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RADICALS OF GROUP ALGEBRAS

YUKIO TSUSHIMA

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1. Introduction. Let k be a field of characteristic $p \neq 0$, G be a finite group whose order is divisible by p and H be its normal subgroup. By $\mathfrak{R}$ and $\mathfrak{N}$ we denote the radical of the group algebra KG and kH respectively. We know $\mathfrak{N} \subset \mathfrak{R}$ by the theorem of Clifford [1]. Hence $\mathfrak{L} = kG \cdot \mathfrak{N} = \mathfrak{N} \cdot kG$ is a two sided ideal of kG contained in $\mathfrak{R}$. We investigate in this note some properties between $\mathfrak{R}$ and $\mathfrak{L}$, (especially when $[G: H]=p$) and also we show if G is p-solvable, $\mathfrak{R}^{p^n}=0$, where p^n is the order of a p-Sylow subgroup of G. Throughout this note, we adhere to the above notation and the following conventions; modules are finitely generated left modules, $\otimes = \otimes_{kH}$, and for a positive integer e and a module M, eM means a direct sum of e copies of M. And finally, if M is a kG-module, M_H is the kH-module obtained by restricting the operators to kH.

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2. Lemma 1. *Let M be an irreducible kG-module. If kH-module N is a composition factor of M_H, then M is a composition factor of $N^G = kG \otimes N$.*

Proof. $\mathrm{Hom}_{kG}(N^G, M) \cong \mathrm{Hom}_{kH}(N, M)$. The right hand side is not 0, since N is a direct summand of M_H. So there is a kG-epimorphism $N^G \to M$, which shows our assertion.

Here we recall the theorem of Clifford [1].

Let N be any kH-module. A conjugate of N means $g \otimes N (\subset kG \otimes N)$, considered naturally as kH-module, where $g \in G$. The inertia group of N, denoted by $H^*(N)$, means $H^*(N) = \{g \in G \mid g \otimes N \cong N$ as kH-modules$\} \supset H$.

Let M be an irreducible kG-module and N be any irreducible kH-submodule of M_H. Then we have $M_H = e(N_1 \oplus N_2 \oplus \cdots \oplus N_r)$, where the N_i's are non isomorphic conjugates of $N_1 = N$, $r = [G: H^*(N)]$, and e is a positive integer.

Lemma 2. *We use the above notation. If $H^*(N)=H$, then we have $N^G \cong M$, equivalently, if the inertia group of an irreducible kH-module N is H itself, then N^G is also irreducible.*

Proof. $r = [G: H]$ by the assumption. From lemma 1 $\dim N^G \geqq \dim M$.

On the other hand, $\dim N^G=[G:H]\dim N$ and $\dim M=er\dim N=e[G:H]$ $\dim N$. Therefore, we have $\dim N^G=\dim M$, that is $N^G\cong M$ and $e=1$.

Proposition 1. *If $[G:H]$ is prime to p, then $\mathfrak{L}=\mathfrak{N}$.*

Proof. It is well known that in this case kG is just a semisimple extension of kH. In other words, any kG-module is (kG, kH)-projective in the sense of Hochschild [5]. And so $kG/\mathfrak{N}$ is also a semisimple extension of $kH/\mathfrak{N}$ by [6]. However, $kH/\mathfrak{N}$ is a semisimple algebra in an usual sense, so is $kG/\mathfrak{N}$. Therefore, $\mathfrak{L}=\mathfrak{N}$.

3. In the section, we assume k is a splitting field for kG and $[G:H]=p$. Hence for any kH-module N, its inertia group is H or G.

Lemma 3. *Let N be any irreducible kH-module. Then N^G is either irreducible or its composition factors are all isomorphic to each other, and the number of them is equal to p. More precisely, the former case holds if $H^*(N)=H$, and the latter holds if $H^*(N)=G$.*

Proof. Anyway, there exists an irreducible kG-module M such that N is a composition factor of M_H. If $H^*(N)=H$, then we have $N^G\cong M$ by lemma 2. If $H^*(N)=G$, then $M=eN$ (since $r=1$). Suppose M appears a times as a composition factor of N^G, then $a\neq 0$ from lemma 1.

We have $\dim N^G\geqq a\dim M\geqq ae\dim N$, that is $p\dim N\geqq ae\dim N$. On the other hand, the group character of N^G, as is easily to be shown, is 0. However the distinct irreducible characters of G are linearly independent over k, since k is a splitting field for G. Hence we have $p\mid a$. Combining with the above inequality, we have $p\geqq ae\geqq p$, that is $a=p$, $e=1$ and $\dim N^G=p\dim NM$. This completes the proof.

REMARK. From the proof, we know for any irreducible kG-module M, M_H is either irreducible or its decomposed into a direct sum of non isomorphic irreducible kH-modules.

Now let $\{U_1\cdots U_s, V_1\cdots V_t\}$ be the full set of non isomorphic irreducible kH-modules in which we assume $H^*(U_i)=H$, and $H^*(V_j)=G$. Then we have $kH/\mathfrak{N}=\oplus\sum f_iU_i\oplus\sum h_jV_j$ and $f_i=\dim U_i$, $h_j=\dim V_j$. We put $kG/\mathfrak{N}=A$. Clearly $A\cong kG\otimes kH/\mathfrak{N}$ as kG-modules. Hence $A\cong f_1U_1^G\oplus f_2U_2^G\oplus\cdots f_sU_s^G\oplus$ $h_1V_1^G\oplus h_2V_2^G\oplus\cdots h_tV_t^G$.

Proposition 2. *V_i^G is either indecomposable or completely reducible as an A-module.*

Proof. Since V_i^G is A-projective, we can decompose $V_1^G=Ae_1\oplus Ae_2\oplus\cdots$ Ae_k, where $\{e_i\}$ are primitive orthogonal idempotents of A. From lemma 3, V_1^G has p number of the composition factors which are isomorphic to each other.

Especially we have $Ae_i \simeq Ae_j$ for all i, j. So if Ae_i is irreducible, then we have $k=p$, and V_i^G is completely reducible. If this is not the case, each Ae_i has the same number of composition factors greater than one. Since p is a prime number, we have $k=1$. This completes our proof.

For a brevity of notations, we put $f_1U_1^G \oplus f_2U_2^G \oplus \cdots \oplus f_sU_s^G = C_0$, $h_jV_j^G = C_j$ and $A \simeq C_0 \oplus C_1 \oplus \cdots \oplus C_t$. We identify each C_i with its isomorphic image in A.

Theorem 1.

(1) *C_0 is a semisimple algebra and each C_i is a block of A ($i \geqq 1$).*

(2) *A is a quasi-Frobenius algebra over k.*

(3) *The composition factors of $\mathfrak{N}/\mathfrak{L}$ are those irreducible kG-modules which are also irreducible as kH-modules. Conversely any irreducible kG-module, say M, which is also irreducible as kH-module appears as composition factor of $\mathfrak{N}/\mathfrak{L}$ with multiplicity $(p-1)$ dim M.*

Proof.

(1) We know from lemma 3 and the remark, for $i \neq j$, C_i and C_j have no composition factor in common. Hence clearly C_i is a block of A for $i \geqq 1$ and C_0 is a semisimple algebra.

(2) For $i \geqq 1$, C_i has only one irreducible module and C_0 is a semisimple algebra. hence our assertion is clear from the definition.

(3) Since $\mathfrak{N}/\mathfrak{R}$ is the radical of A, it is contained in $C_1 \oplus C_2 \oplus \cdots \oplus C_t$. So the first assertion is clear. Let M be an irreducible kG-module which is irreducible as kH-module. Then $M_H \simeq V_i$ for some i. We have $\dim M = \dim V_i = h_i$. M appears $ph_i = p \dim M$ times as a composition factor of A. On the other hand, M appears $\dim M$ times in $kG/\mathfrak{N}$, since k is a splitting field for G. Hence M appears $(p-1)\dim M$ times between $\mathfrak{L}$ and $\mathfrak{N}$.

Lemma 4. $\mathfrak{N}^p \subset \mathfrak{L}$

Proof. Let e be any primitive idempotent of A. Then by lemma 3 and Theorem 1(1), Ae has at most p number of composition factors. Hence we have $(\mathfrak{N}/\mathfrak{L})^p e = 0$ and since e is arbitrary, $(\mathfrak{N}/\mathfrak{L})^p = 0$, that is $\mathfrak{N}^p \subset \mathfrak{L}$.

4. Theorem 2. *If G is p-solvable, then $\mathfrak{N}^{p^n} = 0$.*

Proof. We may assume k is a splitting field for G. If G is a p-group of order p^n, our assertion is clear, since in this case $\dim \mathfrak{N}^n = p^n - 1$ by [2]. (or [3] p. 189) Generally, there exists a normal subgroup of G whose index is p or prime to p. Using proposition 1 and lemma 4, it is easy to prove the theorem by induction on the order of G.

REMARK. It will be necessary to remark that $\dim \mathfrak{N} \geqq p^n - 1$ in general. We may also assume k is a splitting field for G, since in the group algebra the radical is preserved by the extension of the coefficient field[1]. Then there exists a primitive idempotent e of kG such that $(kG)e/\mathfrak{N}e \cong k$. Hence $\dim \mathfrak{N} \geqq \dim \mathfrak{N}e \geqq \dim (kG)e - 1 \geqq p^n - 1$.

OSAKA CITY UNIVERSITY

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1) This is true in general if the structure constants are in a perfect field contained in the coefficient field.