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A NOTE ON THE FIXED RING OF A GALOIS EXTENSION

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M. Harada [5] showed that if A is a central separable C-algebra and a Galois extension of B with group G, and B is a separable $B\cap C$-algebra, then the order of the subgroup of G which leaves C fixed is a unit in C. In this note we obtain a partial converse to this result (Theorem 4 below). The method of approach is to use the modules J_σ associated with automorphisms σ of A. These modules were discovered in [8] and their connection with Galois extensions was recognized in [7].

The author would like to thank the referee for pointing out the reference [4] for the proof of Proposition 2.

We begin by recalling the definition of J_σ:

Definition. Let A be a central separable C-algebra and σ a ring automorphism of A. Then

$$J_\sigma=\{x \text{ in } A \mid \sigma(a)x=xa \text{ for all } a \text{ in } A\}\ .$$

It was shown in [8] that if σ is a C-algebra automorphism of A, then J_σ is a rank one projective C-module. The following useful fact, noted for Galois extensions in [7], can also be extracted from [8]: ($\otimes$ means $\otimes_C$)

Lemma 1. *Let A be a central separable C-algebra, and σ, τ be two C-algebra automorphisms of A. Then the map κ: $J_\sigma\otimes J_\tau\rightarrow J_{\sigma\tau}$ given by $\kappa(x\otimes y)=xy$, x in J_σ, y in J_τ, is an isomorphism.*

It is easy to see that the image of κ is in $J_{\sigma\tau}$, and [8], Lemma 5, shows that there exists an isomorphism from $J_\sigma\otimes J_\tau$ onto $J_{\sigma\tau}$; the proof of Lemma 1 consists, first, in verifying that the sequence of isomorphisms connecting $A\otimes J_\sigma\otimes J_\tau$ and $A\otimes J_{\sigma\tau}$ on the last line of page 1112 of [8] sends $a\otimes x\otimes y$ to $a\otimes xy$, and then, using this fact, noticing that the sequence of isomorphisms on the bottom of page 1111 of [8] which gives the isomorphism of $J_\sigma\otimes J_\tau$ with $J_{\sigma\tau}$ is

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κ. We omit the tedious details.

Proposition 2. *Let A be a central separable C-algebra and G a finite group of C-algebra automorphisms of A. Let $N=\Sigma J_\sigma$, and suppose that as a C-module, the sum is direct. Then N is a separable C-algebra if $|G|$, the order of G, is a unit of C.*

Proof. Since the kernel of the map from N^e to N given by $x\otimes y\to\otimes xy$ is a finitely generated C-module, we have by [1], III, 2.10 that N is a separable C-algebra if $N\otimes C_m=N_m$ is a separable C_m-algebra for all maximal ideals m of C. Moreover, if G' is G acting on $A\otimes C_m=A_m$ via $\sigma'=\sigma\otimes 1$, and $N'=\Sigma\oplus J_{\sigma'}$, where $J_{\sigma'}=\{x' \text{ in } A_m \mid \sigma'(y')x'=x'y' \text{ for all } y' \text{ in } A_m\}$, then $N'=N_m$: in fact $J_{\sigma'}=(J_\sigma)_m$. For

$$(J_\sigma)_m=\left\{\frac{x}{s} \text{ in } A_m \mid \sigma(y)x=xy \text{ for all } y \text{ in } A\right\}, \text{ and}$$

$$J_{\sigma'} =\left\{\frac{x}{s} \text{ in } A_m \mid \exists t \text{ in } C\text{-}m \text{ so that } t(\sigma(y)x-xy)=0\right.$$

$$\left.\text{for all } y \text{ in } A\right\},$$

so clearly $(J_\sigma)_m\subseteq J_{\sigma'}$. On the other hand, if $\frac{x}{s}\in J_{\sigma'}$, let $y_1,\cdots,y_r$ generate A over C, t_i be in C-m such that $t_i(\sigma(y_i)x-xy_i)=0$, and $t=\prod_1^r t_i$. Then $tx\in J_\sigma$, so $\frac{x}{s}=\frac{tx}{ts}$ is in $(J_\sigma)_m$. Now, since $|G|$ is a unit of C if $|G|$ is a unit of C_m for all m, it suffices to prove the theorem assuming C is local.

Assuming C local, $\sigma\in G$ is inner, conjugation by an element u, and $J_\sigma=Cu_\sigma$ ([8]). Since $Cu_\sigma\cdot Cu_\tau=Cu_{\sigma\tau}$, $u_\sigma u_\tau=a_{\sigma,\tau}\, u_{\sigma\tau}$, $a_{\sigma,\tau}$ a unit of C, so $N=\Sigma\oplus Cu_\sigma$ is a twisted group ring (i.e. a crossed product with factor set in the units of C, and with G acting trivially on C). Thus we may apply [4], Lemma 4, to obtain that N is separable over C if $|G|$ is a unit of C, as desired.

Lemma 3. *If A is a central separable C-algebra, G is a finite group of C-algebra automorphisms of A, and $N=\Sigma J_\sigma$, then the fixed ring of G acting on A, A^G, is equal to A^N, the commutator of N in A.*

Proof. If x is in A^N then x is in A^{J_σ} for all σ in G, so $xy_\sigma=y_\sigma x$ for all y_σ in J_σ. But since for all x in A, y_σ in J_σ, we have $\sigma(x)y_\sigma=y_\sigma x$, it follows that if x is in A^N, $(\sigma(x)-x)y_\sigma=0$ for all $y_\sigma$in J_σ and all σ in G. By Lemma 1 $J_\sigma\cdot J_{\sigma^{-1}}=C$, so there exist $y_{\sigma,\nu}$ in J_σ, and $z_{\sigma,\nu}$ in $J_{\sigma^{-1}}$ so that $\sum_\nu y_{\sigma,\nu}z_{\sigma,\nu}=1$. Thus $0=\sum_\nu(\sigma(x)-x)y_{\sigma,\nu}z_{\sigma,\nu}=(\sigma(x)-x)\cdot 1$, so x is in A^G. The converse is trivial.

Now, using Kanzaki's result ([7], Proposition 1) which states that if A is a Galois extension of B with group G, then $N=\Sigma\oplus J_\sigma$, we obtain our main result.

Theorem 4. *Let A be a ring whose center C has no idempotents but* 0 *and* 1. *Suppose A is a Galois extension of B with group G, and A is separable over $B\cap C$. Let H be the subgroup consisting of all elements of G which are the identity on C. Then if the order of H is a unit in C, B is a separable $B\cap C$-algebra.*

Proof. If A is a Galois extension of B with group G, then directly from the definition of Galois extension A is a Galois extension of A^H, the fixed ring of H, with group H. Thus $N=\Sigma\oplus J_\sigma$ by [7], Prop. 1. By Proposition 2, N is a separable C-algebra, so by Lemma 3 and [6], Theorem 2, A^H is separable over C.

Now H is a normal subgroup of G, G restricted to A^H is isomorphic to G/H, as is G restricted to C, and $C^G=B\cap C$. Since A is assumed separable over $B\cap C$, the center C of A is separable over $B\cap C$, so ([3], 1.3) C is a Galois extension of $B\cap C$ with group G/H. Defining the action of G/H on $B\otimes_{B\cap C}C$ via $\sigma(b\otimes c)=b\otimes\sigma(c)$, $B\otimes_{B\cap C}C$ becomes a Galois extension of B with group G/H, just as in [3], 1.7. Also A^H is a Galois extension of B with group G/H. The map from $B\otimes_{B\cap C}C$ to A^H given by $b\otimes c\to bc$ is a G/H-module and B-algebra map, so by a trivial extension of [3], 3.4, it is an isomorphism: $B\otimes_{B\cap C}C\simeq A^H$. Thus, since $B\cap C$ is a $B\cap C$-direct summand of C by [3], 1.6, B is a B-direct summand of A^H, so is separable over $B\cap C$ by [2], IX, 7.1 and the fact that A^H is separable over $B\cap C$. This completes the proof.

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Bibliography

[1] H. Bass: Topics in Algebraic K-Theory, notes of lectures given at the Tata Institute, Bombay, 1965–66.

[2] H. Cartan and S. Eilenberg: Homological Algebra, Princeton, 1956.

[3] S. Chase, D. Harrison and A. Rosenberg: *Galois theory and Galois cohomology of commutative rings,* Mem. Amer. Math. Soc. **52** (1965), 15–33.

[4] M. Harada: *Some criteria for hereditarity of crossed products,* Osaka J. Math. **1** (1964), 69–80.

[5] ————: *Supplementary results on Galois extensions,* Osaka J. Math. **2** (1965), 343–350.

[6] T. Kanzaki: *On commutor rings and Galois theory of separable algebras,* Osaka J. Math. **1** (1964), 103–115.

[7] ————: *On Galois algebras over a commutative ring,* Osaka J. Math. **2** (1965), 309–317.

[8] A. Rosenberg and D. Zelinsky: *Automorphisms of separable algebras,* Pacific J. Math. **11** (1961), 1109–1117.