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Osaka J. Math.
4 (1967), 257-260

ON 6-FOLD TRANSITIVE GROUPS

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(Received October 25, 1967)

There is no known 6-fold transitive group other than $S_n(n\geq 6)$ and $A_n(n\geq 8)$. The purpose of this paper is to prove the following two theorems.

Theorem 1. *Let G be a 6-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. If a stabilizer of six letters in G has a normal Sylow 2 subgroup P and P leaves nine letters invariant, then G must be A_9.*

Theorem 2. *Let G be a 6-fold transitive group. If a stabilizer of five letters in G has a non trivial normal Sylow 2 subgroup, then G must be S_7 or A_9.*

Let G be a 6-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. Since the Mathieu group M_{12} has no transitive extention, by the theorems of M. Hall ([2], Th 5.8.1.) and H. Nagao [5], we have the following two lemmas.

Lemma 1. (M. Hall) *A Sylow 2 subgroup of the stabilizer $G_{12,\cdots,6}$ in G of six letters $1, 2, \cdots, 6$ is not trivial unless G is S_6, S_7, A_8 or A_9.*

Lemma 2. (H. Nagao) *$G_{1,2,\cdots,6}$ fixes exactly six letters unless G is S_7 or A_8.*

Furthermore by a theorem of Witt ([7], Th. 9.4) and Lemma 1, we have

Lemma 3. *Let P be a Sylow 2 subgroup of $G_{1,2,3,4,5,6}$. Then the number of the fixed letters of P is 6, 7, 8 or 9, and the normalizer of P in G operates on the letters fixed by P as S_6, S_7, A_8 or A_9 respectively.*

Proof. See [3], Lemma 1.

By Lemma 2 we may assume that $G_{1,2,3,4,5,6}$ fixes exactly six letters. Then if a Sylow 2-subgroup P of $G_{1,2\cdots,6}$ is normal in $G_{1,2,\cdots,6}$, P must fix six or nine letters.

REMARK 1. If the Mathieu group M_{24} admits a transitive extention G, then G satisfies the assumption of Theorem 1. Therefore Theorem 1 gives another proof of the non existence of a transitive extention of M_{24}.

REMARK 2. Since we make use of the non existence of a transitive extention of M_{12} in the proof of Lemma 2, Theorem 2 does not give a proof of the non existence of M_{12}, though a transitive extention of M_{12}, if it exists, satisfies the assumption of the theorem. But it follows from the theorem in [6] and a remark made by N. Burgoyne and P. Fong [1] that the outer automorphism group of M_{11} is trivial.

NOTATION. For a set X, let $|X|$ denote the number of elements of X. For a set of permutations S on Ω, the set of letters left fixed by S will be denoted by $I(S)$. If a subset Δ of Ω is a fixed block of S, then the restriction of S on Δ will be denoted by S^{Δ}. For a permutation x, let $\alpha_i(x)$ denote the number of i-cycles (cycles of length i) of x.

Proof of Theorem 1

Our method of proving Theorem 1 is a combinatorial one. By a series of steps we shall show the degree n of G must be nine.

Let a be an involution which fixes at least four letters and x an element of order four which fixes at least two letters. Note that G contains such elements since it is 6-fold transitive. Set $|I(a)|=r$ and $|I(x)|=s$. Then

(i) $n=4+\frac{1}{3}(r^2-2r)$.

Proof. We may assume that a is of the following form:

$$a=(1)(2)(34)(i_1)(i_2)\cdots.$$

Let P be a Sylow 2-subgroup of $G_{1,2,3,4,i_1,i_2}$ and set $I(P)=\{1,2,3,4,i_1,i_2,i_3,k,l\}$. Then a normalizes $G_{1,2,3,4,i_1,i_2}$ and hence P. Therefore a induces a permutation on $I(P)$ as follows.

$$a^{I(P)}=(1)(2)(34)(i_1)(i_2)(i_3)(kl).$$

We remark that $a^{I(P)}$ is an even permutation by Lemma 3. Now $I(P)$ is uniquely determined by $G_{1,2,3,4,i_1,i_2}$, therefore $\{i_1, i_2\}$ determines uniquely a 2 cycle (kl) of a, and we have the map

$$\varphi:\ \{i_1, i_2\}\rightarrow(kl)$$

from the family of the subsets of $I(a)-\{1,2\}$ consisting of two letters into 2 cycles of a different from (34). By the definition of φ, it is easy to see that $\varphi(\{i_1, i_2\})=(kl)$ if and only if $G_{1,2,3,4,i_1,i_2}$ and $G_{1,2,3,4,k,l}$ have a Sylow 2-subgroup in common and that φ is onto.

Now suppose that $\varphi(\{i_1, i_2\})=\varphi(\{j_1,j_2\})=(kl)$. Then $G_{1,2,3,4,i_1i_2}$ and $G_{1,2,3,4,k,l}$ have a Sylow 2-subgroup P_1 in common and $G_{1,2,3,4,j_1,j_2}$ and $G_{1,2,3,4,k,l}$ have a Sylow 2-subgroup P_2 in common. Therefore we have $P_1=P_2$ and $\{j_1,j_2\}\subset I(a)\cap I(P_1)=\{i_1, i_2, i_3\}$. Thus we have that each inverse image of φ consists of three subsets of $I(a)-\{1,2\}$ and hence the number of 2 cycles of a different from (34) is $\frac{1}{3}{}_{r-2}C_2$.

In this way, $n=2+r+2\times\frac{1}{3}{}_{r-2}C_2$.

(ii) $\alpha_2(x)\geq 1$, and $\alpha_2(x)=\frac{1}{6}s(s-1)$.

Proof. We may assume x is of the following form:

$$x=(1\ 2\ 3\ 4)\,(i_1)\,(i_2)\cdots.$$

Let P be a Sylow 2-group of $G_{1,2,3,4,i_1,i_2}$ and set $I(P)=\{1, 2, 3, 4, i_1, i_2, i_3, k, l\}$. Then x normalizes $G_{1,2,3,4,i_1,i_2}$ and hence P. Therefore x induces an even permutation on $I(P)$ as follows.

$$x^{I(P)}=(1\ 2\ 3\ 4)\,(i_1)\,(i_2)\,(i_3)\,(kl)\,.$$

Now $I(P)$ is uniquely determined by $G_{1,2,3,4,i_1,i_2}$, therefore $\{i_1, i_2\}$ determines uniquely a 2 cycle (kl) of x, and we have the map

$$\varphi\colon\ \{i_1, i_2\}\rightarrow(kl)$$

from the family of the subsets of $I(x)$ consisting of two letters into 2 cycles of x. It is easy to see φ is onto and that each inverse image of φ consists of three subsets of $I(x)$. Hence $\alpha_2(x)=\frac{1}{3}{}_sC_2=\frac{1}{6}s(s-1)$.

(iii) $\alpha_2(x)$ is also equal to $1+\frac{1}{6}(s-2)(s-3)$.

Proof. We take the same x as in (ii) and assume that x is of the following form:

$$x=(5\ 6)\,(7)\,(8)\cdots.$$

We remark that x has at least one 2 cycle and three fixed letters by (ii).

(α) Case $s\geq 4$.

Set $I(x)=\{7, 8, i_1, i_2, \cdots\}$. Let P be a Sylow 2-group of $G_{5,6,7,8,i_1i_2}$ and set $I(P)=\{5, 6, 7, 8, i_1, i_2, i_3, k, l\}$. Then x normalizes $G_{5,6,7,8,i_1,i_2}$ and hence P. Therefore x induces an even permutation on $I(P)$ as follows.

$$x^{I(P)}=(5\ 6)\,(7)\,(8)\,(i_1)\,(i_2)\,(i_3)\,(kl)\,.$$

Thus the set $\{i_1, i_2\}$ determines uniquely a 2 cycle (kl) of x and we have the map

$$\varphi\colon\ \{i_1, i_2\}\rightarrow(kl)$$

from the family of the subsets of $I(x)$-$\{7, 8\}$ consisting of two letters into 2 cycles of x different from (56). Then φ is onto and each inverse image of φ consists of three subsets of $I(x)$-$\{7,8\}$. Hence $\alpha_2(x)=1+\frac{1}{3}{}_{s-2}C_2=1+\frac{1}{6}(s-2)(s-3)$.

(β) Case s=3.

The equality holds by (ii) in this case.

Now by (ii) and (iii) we have $\frac{1}{6}s(s-1)=1+\frac{1}{6}(s-2)(s-3)$ and hence, $s=3$, $\alpha_2(x)=1$. Since x^2 is an involution which fixes at least four letters, we have $r=s+2\alpha_2(x)=3+2=5$. Then $n=9$ by (i).

REMARK. Theorem 1 follows from the assumption that $|I(P)|=9$ and $I(P)$ is fixed by $G_{1,2,3,4,5,6}$ (see [4], p. 322–p. 324). For the combinatorial analysis which is used in the proof of Theorem 1, the author is indebted to H. Nagao and T. Oyama. He is grateful to them for their helpful advices.

Proof of Theorem 2

Let G satisfy the assumption of the theorem. $G_{1,2,3,4,5}$ fixes exactly five letters. Since a Sylow 2-group P' of $G_{1,2,3,4,5}$ is normal in $G_{1,2,3,4,5}$ and $G_{1,2,3,4,5}$ is transitive on $\Omega-\{1, 2, 3, 4, 5\}$, P' can not fix more than five letters. Thus the degree n of G must be odd. Now assume that G is not S_7 or A_9. Set $P=P'\cap G_{1,2,3,4,5,6}$. Then $P\neq 1$ by Lemma 1, and P is a normal Sylow 2-group of $G_{1,2,3,4,5,6}$. P fixes seven or nine letters by Lemma 3, since n is odd. But P can not fix seven letters since $G_{1,2,3,4,5,6}\triangleright P$ and $|I(G_{1,2,3,4,,5,6})|=6$ by Lemma 2. Hence P fixes nine letters and G must be A_9 by Theorem 1. This is a contradiction.

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