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# ON SPECIAL TYPE OF HEREDITARY ABELIAN CATEGORIES

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In the book of Mitchell [5] he has defined a category of a commutative diagrams over an abelian category $\mathfrak{A}$. Especially he has developed this idea to a finite commutative diagrams and obtained many interesting results on global dimension of this diagram. Among them he has shown in [5], p. 237, Corollary 10. 10 that if $I$ is a linearly ordered set, then gl dim $[I, \mathfrak{A}]=1+$gl dim $\mathfrak{A}$ for an abelian category $\mathfrak{A}$ with projectives. This is a generalization of Eilenberg, Rosenberg and Zelinsky [1], Theorem 8.

On the other hand, the author has studied a semi-primary hereditary ring and shown that it is a special type of generalized triangular matrix ring in [2].

In this note we shall generalize the notion of a generalized triangular matrix ring to an abelian category of generalized commutative diagram $[I, \mathfrak{A}_i]$ over abelian categories $\mathfrak{A}_i$ and obtain the similar results in it to [2], Theorem 1, where $I$ is a finite linearly ordered set. The method in this note is quite similar to [5], IX, §10 and different from that of [2]. Finally we shall show that if the $\mathfrak{A}_i$ are the abelian category of right $R_i$-modules, then $[I, \mathfrak{A}_i]$ is equivalent to a generalized triangular matrix ring over $R_i$ in [2], where $R_i$ is a ring.

The author has shown many applications of generalized triangular matrix ring to semi-primary rings with suitable conditions in [2], [3] and [4]. However we do not study any applications of our results in this note and he hopes to continue this work on some other day.

## 1. Abelian categories of generalized commutative diagrams

Let $I=\{1, 2, \cdots, n\}$ be a linearly ordered set and $\mathfrak{A}_i$ be abelian categories. We consider additive covariant functors $T_{ij}$ of $\mathfrak{A}_i$ to $\mathfrak{A}_j$ for $i<j$. For objects $A_i\in\mathfrak{A}_i$, $A_j\in\mathfrak{A}_j$ we define an arrow $D_{ij}$: $A_i\to A_j$ as follows:

$$(1) \qquad D_{ij} = d_{ij}T_{ij}, \qquad \text{where } d_{ij} \text{ is a morphism in } \mathfrak{A}_j.$$

Using those $D_{ij}$ we can define a category $[I, \mathfrak{A}_i]$ of diagrams over $\{\mathfrak{A}_i\}_{i\in I}$. Namely, the objects of $[I, \mathfrak{A}_i]$ consist of sets $\{A_i\}_{i\in I}$ with $D_{ij}(A_i\in\mathfrak{A}_i)$ and the morphism of $[I, \mathfrak{A}_i]$ consist of sets $(f_i)_{i\in I}(f_i\in\mathfrak{A}_i)$ such that

$$(2) \qquad d'_{ij}\mathrm{T}_{ij}(f_i) = f_j d_{ij}\,,$$

where $f_i$: $A_i A'_i$ and $\mathrm{D}_{ij}=d_{ij}\mathrm{T}_{ij}$, $\mathrm{D}'_{ij}=d'_{ij}\mathrm{T}_{ij}$ are arrows in $\boldsymbol{A}=(A_i)$ and $\boldsymbol{A}'=(A'_i)$, respectively.

Let $\boldsymbol{f}=(f_i)_{i\in I}$ be a morhphism of $\boldsymbol{A}$ to $\boldsymbol{A}'$. Then we define a set $(\mathrm{Im}\, f_i)$, $(\mathrm{coker}\, f_i)$ and so on. If $(\mathrm{Im}\, f_i)$, $(\mathrm{coker}\, f_i)\cdots$coincide with $\mathrm{Im}\, \boldsymbol{f}$, coker $\boldsymbol{f}\cdots$in $[I, \mathfrak{A}_i]$, respectively, we shall call $[I, \mathfrak{A}_i]$ a category induced naturally from $\mathfrak{A}_i$.

**Proposition 1.1.** *Let $I$ and $\mathfrak{A}_i$ be as above. $[I, \mathfrak{A}_i]$ is an abelian category induced naturally from $\mathfrak{A}_i$ if and only if $T_{ij}$ is cokernel preserving.*

Proof. We assume that $\mathrm{T}_{ij}$ is cokernel preserving. Let $f=(f_i)_{i\in I}$: $(A_i)\to(A'_i)$ be a morphism in $\mathfrak{A}=[I, \mathfrak{A}_i]$. Then we can easily see that $(\ker f_i)_{i\in I}$ is Kerf in $\mathfrak{A}$ and that $(\mathrm{coker}\, f_i)_{i\in I}$ is in $\mathfrak{A}$ since $\mathrm{T}_{ij}$ is cokernel preserving. Hence, we know from [1], p. 33, Theorem 20.1 that $\mathfrak{A}$ is an abelian category. Conversely, we assume $\mathfrak{A}$ is an abelian category as above. We may assume $I=(1, 2)$. Let $f$: $A_1\to C_1$ be an epimorphism in $\mathfrak{A}_1$ and $\mathrm{B}_2=\mathrm{im}\,\mathrm{T}(f)$, where $\mathrm{T}=\mathrm{T}_{1,2}$. Put $A=(A_1, \mathrm{T}(A_1))$ $C=(C_1, \mathrm{T}(C_1))$ and $\boldsymbol{f}=(f, \mathrm{T}(f))$. Then $\mathrm{Im}\,\boldsymbol{f}=(C_1, B_2)$, $(\boldsymbol{f}$: $A \xrightarrow{\boldsymbol{f}'} \mathrm{Im}\,\boldsymbol{f} \xrightarrow{i} C)$. By the assumption $\boldsymbol{f}'$ and $i$ are morphisms in $\mathfrak{A}$. Hence, there exists an morphism $d$: $\mathrm{T}(C_1)\to B_2$ in $\mathfrak{A}_2$ such that $d\mathrm{T}$ is an arrow in $\mathrm{im}\,\boldsymbol{f}$. Namely

$$(3) \qquad \begin{array}{ccc} \mathrm{T}(A_1) & \xrightarrow{\mathrm{T}(f)} & \mathrm{T}(C_1) \\ \Big\Downarrow d_{12} & & \Big\downarrow d \\ \mathrm{T}(A_1) & \xrightarrow{\boldsymbol{f}'_2} & B_2 \end{array}$$

is commutative, where $i\boldsymbol{f}'_2=\mathrm{T}(f)$.

Therefore, $\boldsymbol{f}'_2=dT(f)=di\boldsymbol{f}'_2$. Since $\boldsymbol{f}'_2$ is epimorphic $di=\mathrm{I}_{B_2}$. On the other hand, we obtain similarly from an morphism $i$ that $id=\mathrm{I}_{T(C_1)}$. Hence, $d$ is isomorphic and T is an epimorphic functor. Let $A''_1 \xrightarrow{g} A_1 \xrightarrow{f} A_1/\mathrm{g}(A''_1)\to 0$ be exact and $B''_2=\mathrm{im}\,\mathrm{T}(g)$. Put $A=(A''_1, B''_2)$, $\mathrm{C}=(A_1, \mathrm{T}(A_1))$, and $f=(g, i)$, where $\mathrm{T}(g)$: $\mathrm{T}(A'')\to B''_2 \xrightarrow{\mathrm{i}} \mathrm{T}(A_1)$. From the assumption coker $f=(A_1/\mathrm{g}(A''_1), \mathrm{T}(A_1)/B''_2)$. Hence there exists $d$: $\mathrm{T}(A_1/\mathrm{g}(A''_1))\to\mathrm{T}(A_1)/B''_2$ such that $d\mathrm{T}(f)=h$, where $h=\mathrm{coker}\,(B''_2 \xrightarrow{i} \mathrm{T}(A_1))$, (cf. (3)). Hence, $\ker \mathrm{T}(f)\subseteqq B''_2$. $B''_2\subseteq \mathrm{Ker}\,\mathrm{T}(f)$ is clear, since $fg=0$. Therefore, T is cokernel preserving.

From this proposition we always assume that $\mathrm{T}_{ij}$ is cokernel preserving.

We shall define functors $\mathrm{T}_i$: $\mathfrak{A}\to\mathfrak{A}_i$ and $\tilde{S}_i$: $\mathfrak{A}_i\to\mathfrak{A}$ as follows:

Let $A=(A_i)_{i\in I}$

$$(4) \qquad \begin{aligned} \mathrm{T}_i(A) &= A_i \\ \mathrm{T}_j\tilde{S}_i(A_i)) &= 0 \qquad \text{for} \quad \mathrm{j} < \mathrm{i}\,, \end{aligned}$$

$$\mathrm{T}_j\tilde{S}_i(A_i)=\sum_{i<i_1<\cdots<i_k<j}\oplus\mathrm{T}_{i_kj}\mathrm{T}_{i_{k-1}i_k}\cdots\mathrm{T}_{ii_1}(A_i)\qquad\text{for}\quad \mathrm{i}<\mathrm{j}\,,$$

with arrow $\mathrm{D}_{ik}=\mathrm{T}_{jk}$ for j<k.

Then we have a natural equivalence $\eta$: $[\tilde{\mathrm{S}}_i(\mathrm{A}_i),\ \mathrm{D}]\approx[\mathrm{A}_i,\ \mathrm{T}_i(\mathrm{D})]$ for any $\mathrm{A}_i\in\mathfrak{A}_i$ and $\mathrm{D}\in\mathfrak{A}$. Hence, we have from [5], p. 138, Coro. 7.4.

**Proposition 1.2.** *We assume that each $\mathfrak{A}_i$ has a projective class $\mathcal{E}_i$, and $T_{ij}$ is cokernel preserving. Then $\cap T_i^{-1}(\mathcal{E}_i)$ is a projective class in $\mathfrak{A}=[I.\ \mathfrak{A}_i]$, whose projectives are the objects of the form $\bigoplus_{i\in I}\tilde{S}_i(P_i)$ and their retracts, where $P_i$ is $\mathcal{E}_i$-projective for all $i\in I$.*

## 2. Commutative diagrams with special arrows

In the previous section we study a general case of abelian categories of commutative diagrams. However, it is too general to discuss them. Hence, we shall consider the following conditions:

[I] *$T_{ij}$ is cokernel preserving.*

[II] *There exist natural transformations*

$$\psi_{ijk}\colon \mathrm{T}_{jk}\mathrm{T}_{ij}\to\mathrm{T}_{ik}\qquad\textit{for any}\quad i<j<k\,.$$

[III] *For any $i<j<k<l$ and $N$ in $A_i$*

$$\begin{array}{ccc}\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(N) & \xrightarrow{\mathrm{T}_{kl}(\psi_{ijk})} & \mathrm{T}_{kl}\mathrm{T}_{ik}(N)\\ \downarrow{\psi_{jkl}} & & \downarrow{\psi_{ikl}}\\ \mathrm{T}_{jl}\mathrm{T}_{ij}(N) & \xrightarrow{\psi_{ijl}} & \mathrm{T}_{il}(N)\end{array}$$

*is commutative*

[IV] *For arrows $d_{ij}$: $\mathrm{T}_{ij}(A_i)\to A_j$ in $\mathfrak{A}=[I,\ \mathfrak{A}_i]$*

$$\begin{array}{ccc}\mathrm{T}_{jk}\mathrm{T}_{ij}(A_i) & \xrightarrow{\mathrm{T}_{jk}(d_{ij})} & \mathrm{T}_{jk}(A_j)\\ \downarrow{\psi_{ijk}} & & \downarrow{d_{ik}}\\ \mathrm{T}_{ik}(A_i) & \xrightarrow{d_{ik}} & A_k\end{array}$$

is *commutative.*

From now on we always assume I, II and for any arrows in $\mathfrak{A}$, we require the condition IV.

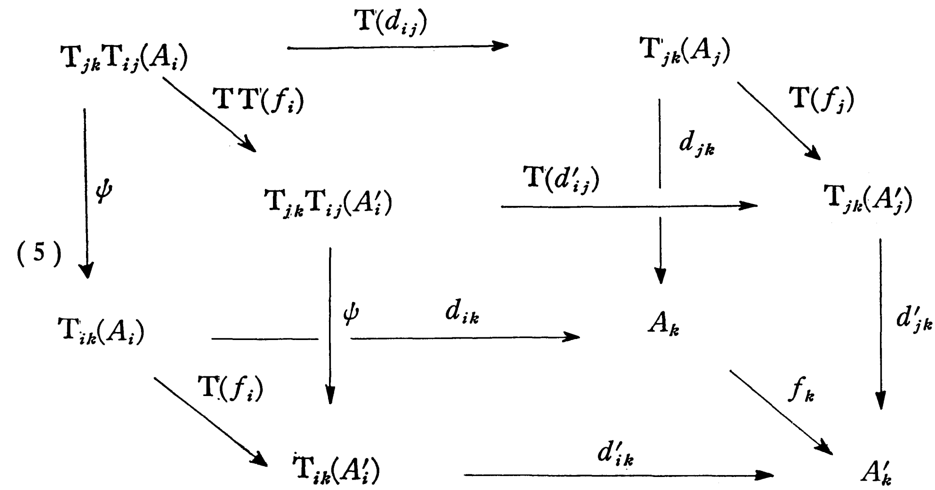
We note that IV implies $\mathrm{D}_{jk}\mathrm{D}_{ij}(A_i)\subseteqq\mathrm{D}_{ik}(A_i)$ for any $A=(A_i)_{i\in I}$ in $\mathfrak{A}$.

First we shall show that $\mathfrak{A}$ is still an abelian category under the assumption I even if we require IV in $\mathfrak{A}$.

**Proposition 2.1.** *Let $(\mathfrak{A}_i)_{i\in I}$ be abelian categories. We assume II. Then*

*$\mathfrak{A}=[I, \mathfrak{A}_i]$ requiring IV is abelian if and only if I is satisfied.*

Proof. Let $f=(f_i)$: $(A_i)\to(A'_i)$ in $\mathfrak{A}$. We consider a diagram




We only prove from Proposition 1.1 that for any morphism $g=(g_i)$, $(\ker g_i)_{i\in I}$ $(\operatorname{coker} g_i)_{i\in I}$ satisfy IV. Put $A_i=\ker g_i$ and $f_i=$inclusion morphism in the above. Then all squares except the rear in (5) are commutative from II, IV and (2). Since $f_k$ is monomorphic, the rear one is commutative. Which shows $(\ker g_i)_{i\in I}$ satisfies IV. Similarly if $A_i=(\operatorname{coker} g_i)$ and $f_i$ epimorphism of cokernel, then $(\operatorname{coker} g_i)$ satisfies IV, since $T_{jk}T_{ij}(f_i)$ is epimorphic from I.

Next, we shall define functors similarly to $\tilde{S}_i$. For $A_i\in\mathfrak{A}_i$ we put

$$(6)\qquad \begin{aligned} &S_i(A_i)=(0, 0, \cdots, A_i, T_{ii+1}(A_i), \cdots, T_{in}(A_i)) \text{ with arrows}\\ &D_{tk}=0 \qquad \text{for } t<i\\ &D_{ik}=T_{ik} \qquad \text{for } k>i\\ &D_{jk}=\psi_{ijk}T_{jk} \qquad \text{for } k>j>i\,. \end{aligned}$$

If $T_{ij}$'s satisfy III, then $S_i(A_i)$ is an object in $[I, \mathfrak{A}_i]$ requiring IV. Furthermore, we can prove easily $[S_i(A_i), D]\approx[A_i, T_i(D)]$ for $D\in[I, \mathfrak{A}_i]$. Hence, we have similarly to Proposition 1.2

**Proposition 1.2′.** *We assume that each $\mathfrak{A}_i$ has a projective class $\mathcal{E}_i$ and I~III are satisfied. Then $\mathfrak{A}=[I, \mathfrak{A}_i]$ requiring IV has a projective class $\cap T_i^{-1}(\mathcal{E}_i)$ whose projectives are the objects of the form $\bigoplus_{i\in I} S_i(P_i)$ and their retracts, where $P_i$ is $\mathcal{E}_i$-projective for all $i\in I$.*

In the rest of the paper we always assume that $[I, \mathfrak{A}_i]$ is an abelian category of the commutative diagrams whose arrows are required IV and that I$\sim$III are satisfied.

**Proposition 2.2.** $(D_{kl}D_{jk})D_{ij}=D_{kl}(D_{jk}D_{ij})$ *for* $i<j<k<l$.

Proof.
$$\begin{aligned}(\mathrm{D}_{kl}\mathrm{D}_{jk})\mathrm{D}_{ij}(A) &= d_{jl}\psi_{jkl}(\mathrm{T}_{kl}\mathrm{T}_{jk})(d_{ij})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A)\\ &= d_{jl}\mathrm{T}_{jl}(d_{ij})\psi_{jkl}\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \quad \text{(naturality of } \psi)\\ &= d_{il}\psi_{ijl}\psi_{jkl}\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \quad \text{(IV)}\\ &= d_{il}\psi_{ikl}\mathrm{T}_{kl}(\psi_{ijk})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \quad \text{(III)}\\ &= d_{kl}\mathrm{T}_{kl}(d_{ik})\mathrm{T}_{kl}(\psi_{ijk})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \quad \text{(IV)}\\ &= d_{kl}\mathrm{T}_{kl}(d_{ik}\psi_{ijk})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A)\\ &= \mathrm{D}_{kl}(\mathrm{D}_{jk}\mathrm{D}_{ij})(A) \qquad \text{for any } A\in\mathfrak{A}_i.\end{aligned}$$

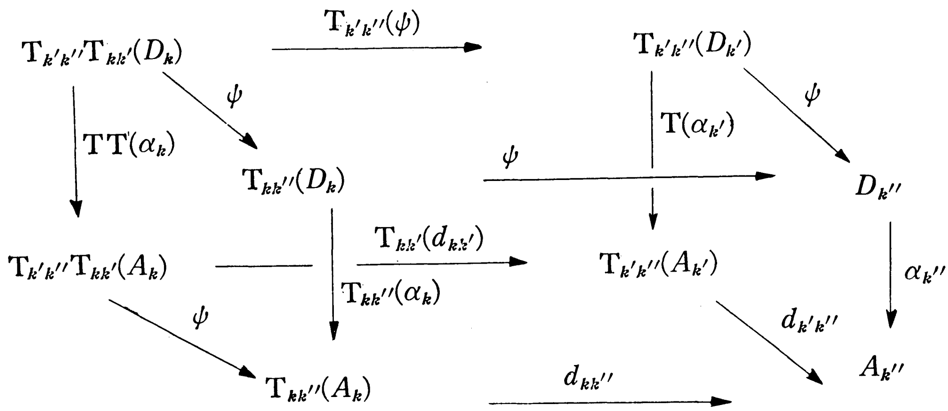
**Theorem 2.3.** (cf. [1], p. 234, Lemma 9.3) *Let* $I=I_1\cup I_2$ *and* $I_1=\{1, 2\cdots, i-1\}$, $I_2=\{i,\cdots,n\}$. *Then* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{A}'=[(1, 2), [I_1\mathfrak{A}_k], [I_2, \mathfrak{A}_{k'}]]$ *with a suitable functor* $\boldsymbol{T}_{12}$: $[I_1, \mathfrak{A}_k]\to[I_2, \mathfrak{A}_{k'}]$.

Proof. First we define a functor $\boldsymbol{T}_{12}$. Let $\boldsymbol{A}_1=(A_i)_{i\in I_1}$. For any $k\geqslant i$ we consider a diagram $\mathrm{D}_k=\{\mathrm{T}_{lk}(A_l),\ \mathrm{T}_{l'k}\mathrm{T}_{ll'}A_l)$ for $l<l'<i<k$ with arrows $\mathrm{T}_{l'k}\mathrm{T}_{ll'}(A_l)\xrightarrow{\psi}\mathrm{T}_{lk}(A_l)$ and $\mathrm{T}_{l'k}\mathrm{T}_{ll'}(A_l)\xrightarrow{\mathrm{T}_{l'k}(d_{ll'})}\mathrm{T}_{l'k}A_{l'})\}$. $\mathrm{D}_k$ has a colimit $A_k$ in $\mathfrak{A}_k$ by [1], p. 46, Coro. 2.5, $(\{D_k\}\xrightarrow{\alpha_k}A_k)$. Put $\boldsymbol{A}_2=(A_i,\cdots,A_n)$. We shall show that $\boldsymbol{A}_2$ is in $[I_2, \mathfrak{A}_{k'}]$. We have to define $\mathrm{D}_{kk'}$ for $i\leq k<k'$. Consider a diagram

$$(7)\qquad \left.\begin{array}{ccc} \mathrm{T}_{kk'}\mathrm{T}_{lk}(A_l) & \xrightarrow{\psi_{lkk'}} & \mathrm{T}_{l'k'}(A_l)\\ \uparrow{\scriptstyle \mathrm{T}(\psi)} & & \uparrow{\scriptstyle \psi}\\ \mathrm{T}_{kk'}\mathrm{T}_{l'k}\mathrm{T}_{ll'}(A_l) & \xrightarrow{\psi_{lkk'}} & \mathrm{T}_{l'k'}\mathrm{T}_{ll'}(A_l)\\ \downarrow{\scriptstyle \mathrm{T}_{k'k}\mathrm{T}_{ll'}(d_{ll'})} & & \downarrow{\scriptstyle \mathrm{T}_{l'k'}(d_{ll'})}\\ \mathrm{T}_{kk'}\mathrm{T}_{l'k}(A_{l'}) & \xrightarrow{\psi_{l'kk'}} & \mathrm{T}_{l'k}(A_{l'}) \end{array}\right\}\to A_{k'}$$

The upper and lower squares are commutative by III and naturality of $\psi$, respectively. Then (7) implies that these exist compatible morphism: $\{\mathrm{T}_{kk'}(D_k)\}\to A_{k'}$. Since $\mathrm{T}_{kk'}$ is colimit preserving by [5], p. 55. Proposition 6.4, we have a unique morphism $d_{kk'}$: $\mathrm{T}_{kk'}(A_k)\to A_{k'}$. Hence we can define $\mathrm{D}_{kk'}=d_{kk'}\mathrm{T}_{kk'}$. Next we show that those $\mathrm{D}_{kk'}$ satisfy IV. For $i\leqslant k<k'<k''$ we have a diagram

(8)




All squares except bottom are commutative by III and the definitions $d_{kk'}$, $d_{kk''}$ and $d_{k'k''}$. On the other hand, it is clear that $\varphi_k$: $T_{k'k''}T_{kk'}(D_k) \xrightarrow{TT(\alpha_k)} T_{k'k''}T_{kk'}(A_k) \xrightarrow{\psi} T_{kk''}(A_k) \xrightarrow{d_{kk''}} \mathfrak{A}_{k''}$ is compatible. Since $T_{k'k''}T_{kk'}$ is colimit preserving, we have a unique morphism $\Phi$: $T_{k'k''}T_{kk'}(A_k) \to A_{k''}$ such that $\psi_k = \Phi TT(\alpha_k)$. Therefore, the bottom square is also commutative, which means II. Thus we have shown that $\boldsymbol{T}_{12}$ is a functor. Let $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ be in $\mathfrak{A}'$, where $\boldsymbol{A}_1 = (A_i)_{i \in I_1}$ and $\boldsymbol{A}_2 = (B_j)_{j \in I_2}$. From the definition of $\boldsymbol{T}_{12}$ we have a morphism: $T_{jk}(A_j) \xrightarrow{\alpha_k} A_k \xrightarrow{d_k} B_k$ for $j \in I_1$, $k \in I_2$, where $(d_i)_{i \in I}$: $\boldsymbol{T}_{12}(\boldsymbol{A}_1) \to \boldsymbol{A}_2$. We put

$$D'_{jk} = d_k \alpha_k T_{jk} \qquad \text{for} \quad j<i<k \text{ and}$$
$$D'_{st} = D_{st} \qquad \text{for} \quad s, t \in I_1 \quad \text{or} \quad T_2 .$$

We shall show that $D'_{ij}$ satisfy IV. Take $j<h<k$. If $j \in I_2$ or $k \in I_1$, then it is obvious. We assume $j \in I_1$ and $h$, $k \in I_2$. Then we have

$$(9) \qquad \begin{array}{ccccc} T_{hk}T_{jh}(A_j) & \xrightarrow{T(\alpha_h)} & T_{hk}(A_h) & \xrightarrow{T(d_h)} & T_{hk}(B_h) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle d_{hk}} & & \downarrow{\scriptstyle d'_{hk}} \\ T_{jk}(A_j) & \xrightarrow{\alpha_k} & A_k & \xrightarrow{d_k} & B_k \end{array}$$

where $d'_{hk}$ is a given morphism in $\boldsymbol{A}_2$. The left side is commutative by the definition of $\boldsymbol{T}_{12}$ and so is the right side, since $h$, $k \in I_2$. Hence, the out side square means IV. We can easily see by the defininition of $\{D_k\}$ that IV is satisfied for $j$, $h \in I_1$ and $k \in I_2$. Hence, $\boldsymbol{T}(\boldsymbol{A}_1, \boldsymbol{A}_2) = (A_1, \cdots, A_{i-1}, B_i, \cdots, B_n)$ is an object in $\mathfrak{A}$. Conversely, for $\boldsymbol{A} = (A_1, \cdots, A_n)$ we put $\boldsymbol{S}(\boldsymbol{A}) = ((A_1, \cdots, A_{i-1}), (A_i, \cdots, A_n))$. Then it is clear that $\boldsymbol{S}(\boldsymbol{A}) \in \mathfrak{A}'$ and $\boldsymbol{TS} = I_{\mathfrak{A}}$, $\boldsymbol{ST} = I_{\mathfrak{A}'}$. This shows that $\boldsymbol{T}_{12}$ is cokernel preserving by Proposition 1.1.

## 3. Hereditary categories

In this section, we always assume that I$\sim$IV are satisfied and every $\mathfrak{A}_i$ has projectives and hence $\mathfrak{A}=[I, \mathfrak{A}_i]$ has projectives by Proposition 1.2′.

If every object in an abelian category $\mathfrak{B}$ is projective, we call $\mathfrak{B}$ a semi-simple category, which is equivalent to a fact gl dim $\mathfrak{B}=0$. If gl dim $\mathfrak{B}\leqq 1$ we call $\mathfrak{B}$ hereditary.

**Proposition 3.1.** ([5], p. 235, Coro. 10.3). *We assume that $\mathfrak{A}_i$ has projectives and that $T_{ij}$ is projective preserving. Let $D=(D_i)_{i\in I}$ be an object in $[I, \mathfrak{A}_i]$* and $m=\max\,(hd\ D_i)$, *$n=$the number of elements of $I$. Then $hd\ D\leqslant n+m-1$.*

Since $\mathrm{T}_{ij}$ is projective preserving, we can prove it similarly to [1], p. 235.

**Corollary.** *Let $I=(1,2)$ and $T_{12}$ be projective preserving. Then*

$$\max\,(\mathrm{gl\ dim}\ \mathfrak{A}_1,\ \mathrm{gl\ dim}\ \mathfrak{A}_2)\leqq \mathrm{gl\ dim}\ [(1,\ 2),\ \mathfrak{A}_1,\ \mathfrak{A}_2]\leqslant \max\,(\mathrm{gl\ dim}\ \mathfrak{A}_i)+1\ .$$

Proof. The right side inequality is clear from Proposition 3.1. Let $A$ be an object in $\mathfrak{A}_1$. It is clear that $\mathrm{hd}(A, 0)\geqq \mathrm{hd}\ A$. Since $\mathrm{T}_{12}$ is projective preserving, we have similarly $\mathrm{hd}(0, A')\geqq \mathrm{hd}A'$ for $A'\in\mathfrak{A}_2$.

**Lemma 3.2.** *Let $\mathfrak{A}=[(1, 2), \mathfrak{A}_1, \mathfrak{A}_2]$. If* gl dim $\mathfrak{A}\leqq 1$, *then $T_{12}$ is projective preserving.*

Proof. Let $P_1$ be projective in $\mathfrak{A}_1$. Then $(P_1, \mathrm{T}_{12}(P_1))$ is projective in $\mathfrak{A}$ by Proposition 1.2. Let $0\leftarrow \mathrm{T}_{12}(P_1)\leftarrow Q$ be an exact sequence in $\mathfrak{A}_2$ with $Q$ projective. Then $(0.\ 0)\leftarrow(P_1, 0)\leftarrow(P_1, \mathrm{T}_{12}(P_1))\leftarrow(0, Q)$ is exact in $\mathfrak{A}$. Since gl dim $\mathfrak{A}\leqq 1$, $(0, \mathrm{T}_{12}\ (P_1))$ is projective in $\mathfrak{A}((0, \mathrm{T}_{12}\ (P_1))\subset(P_1, \mathrm{T}_{12}(P_1))$. Hence, $\mathrm{T}(P_1)\leftarrow Q$ is retract and $\mathrm{T}_{12}(P_1)$ is projective in $\mathfrak{A}_2$.

Similarly to the category of modules we have

**Lemma 3.3.** *Let $A$ be an abelian category. If $A\oplus B=A'\oplus C$ and $A\supset A'$, then $A'=A\oplus A''$, $A''=A\cap C$ and $C=A''\oplus C'$.*

**Lemma 3.4.** *Let $I=(1, 2)$ and $\mathfrak{A}=[I, \mathfrak{A}_i]$. If $T_{12}$ is projective preserving, then every projective object $A$ in $\mathfrak{A}$ is of a form $(P_1, T_{12}(P_1)\oplus P_2)$ and the arrow $d_{12}$ in $A$ is monomorphic, where $P_i$ is projective in $\mathfrak{A}_i$.*

Proof. Since $\boldsymbol{A}=(A_1, A_2)$ is a retraction of an object of a form $\boldsymbol{P}=(P_1, \mathrm{T}_{12}(P_1)\oplus P_2)$ with $P_i$ projective in $\mathfrak{A}_i$. Hence, $0\to\boldsymbol{A}\to\boldsymbol{P}$ splits. Let $P_1=A_1\oplus Q_1$. Then $\mathrm{T}_{12}(P_1)=\mathrm{T}_{12}(A_1)\oplus\mathrm{T}_{12}(Q_1)$ and $A_2$ is a coretract of $\mathrm{T}_{12}(A_1)\oplus\mathrm{T}_{12}(Q_1)\oplus P_2$. Furthermore, $\mathrm{T}_{12}(A_1)\xrightarrow{d_{12}} A_2\to\mathrm{T}_{12}(P_1)\oplus P_2=\mathrm{T}_{12}(A_1)\to\mathrm{T}_{12}(P_1)\oplus P_2$, and the right side is monomorphic. Hence, $d_{12}$ is monomorphic. Thus we may assume $T_{12}(A_1)\subset A_2\subset T_{12}(P_1)\oplus P_2$. Therefore, $A_2=T_{12}(A_1)\oplus A_2'$ by Lemma 3.3. Since $P_1$ is projective and $T_{12}$ is projective preserving, $T_{12}(P_1)\oplus P_2$ is projective in $\mathfrak{A}_2$. Hence, $A_2'$ is projective by Lemma 3.3.

**Lemma 3.5.** *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be hereditary and $T_{12}$ projective preserving. If $T_{12}(P_2)$ is a coretract of $T_{12}(P_1)$ for any projective objects $P_1\supset P_2$ in $\mathfrak{A}_1$, then $\mathfrak{A}=[(1, 2), \mathfrak{A}_1, \mathfrak{A}_2]$ is hereditary.*

Proof. Let $(A_1, A_2)$ be any object in $\mathfrak{A}$ and $0\leftarrow(A_1, A_2)\overset{f}{\leftarrow} P$ be exact, where $P$ $\mathfrak{A}$-projective. Then $P=(P_1, T_{12}(P_1)\oplus P_2)$ with $P_i$ projective by Lemma 3.4. Put $\ker f=(K_1, K_2)$. Since $\mathfrak{A}_1$ is hereditary, $K_1$ is projective. Hence, $T_{12}(K_1)$ is a coretract of $T_{12}(P_1)$ by the assumption. Hence, $K_2=T_{12}(K_1)\oplus K_2'$ by Lemma 3.3. Since $K_2$ is projective, $(K_1, K_2)$ is $\mathfrak{A}$-projective.

**Theorem 3.6.** *Let $I=(1, 2, \cdots, n)$ be a linearly ordered set, $\mathfrak{A}_i$ abelian categories with projectives. Let $\mathfrak{A}=[I, \mathfrak{A}_i]$ be the abelian category of commutative diagrams over $\mathfrak{A}_i$ with functors $T_{ij}$ satisfying $I\sim IV$. If $\mathfrak{A}$ is hereditary, then we have*:

i) *Every projective object of $\mathfrak{A}$ is of a form $\bigoplus_{i\in I} S_i(P_i)$, where $P_i$ is projective in $\mathfrak{A}_i$.*

ii) *$T_{ij}$ is projective preserving for any $i<j$.*

iii) *$T_{ij}(P_2)$ is a coretract of $T_{ij}(P_1)$ for any projective objects $P_1\supset P_2$ in $\mathfrak{A}_i$.*

iv) *$[(i_1, i_2, \cdots, i_t), A_{i_1}, A_{i_2}, \cdots, A_{i_t}]\equiv\mathfrak{A}(i_1, i_2, \cdots, i_t)$ is hereditary for any $i_1<i_2<\cdots<i_t$.*

v) *If $P=(P_i)_{i\in I}$ is projective in $\mathfrak{A}$, then every $d_{ij}$ in $P$ is a coretract. $(P_{i_1}, P_{i_2}, \cdots, P_{i_t})$ is $\mathfrak{A}(i_1, i_2, \cdots, i_t)$-projective.*

Proof. We shall prove the theorem by the induction on the number $n$ of element of $I$. We obtain $\mathfrak{A}\approx[(1,2), \mathfrak{A}_1, \mathfrak{A}(I-1)]\equiv A'$ from Theorem 3.2. Then $\mathfrak{A}(I-1)$ is hereditary by Lemma 3.2 and Corollary to Proposition 3.1. Furthermore, $\boldsymbol{T}_{12}$ in $\mathfrak{A}'$ is projective preserving. i) Let $P=(P_i)_{i\in I}$ be projective in $\mathfrak{A}$. Then $P=(P_1, \boldsymbol{T}_{12}(P_1)\oplus\boldsymbol{P}_2)$ by Lemma 3.4, where $\boldsymbol{P}_2$ is projective in $\mathfrak{A}(I-1)$. We obtain, by the definition of $\boldsymbol{T}_{12}$, that $\boldsymbol{T}_{12}(P_1)=(T_{1i}(P_1))_{i\in I-1}$. Hence, $P=\bigoplus_{i\in I} S_i(P_i)$ by the induction hypothesis. ii) Every component of projective object in $\mathfrak{A}(I-1)$ is projective by the induction. Hence, $T_{1i}(P_1)$ is projective in $\mathfrak{A}_i$. iii) Let $P_1\supset P_2$ be projective in $\mathfrak{A}_1$. Put $A=(P_1/P_2, 0, \cdots, 0)$. Then we have an exact sequence $0\leftarrow A\leftarrow(P_1, \boldsymbol{T}_{12}(P_1))$. Since $\mathfrak{A}$ is hereditary, its kernel $(P_2, \boldsymbol{T}_{12}(P_1))$ is projective. Therefore, $T_{1i}(P_2)$ is a coretract from i). iv) We may show that $\mathfrak{A}(I-i)$ is hereditary for any $i$. $\mathfrak{A}\approx[I_1, i, I_2, \mathfrak{A}_1', \mathfrak{A}_i, \mathfrak{A}_2']$, where $I_1=(1, \cdots, i-1)$, $I_2=(i+1,\cdots, n)$, $\mathfrak{A}_1=\mathfrak{A}(I_1)$ and $\mathfrak{A}_2=\mathfrak{A}(I_2)$. From Lemma 3.2 $\boldsymbol{T}_{13}$ is projective preserving and hence $\mathfrak{A}(I-i)$ is hereditary from iii) and Lemma 3.5 and the definition of $\boldsymbol{T}_{13}$. v) Since $P=(P_1, \boldsymbol{T}_{12}(P_1)\oplus P_2)$, $d_{1i}$: $T_{1i}(P_1)\to P_i$ is a coretract.

$P\approx(\boldsymbol{P}_1', P_2, \boldsymbol{P}_3')$, where $\boldsymbol{P}_1'=(P_j)_{j\in I_1}$ and $P_3'=(P_j)_{j\in I_2}$. Then it is clear from i) and induction that $(\boldsymbol{P}_1', \boldsymbol{P}_3')$ is $\mathfrak{A}(I-i)$-projective.

Next we shall study a condition of every projective objects in $\mathfrak{A}$ being of a form $\oplus \mathrm{S}_i(P_i)$, when $\mathrm{T}_{ij}$ is projective preserving.

**Lemma 3.7.** *Let $\mathfrak{A}$ and $\mathfrak{A}_i$ be as above and $T_{ij}$ projective preserving. If we have*

$$(*)\qquad \mathrm{T}_{ij}(P_i) = \mathrm{T}_{i+1j}\mathrm{T}_{i,i+1}(P_i)\oplus\mathrm{T}_{i+2j}(\mathrm{K}^{i+2}(P_i))\oplus\cdots\oplus\mathrm{T}_{j-1j}(\mathrm{K}^{j-1}(P_i))\oplus\mathrm{K}^j(P_i)$$

*for any projective object $P_i$ in $\mathfrak{A}_i$ for all $i$, then every object $A=(A_i)_{i\in I}$ in $\mathfrak{A}$ is of a form $\oplus S_i(Q_i)$ whenever $A$ is subobject of $P=(Q_i')_{i\in I}$ and $A_i$ is a coretract of $Q_i'$ for all $i$, where $K^j(P_i)$ is an object in $\mathfrak{A}_j$, $Q_i$ and $Q_i'$ are $\mathfrak{A}_i$-projective, and the equality in (*) is given by taking suitable transformation from the right side to the left in (*).*

Proof. We may assume $P=\bigoplus_{i\in I}\mathrm{S}_i(P_i)$ and $P_i$ is $\mathfrak{A}_i$-projective. Put $P=(\boldsymbol{P}_i)_{i\in I}$. From the assumption $\boldsymbol{P}_1=A_1\oplus Q_1$. We shall show the following fact by the induction on $i$.

i) $$A_i = \mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{2i}(K^2)\oplus\cdots\oplus\mathrm{T}_{i-1i}(K^{i-1})\oplus K^i$$

ii) $$K^i\oplus Q_i = P_i\oplus\mathfrak{K}^i(Q_1)\oplus\mathfrak{K}^i(Q_2)\cdots\oplus\mathrm{K}^1(Q_{i-2})\oplus\mathrm{T}_{i-1i}(Q_{i-1}),$$

and this is a coretract of $P_i$, where $\mathrm{K}^i(Q_j)$ is the object in (*) for projective $Q_i$ and the equalities are considered in $P_i$ by suitable imbedding mappings. If $i=1, 2$, i) and ii) are clear (see the proof of Lemma 3.4). We assume i) and ii) are true for $k<i$. Using this assumption we first show for $2<j<i-1$ that

iii) $$\begin{aligned}\boldsymbol{P}_i = {} & \mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{2i}(K^2)\oplus\cdots\oplus\mathrm{T}_{ji}(K^j)\\ & \oplus\mathrm{T}_{j+1i}(P_{j+1}\oplus(\mathrm{K}^{j+1}(Q_1)\oplus\cdots\oplus\mathrm{K}^{j+1}(Q_{j-1})\oplus\mathrm{T}_{jj+1}(Q_j))\\ & \oplus\mathrm{T}_{j+2i}(P_{j+2}\oplus\mathrm{K}^{j+2}(Q_1)\oplus\cdots\oplus\mathrm{K}^{j+2}(Q_{j-1})\oplus\mathrm{K}^{j+2}(Q_j))\\ & \oplus\cdots\cdots\cdots\cdots\\ & \oplus\mathrm{T}_{i-1i}(P_{i-1}\oplus(\mathrm{K}^{i-1}(Q_1)\oplus\cdots\oplus\mathrm{K}^{i-1}(Q_{j-1})\mathrm{K}^{i-1}(Q_j))\\ & \oplus P_i\oplus\mathrm{K}^i(Q_1)\oplus\cdots\oplus\mathrm{K}^i(Q_{j-1})+\mathrm{K}^i(Q_j).\end{aligned}$$

Now
$$\begin{aligned}\boldsymbol{P}_i &= \mathrm{T}_{1i}(P_1)\oplus\mathrm{T}_{2i}(P_2)\oplus\cdots\oplus\mathrm{T}_{i-1i}(P_{i-1})\oplus P_i\\ &= \mathrm{T}_{1i}(P_1)\oplus\mathrm{T}_{2i}(P_2)\oplus\boldsymbol{P}_i' \qquad (\boldsymbol{P}_i' = \mathrm{T}_{3i}(P_3)\oplus\cdots\oplus P_i)\\ &= \mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{1i}(Q_1)\oplus\mathrm{T}_{2i}(Q_1)\oplus\mathrm{T}_{2i}(P_2)\oplus\boldsymbol{P}_i'\\ &= \mathrm{T}_{1i}(A_1)\oplus(\mathrm{T}_{2i}\mathrm{T}_{12}(Q_1)\oplus\mathrm{T}_{3i}(\mathrm{K}^3(Q_1))\oplus\cdots\oplus\mathrm{T}_{i-1i}(\mathrm{K}^{i-1}(Q_1))\\ &\qquad\oplus\mathrm{K}^i(Q_1))\oplus\mathrm{T}_{2i}(P_2)\oplus\boldsymbol{P}_i' \qquad ((*))\\ &= \mathrm{T}_{1i}(A_1)\oplus(\mathrm{T}_{2i}(P_2\oplus\mathrm{T}_{12}(Q_1))\oplus(\mathrm{T}_{3i}(\mathrm{K}^3(Q_1))\oplus\cdots\oplus\mathrm{K}^i(Q_1)+\boldsymbol{P}_i'\\ &= \mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{2i}(K^2)\\ &\qquad\oplus\mathrm{T}_{3i}(P_3\oplus\mathrm{K}^3(Q_1)\oplus\mathrm{T}_{23}(Q_2))\end{aligned}$$

$$\oplus \mathrm{T}_{4i}(P_4 \oplus \mathrm{K}^3(Q_1) \oplus \mathrm{K}^4(Q_2)) \oplus \cdots$$
$$\oplus \mathrm{T}_{i-1i}(P_{i-1} \oplus \mathrm{K}^{i-1}(Q_1) \oplus \mathrm{K}^{i-1}(Q_2))$$
$$\oplus P_i \oplus \mathfrak{K}^i(Q_1) \oplus \mathrm{K}^i(Q_2)\,.$$

This is a case of $j=2$ in iii). We assume iii) is true for $k \leqq j$. Since $j+1<i$, we obtain from ii) and taking $\mathrm{T}_{j+1i}$

$$\mathrm{T}_{j+1i}(K^{j+1}) \oplus \mathrm{T}_{j+1i}(Q_{j+1}) = \mathrm{T}_{j+1i}(P_{j+1} \oplus \mathrm{K}^{j+1}(Q_1) \oplus \mathrm{K}^{j+1}(Q_2) \oplus \cdots \oplus \mathrm{K}^{j+1}(Q_{j-1}) \oplus \mathrm{T}_{jj+1}(Q_j))\,.$$

On the other hand,

$$\mathrm{T}_{j+1i}(Q_{j+1}) = \mathrm{T}_{j+2i}\mathrm{T}_{j+1j+2}(Q_{j+1}) \oplus \mathrm{T}_{j+3i}(\mathrm{K}^{j+3}(Q_{j+1})) \oplus \cdots \oplus \mathrm{T}_{i-1i}(\mathrm{K}^{i-1}(Q_{i+1})) \oplus \mathrm{K}^i(Q_{j+1})$$

Since $Q_{j+1}$ is a coretract of $\boldsymbol{P}_{j+1}$ and $\mathrm{T}_{j+1i}(\boldsymbol{P}_{j+1})$ is a coretract of $\boldsymbol{P}_i$ by the following Lemma 3.8, we may regard the above objects on the both sides as sub objects in $\boldsymbol{P}_i$. Hence, we obtain

$$\begin{aligned}\boldsymbol{P}_i = {} & \mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{ji}(K^i) \oplus \mathrm{T}_{j+1i}(K^{j+1}) \\ & \oplus \mathrm{T}_{j+2i}(P_{j+2} \oplus \mathrm{K}^{j+2}(Q_1) \oplus \cdots \oplus \mathrm{K}^{j+2}(Q_j)) \oplus \mathrm{T}_{j+1j+2}(Q_{j+1})) \oplus \cdots \\ & \oplus \mathrm{T}_{i-1i}(P_i \oplus \mathrm{K}^{i-1}(Q_1) \oplus \cdots \oplus \mathrm{K}^{i-1}(Q_j)) \oplus \mathrm{K}^{i-1}(Q_{j+1})) \\ & \oplus P_i \oplus \mathrm{K}^i(Q_1) \oplus \cdots \oplus \mathrm{K}^i(Q_j) \oplus \mathrm{K}^i(Q_{j+1})\,.\end{aligned}$$

Thus we obtain from i) and ii)

$$\begin{aligned}\boldsymbol{P}_i &= \mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{i-2i}(K^{i-2}) \oplus \mathrm{T}_{i-1i}(P_{i-1} \oplus \mathrm{K}^{i-1}(Q_1) \oplus \cdots \\ &\quad \oplus \mathrm{K}^{i-1}(Q_{i-3}) \oplus \mathrm{T}_{i-2i-1}(Q_{i-2})) \oplus (P_i \oplus \mathrm{K}^i(Q_1)) \oplus \cdots \oplus \mathrm{K}^i(Q_{i-2})) \\ &= \{\mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{i-2i}(K^{i-2}) \oplus \mathrm{T}_{i-1i}(K^{i-1})\} \oplus \{P_i \oplus \mathrm{K}^i(Q_1) \oplus \cdots \\ &\quad \oplus \mathrm{K}^i(Q_{i-2}) \oplus \mathrm{T}_{i-1i}(Q_{i-1})\}\,.\end{aligned}$$

Since $A_l \supset K^l$ and $A_i \supset \mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{i-1i}(K^{i-1}) = A'_i$, we obtain $A_i = A'_i \oplus K^i$ and $Q_i$ in $\mathfrak{A}_i$ such that

$$K^i \oplus Q_i = P_i \oplus \mathrm{K}^i(Q_1) \oplus \cdots \oplus \mathrm{K}^i(Q_{i-2}) \oplus \mathrm{T}_{i-1i}(Q_{i-1})\,,$$

and hence, $K^i \oplus Q_i$ is a coretract of $\boldsymbol{P}_i$. Therefore, $A = \bigoplus_{i \geqslant 2} \mathrm{S}_i(K^i) \oplus \mathrm{S}_1(A_1)$. Since $\mathrm{T}_{ij}$ is projective preserving, each $K^i$ is $\mathfrak{A}_i$-projective.

**Lemma 3.8.** *Let $\mathfrak{A}$ and $\mathfrak{A}_i$ and $T_{ij}$ be as above. We assume that $T_{ij}$ satisfies the condition (*). Then $T_{ij}(P_i)$ is a coretract of $P_j$ for any projective object $P=(P_i)_{i \in I}$.*

Proof. We may assume $P = \bigoplus_{i \in I} \mathrm{S}_i(Q_i)$ by Lemma 3.3, where $Q_i$ is $\mathfrak{A}_i$-projective. Then $P_i=\sum_{i=1}^{k-1}\oplus T_{ki}(Q_k)\oplus Q_i$. We shall show under the assumption of Lemma 3.8 that $T_{jl}T_{ij}(P_i)\xrightarrow{\psi_{ijl}}T_{il}(P_i)$ is a coretract. Let $t=l-i$. If $t=2$, then the fact is clear from (*). We assume it for $t<k$ and $k=l-i$. $T_{jl}T_{ij}(P_i)=T_{jl}T_{i+1j}T_{ii+1}(R_i)\oplus T_{jl}(T_{i+2j}(K^{i+2}(P_i))\oplus\cdots\oplus T_{j-1j}(K^{j-1}(P_i))\oplus K^j(P_j))$ and

$$T_{il}(P_i)=T_{i+1l}T_{ii+l}(P_i)\oplus T_{i+2l}(K^{i+2}(P_i))\oplus\cdots\oplus T_{jl}(K^j(P_i))$$
$$+T_{j+1l}(K^{j+1}(P_i)\oplus\cdots\oplus K^l(P_i)\,.$$

Hence, we obtain $\psi_{ijl}$ is a coretract from the assumption III, naturality of $\psi$ and induction hypothesis. From those facts we can easily prove Lemma 3.8.

**Lemma 3.9.** *Let $\mathfrak{A}_i$ and $\mathfrak{A}$ be as above, and $I'$a subset of $I$. Then there exist functors $M$: $[I',\mathfrak{A}]\to[I,\mathfrak{A}]$, $F$: $[I,\mathfrak{A}]\to[I',\mathfrak{A}_i]$ such that $FM=I_{[I',\mathfrak{A}_i]}$, where $F$ is the restriction functor.*

Proof. We may assume $I=I'\cup\{i\}$ by the induction. Let $I_1=\{j\mid\in I, j<i\}$ $I_2=\{j\mid\in I, j>i\}$ and $A=(A_j)_{j\in I'}$. If $I_1=\phi$, we put $A_1=0$. We assume $I_1=\phi$. We consider a family $D_i=\{T_{ki}(A_k), T_{k'i}T_{kk'}(A_k)\xrightarrow{\psi_{kk'i}}T_{ki}(A_k)$ and $T_{ki}T_{kk'}(A_k)$ $\xrightarrow{T_{ki}(d_{kk'})}T_{ki}(A_k)$ for $k<k'<i\}$. Put $A_i$ is a colimit of $D_i$. Then we have defined arrows $D_{ki}$ and $D_{il}$ for $k\in I_1$, $l\in I_2$ from (7). It is easily seen from the definition of colimit that those $D_{ij}$ satisfy IV. Then $M(A)=(A_k)_{k\in I}$ is a desired functor.

REMARK. We note that if $A=(A_k)$ is a coretract of $B=(B_k)_{k\in I'}$, then $M(A)$ is a coretract of $M(B)$, (cf. [5], p. 47, Coro. 2.10).

**Proposition 3.10.** *Let $\{\mathfrak{A}_i\}_{i\in I}$ be abelian categories with projective class $\mathcal{E}_i$ and $\mathfrak{A}(I)=[I,\mathfrak{A}_i]$. We assume $T_{ij}$ is projective preserving. Then every projective object $P=(P_i)_{i\in I'}$ in $\mathfrak{A}(I')$ is of a form $\bigoplus_{i\in I'}S_i(Q_i)$ with $Q_i$ projective in $\mathfrak{A}_i$ for any subset $I'$ of $I$ and $(P_j)_{j\in I''}$ is $\mathfrak{A}(I'')$-projective for any subset $I''$ of $I'$ if and only if (*) is satisfied.*

Proof. "only if". Let $P_i$ be projective in $\mathfrak{A}$. Then $S_i(P_i)$ is $\mathfrak{A}$-projective, and hence, $\boldsymbol{P}'=(T_{ii+1}(P_i),\cdots,T_{in}(P_i))$ is $\mathfrak{A}(I\text{-}\{1,\cdots,i\}]$-projective. Therefore, the fact $\boldsymbol{P}'=\bigoplus_{k\geqslant i+1}S_k(Q_k)$ from the assumption is equivalent to (*). "if". Let $\boldsymbol{P}'=(P'_k)_{k\in I'}$ be projective in $\mathfrak{A}(I')$. Then $\boldsymbol{P}'$ is a retract of $\bigoplus_{t\in I'}\bar{S}_t(P_t)$, where $P_t$ is $\mathfrak{A}_t$-pojectrive and $\bar{S}_t$ is functor: $\mathfrak{A}_t\to\mathfrak{A}(I')$ in (6). Let M be a functor in Lemma 3.9. Then $M(\bigoplus_{t\in I'}\bar{S}_t(P_t))=\bigoplus_{i\in I'}S_t(P_t)$ from the construction of $M_t$ and $M(P')$ is its retract from the above remark. Hence, $M(P')$ is $\mathfrak{A}$-projective.

Therefore, $\mathrm{M}(P')=\bigoplus_{i\in I}\mathrm{S}_i(Q_i)$ with $Q_i$ projective in $\mathfrak{A}_i$ by Lemma 3.7. Let $I'=\{i_1,\cdots,i_t\}$. We shall show $A_{i_k}=(\mathrm{T}_{i_k'i_k}(Q_{i_k'}))^t{}_{k=k'}=\sum_{k=k'}^{t}\oplus\bar{S}_{i_k}(P''_{i_k'})$, where $\mathrm{T}_{i_k'i_k'}=I_{\mathfrak{A}_{i_k'}}$ and $P''_{i_k}$ is $\mathfrak{A}_{i_k}$-projective. We obtain from Lemma 3.7 that $\mathrm{T}_{i_k'i_{k-1}}(Q_{i_k'})=\mathrm{T}_{i_{t-1}i_k}\mathrm{T}_{i_k'i_{t-1}}(Q_{i_k'})\oplus P'_{i_k}$ and $\mathrm{T}_{i_k'i_{t-1}}(Q_{i_k'})=\mathrm{T}_{i_{t-2}i_{t-1}}\mathrm{T}_{i_k'i_{t-2}}(Q_{i_k'})\oplus P'_{i_{t-1}}$. Hence,

$$\begin{aligned}\mathrm{T}_{i_k'i_t}(Q_{i_k'}) &= \mathrm{T}_{i_{t-1}i_t}\mathrm{T}_{i_{t-2}i_{t-1}}\mathrm{T}_{i_k'i_{t-2}}(Q_{i_k'})\oplus\mathrm{T}_{i_{t-1}i_t}(P'_{i_{t-1}})\oplus P'_{i_t}\\ &= \mathrm{T}_{i_{t-2}i_t}\mathrm{T}_{i_k'i_{t-2}}(Q_{i_k'})\oplus\mathrm{T}_{i_{t-1}i_t}(P'_{i_{t-1}})\oplus P'_{i_t}\end{aligned}$$

from III. Repeating this argument we have $A_{i_k'}=\sum_{k=k'}^{t}\oplus\bar{S}_{i_k'}(P'_{i_k})$. Therefore, $\boldsymbol{P}=\sum_{k=1}^{t}\oplus A_{i_k'}=\bigoplus_{i_k'\in I'}\mathrm{S}_{i_k'}(P'''_{i_k})$. This completes the proof.

**Proposition 3.11.** *Let $\mathfrak{A}$ and $\mathfrak{A}_i$ be as above. We assume $T_{ij}$ is projective preserving and satisfies (*), then for $D=(D_i)_{i\in I}$ in $\mathfrak{A}$*

$$hd\, D\leqq\max\,(hd\, D_i)+1\,.$$

Proof. Put $n=\max\,(hd\, D_i)$. Let $0\leftarrow D\leftarrow P_0\leftarrow\cdots\leftarrow P_{n-1}\xleftarrow{d_n}P_n$ be a projective resolution of $D$ and $K_n=\ker d_n$. Since $n\geqq \mathrm{hd}\, D_i$, every component of im $d_n$ is projective. Hence, $K_n$ is $\mathfrak{A}$-projective by Lemma 3.7.

**Corollary.** *Let $A_i$, $A$ and $T_{ij}$ be as above. Then*

$$\mathrm{gl\,dim}\,\mathfrak{A}\geqq\mathrm{gl\,dim}\,\mathfrak{A}(I')$$

*for any subset of $I'$ and* $\mathrm{gl\,dim}\,\mathfrak{A}\leqq\max\,(\mathrm{gl\,dim}\,\mathfrak{A}_i)+n-1$.

Proof. Let $A$ be in $\mathfrak{A}(I')$ and $0\leftarrow\mathrm{M}(A)\leftarrow P_1\leftarrow P_2\leftarrow\cdots$ be a projective resolution of $\mathrm{M}(A)$ in $\mathfrak{A}$. Then $0\leftarrow A\leftarrow\mathrm{F}(P_1)\leftarrow\mathrm{F}(P_2)\leftarrow$ is a projective resolution of $\mathfrak{A}$ in $\mathfrak{A}(I')$ from Proposition 3.10.

We recall that $\mathfrak{A}$ is *semi-simple* if and only if every object of $\mathfrak{A}$ is projective.

**Theorem 3.12.** *Let $\mathfrak{A}_i$ be semi-simple abelian categories and $I$ a linearly ordered finite set. Then $\mathfrak{A}=[I,\mathfrak{A}_i]$ with $T_{ij}$ satisfying I~IV is hereditary if and only if*

$$\mathrm{T}_{ij}(M)=\mathrm{T}_{i+1j}\mathrm{T}_{ii+1}(M)\oplus\mathrm{T}_{i+2j}(\mathrm{K}^{i+2}(M))\oplus\cdots\oplus\mathrm{T}_{j-1j}(\mathrm{K}^{j-1}(M))\oplus\mathrm{K}^{j}(M)$$

*for every object $M$ in $\mathfrak{A}$ for all $i$, where $K^t(M)\in\mathfrak{A}_t$. Furthermore,* $\mathrm{gl\,dim}\,\mathfrak{A}=1$ *if and only if there exists not a zero functor $T_{ij}$*, (cf. [2], Theorem 1).

Proof. The first half is clear from Lemmas 3.7 and 3.8 and Proposition 3.11. If $\mathrm{T}_{ij}$ is not a zero functor, then $\boldsymbol{A}=(A,0)$ is not projective in $\mathrm{A}(i,j)$ for any $\mathfrak{A}$ such that $\mathrm{T}_{ij}(\mathfrak{A})\neq 0$ by Proposition 3.10. Hence, $\mathrm{gl\,dim}\,\mathfrak{A}\geqq\mathrm{gl\,dim}\,\mathfrak{A}(i,j)\geqq 1$. If $\mathrm{T}_{ij}$ is a zero functor for all $i<j$, then $\mathfrak{A}=\sum\oplus\mathfrak{A}_i$. Hence, $\mathrm{gl\,dim}\,\mathfrak{A}=0$.

Let $\{R_i\}_{i\in I}$ be rings. Finally we assume that $\mathfrak{A}_i$ is the abelian category of right $R_i$-modules. By [5], p. 121., Propo. 1.5 we know $U=\bigoplus_i S_i(R_i)$ is a small, projective generator in $\mathfrak{A}$. Put $R=[U, U]$. Let $r, r'$ be elements in $R_i$ and $T_{ij}(R_i)$, respectively. By $r_l, r'_l$ we denote morphisms in $[R_i, R_i]$ and $[R_j, T_{ij}(R_i)]$ such that $r_l(x_i)=rx_i$ and $r'_l(x_j)=r'x_j$, respectively where $x_t\in R_t$. We can naturally regard $T_{ij}(R_i)$ a left $R_i$-module by setting $\bar{r}y=T_{ij}(r_l)y$ for any $r\in R_i$ and $y\in T_{ij}(R_i)$. Furthermore, we define $\bar{r}'_l z=\psi_{ijk}T_{ik}(r_l)$ for any $k>j$ and $z\in T_{jk}(R_j)$, where we assume $T_{ii}=I_{\mathfrak{A}_i}$. Then we identify $R$ with the set

$$R=\left\{\begin{pmatrix} r_1 r_{12}\cdots\cdots\cdots r_{1n} \\ \quad r_2 r_{22}\cdots\cdots r_{2n} \\ \cdots\cdots \vdots \\ 0 \qquad\qquad \vdots \\ \qquad\qquad r_n \end{pmatrix},\quad r_{ij}\in T_{ij}(R_i),\ r_i\in R_i\right\}.$$

**Lemma 3.13.** *$\bar{r}_{ij}\bar{r}_{jk}=\bar{r}_{ij}(r_{jk})$ and $\bar{r}_{ij}\bar{r}_j=\overline{r_{ij}r_j}$, $\bar{r}_i\bar{r}_{ij}=\overline{r_i r_{ij}}$.*

Proof. For any $k\geqq j$ we have $\bar{r}_{ij}\bar{r}_j=\psi_{ijk}T_{jk}((r_{ij})_l)T_{jk}((r_j)_l)=\psi_{ijk}T_{jk}(r'_{ij}r)_l)$ $=\overline{r_{ij}r}$, and

$$\begin{aligned}\bar{r}_i\bar{r}_{ij} &= T_{ik}((r_i)_l)\psi_{ijk}T_{jk}((r_{ij})_l) = \psi_{ijk}T_{jk}T_{ij}((r_i)_l)T_{jk}((r_{ij})0_l) \quad \text{(naturality of } \psi)\\ &= \psi_{ijk}T_{jk}(T_{ij}((r_i)_l)(r_{ij})_l)\\ &= \psi_{ijk}T_{jk}((r_i r_{ij})_l) \quad \text{(definition of } R_i \text{ module } T_{ij}(R_i)\,.\\ &= \overline{r_i r_{ij}}\,.\end{aligned}$$

$$\begin{aligned}\bar{r}_{ij}\bar{r}_{jk} &= \psi_{ijt}T_{jt}((r_{ij})_l)\psi_{ikl}T_{kt}(r_{jk})_l)\\ &= \psi_{ijt}\psi_{jkl}T_{kt}(T_{jk}(r_{ij})_l)T_{kt}((r_{jk})_l) \quad \text{(naturality of } \psi)\,.\end{aligned}$$

On the other hand we put

$$\begin{aligned} r_{ik} &= \bar{r}_{ij}(r_{jk}) = (\psi_{ijk}T_{jk}(r_{ij})_l))(r_{jk})(r_{ik})_l : R_k \xrightarrow{(r_{jk})_l} T_{jk}(R_j)\\ &\xrightarrow{T_{jk}(r_{ij})} T_{jk}T_{ij}(R_i) \xrightarrow{\psi} T_{ij}(R_i). \quad \text{Hence,}\\ \bar{r}_{ik} &= (\psi_{ijt}T_{kt})(\psi_{ijk}T_{jk}((r_{ij})_l)(r_{jk})_l)\,.\end{aligned}$$

Therefore, $\bar{r}_{ij}\bar{r}_{jk}=\overline{\bar{r}_{ij}(r_{jk})}$ by the assumption III.

If we define a multiplication on $\boldsymbol{R}$ by setting

$$(*)\qquad r_{ij}r_{jk}=\bar{r}_{ij}(r_{jk})$$

we have from [5], p. 104, Theorem 4.1 and p. 106, Theorem 5.1

**Theorem 3.14.** *Let $\mathfrak{G}^{R_i}$ be the abelian category of right $R_i$-module. Then $[I, \mathfrak{G}^{R_i}]$ is equivalent to the abelian category of a left R-module, where*

$$R = \begin{pmatrix} R_1 T_{12}(R_1) \cdots\cdots T_{1n}(R_1) \\ R_2 \cdots\cdots T_{2n}(R_2) \\ \ddots \quad \vdots \\ 0 \quad \ddots \quad \vdots \\ R_n \end{pmatrix} \quad \textit{with product } (**) .$$

*And* $T_{ij}(M_i) \approx M \otimes T_{ij}(R_i)$ *for any* $M_i \in A_i$ *(**) is given by an* $R_i$-$R_j$ *homomorphism* $\psi_{ik} T_{ij}(R_i) \underset{R_j}{\otimes} T_{jk}(R_j) \to T_{ik}(P_i)$ (cf. [2], Theorem 1).

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