+1)c(l+1)\psi(1/b(c(l))c(p))\exp\left(\frac{-h^2}{2\psi^2(1/b(c(l))c(p))}\right).$$

Let $$h=h(l,p)=\sqrt{2\log b(c(l+1))c(p+1)c(l+1)}\,\psi(1/b(c(l))c(p))\,.$$

Then, we see

(10) $$EZ(l,p)\leq 2h(l,p)\,.$$

Recalling the definition $b(n)$ and $c(p)$, we have

(11) $$\sqrt{\log b(c(l+1))c(p+1)c(l+1)}\leq d(2^l+2^{p/2})\,,$$

with a properly chosen constant d which may depend on ε. On the other hand, by (2),

(12) $$\sum_{p=1}^{\infty}2^{p/2}\psi(1/b(c(l))c(p))\leq\sum_{p=1}^{\infty}2^{p/2}\psi(1/c(p))$$
$$\leq 3\int_0^{\infty}\psi(2^{-x^2})\,dx<\infty\,,\qquad l=1,2,\cdots.$$

Furthermore, by (8), with a properly chosen constant d'

$$\psi(1/b(c(l))c(p)) \leq d'(2^{2l}+2^{p})^{-1/2}. \tag{13}$$

Using the inequality, for $\alpha \in (0, 1)$,

$$x+y \geq x^{\alpha}y^{1-\alpha}+x^{1-\alpha}y^{y} \geq x^{\alpha}y^{1-\alpha}, \qquad \text{for} \quad x<0,\ y<0,$$

we have,

$$\psi(1/b(c(l))c(p)) \leq d' 2^{-(2/3)l} 2^{-(1/6)p}.$$

Hence, combining this with (11) and (12), we have

$$\sum_{p=1}^{\infty} h(l, p) \leq dd' 2^{l/3} \sum_{p=1}^{\infty} 2^{-p/6} + 3d \int_0^{\infty} \psi(2^{-x^2})\, dx.$$

Therefore, by (10),

$$\sum_{p=1}^{\infty} 2^{-l/2} \sum_{p=1}^{\infty} EZ(l, p) < \infty.$$

For any $\varepsilon>0$,

$$\begin{aligned} &P\Big(\sum_{p=1}^{\infty} Y(n, p) \geq \varepsilon\sqrt{2\log n}, \qquad \text{for} \quad \text{some } n \in (c(l), \cdots c(l+1))\Big) \\ &\leq P\Big(\sum_{p=1}^{\infty} Z(l, p) \geq \varepsilon\sqrt{2\log c(l)}\Big) \\ &\leq \frac{\sum_{p=1}^{\infty} EZ(l, p)}{\varepsilon\sqrt{2\log c(l)}}. \end{aligned}$$

Hence, we have

$$\sum_{l=1}^{\infty} P\Big(\sum_{p=1}^{\infty} Y(n, p) \geq \varepsilon\sqrt{2\log n}, \quad \text{for} \quad \text{some } n \in (c(l), \cdots c(l+1))\Big) < \infty. \tag{14}$$

Since X has continuous paths, for $t \in \left[n+\frac{k}{1+a(n)}+\frac{j}{b(n)}, n+\frac{k}{1+a(n)}+\frac{j+1}{b(n)}\right]$,

$$|X(t)| \leq \sum_{p=1}^{\infty} Y(n, p) + |\eta(n, k, j)| + |\xi(n, k)|.$$

Therefore, recalling (7) (9) and (14), for almost all ω, we can choose a finite $N_0(\omega)$ so that, for $n=N_0(\omega), N_0(\omega)+1, \cdots$.

$$\sup_{t\in[n, n+1]} |X(t, \omega)| \sqrt{2\log n}\,(1+3\varepsilon).$$

This completes the proof of statement A.

To prove Theorem 2, it is enough to show the statement.

B. For any $\varepsilon>0$, we can find a finite $T_0(\varepsilon)$ such that the inequality

$$\liminf_{N\uparrow\infty} \frac{\max_{j=1,\cdots,N} X(jT_0,\omega)}{\sqrt{2\log N}} > 1-\varepsilon$$

holds, with probability 1.

Without any difficulty, we can carry out the same method as in [8] (pp. 203–204). We take T so that $\sup_{|t-s|>T} \rho(t,s)<\varepsilon$. Let $\{\xi, \eta_n, n=1, 2, \cdots\}$ be a system of independent Gaussian random variables with $E\xi=E\eta_n=0$, $E\xi^2=\varepsilon$ and $E\eta_n^2=1-\varepsilon$. Put $Y_l=\xi+\eta_l$. Then

$$EY_l^2 = EX^2(Tl) = 1$$

(15) and

$$EY_lY_j \geq EX(lT)X(jT)\,.$$

On the other hand, let $R(=\{r_{ij}\})$ be a $N\times N$ symmetric positive definite matrix with 1's along the diagonal. Define

$$Q(c;\{r_{ij}\}) \equiv \int_{-\infty}^{c}\int_{-\infty}^{c} \frac{1}{(2\pi)^{N/2}\sqrt{\det R}} \exp\left(\frac{1}{2}(x_1,\cdots x_N)R^{-1}(x_1,\cdots x_n)^t\right)dx_1\cdots dx_N\,.$$

Then $Q(c;\{r_{ij}\})$ is an increasing function of the arguments $\{r_{ij}\}$, ([2], p. 508). Combining this with (15), we get

$$P(\max_{k=1,\cdots N} X(Tk) \leq c) \leq P(\max_{k=1,\cdots N} Y_k \leq c)\,. \tag{16}$$

For any ε', $(0<\varepsilon'<1)$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} P(\max_{k=1,\cdots 2^n} Y_k \leq \sqrt{2(1-\varepsilon)\log 2^n}(1-\varepsilon')) \\ &\leq \sum_{n=1}^{\infty} P\left(\xi \leq -\frac{\varepsilon'}{2}\sqrt{2(1-\varepsilon)\log 2^n}\right) \\ &\quad + \sum_{n=1}^{\infty} P\left(\max_{k=1,\cdots 2^n} \eta_k \leq \left(1-\frac{\varepsilon'}{2}\right)\sqrt{2(1-\varepsilon)\log 2^n}\right) < \infty\,, \end{aligned} \tag{17}$$

by the inequality of (6). Therefore, using (16) and (17), we have

$$\liminf_{N\uparrow\infty} \frac{\max_{k=1,\cdots N} X(kT)}{\sqrt{2\log N}} > \sqrt{1-\varepsilon}\,(1-\varepsilon'), \quad \text{a.s.}$$

Since ε' is arbitrary, we get statement B.

To prove Corollary, we shall express $\boldsymbol{X}$ by the sum of mutually independent Gaussian processes so that

$$X(t) = \xi(t) + \sum_{n=0}^{\infty} \eta_n \cos\lambda_n t + \sum_{n=0}^{\infty} \zeta_n \sin\lambda_n t \tag{18}$$

where $E\xi(t)=E\eta_n=E\zeta_n=0$ and the stationary Gaussian process ξ has the continuous spectral measure dF_c, ([7]). We define $X_k(t)$ by

$$X_k(t) = X(t) - \sum_{n=0}^{k-1} \eta_n \cos \lambda_n t - \sum_{n=0}^{k-1} \zeta_n \sin \lambda_n t \,. \tag{19}$$

Then it is easily seen that

$$E\,|X_k(t)-X_k(s)|^2 \leq E\,|X(t)-X(s)|^2$$

and the process X_k also satisfies condition A. Therefore, by Theorem 1, we have

$$\limsup_{T\uparrow\infty} \frac{\sup_{t\in[0,T]} |X_k(t)|}{\sqrt{2v_k \log T}} \leq 1\,, \qquad \text{a.s.}\,,$$

where $v_k=EX_k^2(t)$. Since almost all sample paths of $\sum_{n=0}^{k-1}\eta_n \cos \lambda_n t + \sum_{n=0}^{k-1}\zeta_n \sin \lambda_n t$ are bounded functions, we have

$$\limsup_{T\uparrow\infty} \frac{\sup_{t\in[0,T]} |X_k(t)|}{\sqrt{2v_k \log T}} = \limsup_{T\uparrow\infty} \frac{\sup_{t\in[0,T]} |X(t)|}{\sqrt{2v_k \log T}} \qquad \text{a.s.}$$

Therefore, we obtain the former half of Corollary, since v_k tends to v_c.

As to the latter half, condition B implies

$$\liminf_{T\uparrow\infty} \frac{\sup_{t\in[0,T]} |X(t)|}{\sqrt{2v_c \log T}} \geq \sqrt{\frac{v}{v_c}} \geq 1$$

by Theorem 2. Hence, we have $v=v_c$. Therefore under conditions A and B, we complete the proof of Corollary.

4. Proof of Theorem 3 and 4

To prove Theorem 3, we shall firstly derive the following inequality from assumption (4),

$$\limsup_{t\downarrow 0} \frac{|X(t)-X(0)|}{\sigma(t)\sqrt{2\log\log\frac{1}{t}}} \leq 1\,, \qquad \text{a.s.} \tag{20}$$

We shall introduce an auxiliary Gaussian process Y by

$$Y(n+t) = \frac{X(2^{-n}-t2^{-n-1})-X(0)}{\sigma(2^{-n}-t2^{-n-1})}\,, \qquad t\in[0,1],\ \ n=0, 1, \cdots.$$

Since X has continuous paths by (4), ([1], [6]), Y is also a continuous Gaussian process with $EY(t)=0$ and $EY^2(t)=1$. Moreover, using (4), we have

$$
\begin{aligned}
&E\,|\,Y(n+t)-Y(n+s)\,|^2 \\
&\quad=\frac{\sigma^2((t-s)2^{-n-1})-(\sigma(2^{-n}-t2^{-n-1})-\sigma(2^{-n}-s2^{-n-1}))^2}{\sigma(2^{-n}-t2^{-n-1})\,\sigma(2^{-n}-s2^{-n-1})} \\
&\quad\le\frac{\sigma^2((t-s)2^{-n-1})}{\sigma(2^{-n}-t2^{-n-1})\,\sigma(2^{-n}-s2^{-n-1})} \\
&\quad\le L^2(t-s)^{2\beta}\,,\qquad \text{for}\quad t,\ s\in[0,1]\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E\,|\,Y(n)-Y(n-s)\,|^2 \\
&=E\,|\,Y(n-1+1)-Y(n-1+1-s)\,|^2 \\
&\le L^2s^{2\beta}\,,\qquad \text{for}\quad s\in[0,1]\,.
\end{aligned}
$$

Therefore, we have

$$E\,|\,Y(t)-Y(s)\,|^2\le 4L^2\,|\,t-s\,|^{2\beta}\,,\qquad \text{for}\quad |\,t-s\,|\le 1\,.$$

On the other hand $E\,|\,Y(t)-Y(s)\,|^2\le 4$. Hence, $\boldsymbol{Y}$ satisfies condition A. So, Theorem 1 tells us that

$$\limsup_{T\uparrow\infty}\frac{\max_{t\in[0,T]}|\,Y(t)\,|}{\sqrt{2\log T}}\le 1\,,\qquad \text{a.s.}\,,$$

holds. Therefore

$$\limsup_{T\uparrow\infty}\frac{|\,Y(T)\,|}{\sqrt{2\log T}}\le 1\,,\qquad \text{a.s.}$$

Hence, we have

$$\limsup_{t\downarrow 0}\frac{|\,X(t)-X(0)\,|}{\sigma(t)\sqrt{2\log\varphi(t)}}\le 1\,,\qquad \text{a.s.}\,,$$

where φ is defined by $\varphi(2^{-n}-\tau 2^{-n-1})=n+\tau$ for $\tau\in[0,1]$ and $n=0,1,\cdots$. Since

$$\left(\log\log\frac{1}{t}\right)\Big/\log\varphi(t)\to 1\,,\quad \text{as}\quad t\to 0\,, \tag{21}$$

we obtain (20).

By virtue of (4) and (5), we shall show the converse inequality of (20). For $n<m$, we have

$$
\begin{aligned}
&EY(n+t)\,Y(m+s) \\
&\quad=\frac{1}{2}\,\frac{\sigma^2(2^{-n}-t2^{-n-1})+\sigma^2(2^{-m}-s2^{-m-1})-\sigma^2(2^{-n}-t2^{-n-1}-2^{-m}+s2^{-m-1})}{\sigma(2^{-n}-t2^{-n-1})\,\sigma(2^{-m}-s2^{-m-1})} \\
&\quad\le\frac{1}{2}(L+1)\,\frac{\sigma^2(2^{-m}-s2^{-m-1})}{\sigma(2^{-n}-t2^{-n-1})\,\sigma(2^{-m}-s2^{-m-1})} \\
&\quad\le \text{const.}\ 2^{-\beta(m-n)}\,.
\end{aligned}
$$

So, $\boldsymbol{Y}$ satisfies condition B. Hence, for $\varepsilon>0$ and for almost all ω, we can choose a finite $T_0(\varepsilon, \omega)$ so that, for any T greater than T_0, the inequality

$$\frac{\max_{t\in[0,T]} Y(t, \omega)}{\sqrt{2\log T}} > 1-\varepsilon$$

holds. For any v smaller than $\varphi^{-1}(T_0)$(3), the inequality

$$\max_{u\in[v,1]} \frac{X(u, \omega)-X(0, \omega)}{\sigma(u)\sqrt{2\log\varphi(v)}} > 1-\varepsilon$$

holds. Since $\frac{X(u, \omega)-X(0, \omega)}{\sigma(u)}$ is continuous on $(0, 1]$, for $\delta>0$, we can take $S_0(\omega)$, smaller than $\varphi^{-1}(T_0)$, so that, for any s smaller than $S_0(\omega)$,

$$\max_{u\in[s,\delta]} \frac{X(u, \omega)-X(0, \omega)}{\sigma(u)\sqrt{2\log\varphi(u)}} > 1-\varepsilon\,.$$

Therefore, for any $\delta>0$, and for almost all ω,

$$\sup_{u\in(0,\delta]} \frac{X(u, \omega)-X(0, \omega)}{\sigma(u)\sqrt{2\log\varphi(u)}} > 1-\varepsilon\,.$$

Combining this with (21), we get the converse inequality of (20).

To prove Theorem 4, we shall fix a positive ε arbitrarily and define by

$$\xi(n, k, l) = \left(X\left(\frac{k+l}{b(n)}\right)-X\left(\frac{k}{b(n)}\right)\right)\Big/\sigma\left(\frac{l}{b(n)}\right)$$

$$l=1, 2, \cdots a(n), \quad k=0, 1, \cdots b(n), \quad n=1, 2, \cdots,$$

where $a(n)=[2^{n\varepsilon}]$ and $b(n)=2^n a(n)$. Using (6), we have

$$\sum_{n=1}^{\infty}\sum_{k=0}^{b(n)}\sum_{l=1}^{a(k)} P(|\xi(n, k, l)| \geq (1+\varepsilon)\sqrt{2\log 2^n}) < \infty\,. \tag{22}$$

Define a continuous Gaussian process $\{Y(s), s\geq 0\}$ by

$$Y(N(n, k)+t) = \begin{cases} \left(X\left(\frac{k+t}{b(n)}\right)-X\left(\frac{k}{b(n)}\right)\right)\Big/\sigma\left(\frac{1}{b(n)}\right), & \text{for } t\in[0, 1]\,, \\ \left(X\left(\frac{k+2-t}{b(n)}\right)-X\left(\frac{k}{b(n)}\right)\right)\Big/\sigma\left(\frac{1}{b(n)}\right), & \text{for } t\in[1, 2]\,, \end{cases}$$

$$k=0, 1, \cdots, b(n), \quad n=1, 2, \cdots,$$

where $N(n, k)=2\sum_{j=1}^{n-1} b(j)+2k$. Then, using (4), we have

(3) φ^{-1} means the inverse function of φ.

$$E|Y(N(n,k)+t)-Y(N(n,k)+s)|^2 = \frac{\sigma^2((t-s)/b(n))}{\sigma^2(1/b(n))} \le L^2|t-s|^{2\beta}, \qquad \text{for} \quad t, s\in[0, 1].$$

Hence

$$E|Y(N(n,k)+t)-Y(N(n,k)+s)|^2 \le L^2|t-s|^{2\beta}, \qquad \text{for} \quad t, s\in[0, 2].$$

Since, $Y(2j)=0$, for $j=0, 1, 2, \cdots$, we get

$$E|Y(u)-Y(u)|^2 \le 4L^2|u-v|^{2\beta} \tag{23}$$

and

$$E|Y(u)|^2 \le L^2. \tag{24}$$

Let η be a standard Gaussian variable which is independent to $\{Y(u), u\ge 0\}$. We shall define a Gaussian process $\boldsymbol{Z}$ by

$$Z(u) = Y(u)+\sqrt{1+L^2-EY^2(u)}\,\eta$$

and show that $\boldsymbol{Z}$ satisfies condition A.

$$E(Z(u)-Z(v))^2 \le 4L^2|u-v|^{2\beta}+(1+L^2-EY^2(v))\left|\sqrt{1+\frac{EY^2(v)-EY^2(u)}{1+L^2-EY^2(v)}}-1\right|^2. \tag{25}$$

By (23) and (24), we have

$$|EY^2(u)-EY^2(v)| \le 4L^2|u-v|^{\beta}.$$

Hence, using the inequality $|\sqrt{1+x}-1| \le |x|$ for $|x|\le 1$, we see that the second term of the right side of (25) is less than $16L^4|u-v|^{2\beta}$ for $|u-v|\le(4L^2)^{-1/\beta}$. So, the process $\boldsymbol{Z}$ satisfies condition A. Therefore, by $EZ(t)=0$ and $EZ^2(t)=L^2+1$, we have

$$\limsup_{T\uparrow\infty} \frac{\max_{u\in[0,T]}|Z(u)|}{\sqrt{L^2+1}\sqrt{2\log T}} \le 1, \qquad \text{a.s.}$$

This implies that

$$\limsup_{T\uparrow\infty} \frac{\max_{u\in[0,T]}|Y(u)|}{\sqrt{L^2+1}\sqrt{2\log T}} \le 1, \qquad \text{a.s.},$$

because the second component of $\boldsymbol{Z}$ is bounded in u, for almost all ω. Recalling the definition of $\boldsymbol{Y}$, we have

$$\limsup_{n\uparrow\infty} \frac{\max_{k=0,\cdots,b(n)}\max_{t\in[0,T]}\left|X\left(\frac{k+t}{b(n)}\right)-X\left(\frac{k}{b(n)}\right)\right|}{\sqrt{2\log N(n+1,0)}\,\sigma\left(\frac{1}{b(n)}\right)} \le \sqrt{L^2+1}, \qquad \text{a.s.}$$

On the other hand, $\frac{\log N(n+1, 0)}{\log 2^n}$ tends to $1+\varepsilon$ when n tends to ∞. Therefore, for almost all ω, there is $n_0(\omega)$ such that, for any integer n greater than $n_0(\omega)$, the inequality

$$\max_{k=0,\cdots,b(n),\, t\in[0,1]} \left| X\left(\frac{k+t}{b(n)}, \omega\right) - X\left(\frac{k}{b(n)}, \omega\right)\right| \le (1+2\varepsilon)\sqrt{L^2+1}\,\sigma\left(\frac{1}{b(n)}\right)\sqrt{2\log 2^n} \tag{26}$$

holds. On the other hand, we have

$$\frac{\sigma\left(\frac{1}{b(n)}\right)}{\sigma(\tau)} \le L2^{\beta-\beta\varepsilon n}, \qquad \text{for} \quad \tau\in[2^{-n-1}, 2^{-n}].$$

Moreover, for small positive τ, we take integer n and i so that

$$2^{-n-1} < \tau \le 2^{-n} \quad \text{and} \quad \frac{i}{b(n)} \le \tau < \frac{i+1}{b(n)}.$$

Then, we have, by the concavity of σ^2,

$$\frac{\sigma\left(\frac{i}{b(n)}\right)}{\sigma(\tau)} \le 1,$$

and, for any positive $s(<1-\tau)$,

$$|X(s+i)-X(s)| \le \max_{j=0,\cdots,b(n)} \max_{l=1,\cdots,a(n)} |\xi(n, j, l)|\,\sigma(i/b(n))$$
$$+2\max_{k=0,\cdots,b(n)} \max_{u\in[0,1]} \left|X\left(\frac{k+u}{b(n)}\right) - X\left(\frac{k}{b(n)}\right)\right|.$$

Therefore, appealing to (22) and (26), we see that, for almost all ω,

$$|X(s+i)-X(s)| \le (1+2\varepsilon)\sigma(\tau)\sqrt{2\log\frac{1}{\tau}}, \qquad \text{for} \quad \text{small } \tau. \tag{27}$$

We shall derive the converse inequality of (27) from the concavity of $\sigma^2(t)$. Define a separable Gaussian process $\boldsymbol{Y}$ by

$$Y(2^n+k+t) = \frac{K((k+1)2^{-n}) - X(k2^{-n})}{\sigma(2^{-n})}, \quad t\in[0, 1],$$
$$k=0, 1, \cdots, 2^n-1, \quad n=1, 2, \cdots.$$

Then, by the convexity of the covariance function of $\boldsymbol{X}$, we have

$$EY(2^l+k+t)Y(2^m+j+s)$$
$$=\frac{1}{\sigma(2^{-l})\sigma(2^{-m})}\{\gamma(k2^{-l}-j2^{-m})-\gamma(k2^{-l}-(j+1)2^{-m})$$
$$-\gamma((k+1)2^{-l}-j2^{-m})+\gamma((k+1)2^{-l}+(j+1)2^{-m})\}\leq 0\,,$$
$$\text{for}\quad (j+1)2^{-m}\leq k2^{-l}\,.$$

Hence $\boldsymbol{Y}$ satisfies condition B. So, for any $\varepsilon>0$ and for almost all ω, there exists an integer $n_0(\varepsilon, \omega)$ such that

$$\max_{m=1,\cdots,n}\frac{\max_{k=,\cdots,2^m-1}|X((k+1)2^{-m},\omega)-X(k2^{-m},\omega)|}{\sigma(2^{-m})\sqrt{2\log(2^{n+1}-2)}}>1-\varepsilon\,,$$
$$\text{for}\quad n\geq n_0(\varepsilon,\omega)\,.$$

Hence, for any integer l,

$$\sup_{m=l,l+1,\cdots}\frac{\max_{k=0,\cdots,2^m-1}|X((k+1)2^{-m})-X(k2^{-m})|}{\sigma(2^{-m})\sqrt{2\log 2^m}}>1-\varepsilon\,,\qquad\text{a.s.}$$

Consequently, we have the following required inequality

$$\limsup_{h\downarrow 0}\frac{\sup_{t,s\in[0,1],|t-s|=h}|X(t)-X(s)|}{\sigma(h)\sqrt{2\log\frac{1}{h}}}>1-\varepsilon\,,\qquad\text{a.s.}$$

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