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Osaka J. Math.
4 (1967), 151-156

NOTE ON ORDERS OVER WHICH AN HEREDITARY ORDER IS PROJECTIVE

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(Received December, 21, 1966)

Let R be an integral noetherian domain with quotient field K, and Σ a semi-simple K-algebra with finite dimension. An R-algebra Λ in Σ is called an R-order when Λ is R-finitely generated and $\Lambda K=\Sigma$.

The author has studied essential properties of hereditary orders in [2] and [3], (briefly h-order), which has the property that every one-sided ideal in Λ is Λ-projective.

In this short note we shall show the following theorem as a corollary to [3], Proposition 5.1.

Theorem. *Let R be an integral noetherian domain with quotient field K and Σ a K-separable semi-simple algebra. Let Λ be an R-order in Σ. We assume that there exists an h-order Γ over R which properly containing Λ and is left Λ-projective. Then there exsits an h-order Γ_0 such that $\Gamma\supsetneqq\Gamma_0\supseteqq\Lambda$.*

From this theorem we shall give some criteria of order being hereditary, which contain [6], Theorem 5.1: If every maximal order containing Λ is Λ-projective, then Λ is hereditary, whenever Σ is a central simple K-algebra.

Finally, we shall give another criterion in a special case that Λ is an h-order if and only if Λ has a unique irredundant representation by maximal orders, (see the below for the definition). Hijikata has already shown the above fact in [5] by a direct computation. We give here a proof by the method in [3].

First we assume that R is an integral noetherian domain with quotient field K and Σ a semi-simple K-algebra. Let Γ be an h-order in Σ and Z the center of Γ.

Lemma 1. *Let R, Γ, Z and Σ be as above, and Λ an R-order contained properly in Γ. If Γ is left Λ-projective, then $\Gamma\neq\Lambda Z$ and Γ is left ΛZ-projective.*

Proof. Let $C=\mathrm{Hom}^l_\Lambda(\Gamma, \Lambda)$. Then C is right Λ-projective, since Γ is left Λ-projective. We put $\Lambda'=\Lambda Z$. Then C is right Λ'-projective and Γ is left Λ'-projective from [2], Lemma 1.3. Furthermore, $C'=\mathrm{Hom}^l_{\Lambda'}(\Gamma, \Lambda')$

is also right Λ'-projective. We know from [2], Lemma 1.3 and Proposition 1.6 that $\Gamma=\mathrm{End}^r_\Lambda(C)=\mathrm{End}^r_{\Lambda'}(C)=\mathrm{End}^r_{\Lambda'}(C')$ and hence $C=C'$ from [2], Lemma 1.5 and Proposition 1.6. If $\Gamma=\Lambda'$, $C'=\Lambda'$, which would imply $\Lambda=\Gamma$.

Let $\Sigma=\sum_{i=1}^n \oplus\Sigma_i$ be the decomposition of Σ into simple algebras Σ_i and $1=\sum_{i=1}^n e_i$, where the e_i's are identities in Σ_i. Then every h-order Λ is written as $\Lambda=\sum\oplus\Lambda e_i$. Therefore, if we are interested in an order Λ which is contained in a Λ-projective *h*-order Γ, we may assume from Lemma 1 that Σ is a simple *K*-algebra.

Proof of Theorem. From Lemma 1 and the above remark we may assume that Σ is a central simple *K*-algebra and Λ is an order over *Z*. Furthermore, we may restrict ourselves to a case of *R* being a discrete rank one, valuation ring by [2], Theorem 2.6. Let $C=\mathrm{Hom}^l_\Lambda(\Gamma, \Lambda)$, then $C\Gamma=\Gamma$, since Γ is Λ-projective. Let N be the radical of Γ, and $A=C+N$. Since $C\Gamma=\Gamma$ and $C\neq\Gamma$, A is not a two-sided ideal in Γ. Hence, $\Gamma_0=\Gamma\cap\mathrm{End}^l_\Gamma(A)$ is an h-order by [3], Proposition 5.1. Furthermore, since A is a right Λ-module, $\Gamma_0\supseteqq\Lambda$. It is clear that $\Gamma\supsetneqq\Gamma_0$.

Corollary 1. *We assume that Σ is a central simple K-algebra and R is a discrete rank one, valuation ring. If an R-order Λ is contained in a minimal h-order*[1], *which is left Λ-projective, then Λ is hereditary.*

We have defined a rank of h-order Λ in [3], p. 10 under the assumption of Corollary 1 which is equal to the number of maximal two-sided ideals in Λ.

Proposition 1. *Let Σ and R be as in Corollary 1. We assume that an h-order Γ of rank r contains an R-order Λ and that Γ is left Λ-projective. If $C\cap N/CN$ contains s non-isomorphic irreducible components as a right Λ/N-module, then Λ is contained in an h-order of rank r+s, where $C=\mathrm{Hom}^l_\Gamma(\Gamma, \Lambda)$ and Nis the radical of Λ.*

Proof. Let $\{\Omega_i\}_{i=1}^r$ be the set of maximal orders containing Γ and $C_i=\mathrm{Hom}^l_\Lambda(\Omega_i, \Lambda)$. Since Ω_i is left Γ-projective, Ω_i is left Λ-projective. Hence, C_i is a minimal two-sided idempotent ideal in Λ by [2], Lemma 5,1. It is well known that $\Gamma=\mathrm{End}^r_\Lambda(C)$ and $\Gamma/N_\Gamma=\mathrm{End}^r_{\Lambda/N}(C/CN)$, where N_Γ is the radical of Γ, (see [2], Propositions 1.6 and 4.4). Furthermore, $C/CN\approx C+N/N\oplus C\cap N/CN$ as a right Λ-module and $C+N/N\supseteqq C_1+N/N\oplus\cdots\oplus C_r+N/N$ by [3], Lemma 3.2. Since Γ/N_Γ contains only *r* simple components, $C+N/N=C_1+N/N\oplus\cdots\oplus C_r+N/N$. Let $A=C+N_\Gamma$ and $A/N_\Gamma=\sum_{i=1}^r\oplus\bar{L}_i$,

1) See [3], p. 3 for the definition of minimal order.

where $\bar{L}_i$ is a left ideal in a simple component $\bar{\Gamma}_i$ of $\bar{\Gamma}/N\bar{\Gamma}_i$. Since $C=\sum_{i=1}^{r} C_i$ and $\Gamma/N_\Gamma \approx \mathrm{End}_{\Lambda/N}(C/CN)$, $\bar{L}_i \neq (0)$ for all i. Furthermore, since $C(C\cap N/CN)=(0)$, the number of non-isomorphic irreducible components of $C\cap N/CN$ is equal to the number of $\bar{L}_i$ such that $\bar{L}_i \neq \bar{\Gamma}_i$. Hence, the rank of $\Gamma_0=\Gamma \cap \mathrm{End}^r_\Lambda(A)$ is equal to $r+s$ by [3], Theorem 5.3.

REMARK. Corollary 1 is not true if we replace a minimal h-order by an arbitrary *h*-order containing Λ, (see the example below).

Lemma 2. *Let R be a Dedekind domain and Ω an order containing an order Λ. Then there exists only a finite many of orders between Ω and Λ.*

Proof. Since $r\Omega \subseteqq \Lambda$ for some $r\neq 0$ in R, if a prime ideal p in R does not divides r, then $\Omega_p=\Lambda_p$. If p divides r, then $\Omega_p/r\Omega_p=\Sigma\oplus R_p/rR_p$. Hence, we have the lemma.

Let Λ and Λ' be orders. We have called in [2], p. 281 that Λ and Λ' are the same type if there exists $\Lambda-\Lambda'$ ideal A such that $\mathrm{End}^l_\Lambda(A)=\Lambda'$ and $\mathrm{End}^r_{\Lambda'}(A)=\Lambda$.

Proposition 2. *Let R be a Dedekind domain and Σ a separable, simple K-algebra. Let Λ be an order over R. If every h-order belonging to the same type and containing Λ is left Λ-projective, then Λ is hereditary.*

Proof. Let Γ be an *h*-order containing Λ and Z the center of Γ. We note that Z is also a Dedekind domain. We put $\Lambda'=\Lambda Z$. Then there exists a minimal one Γ' among h-orders over Z containing Λ' by Lemma 2. Let p be a prime in Z, and $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be the set of h-orders containing Λ which belong to the same type as Γ_p. Then $\tau^l_{\Lambda_{p'}}(\sum_{i=1}^{n}\oplus\Gamma_i)=\Gamma_{p'}$ by [2], Theorems 3.3 and 4.3. Hence, $\tau^l_{\Gamma'}(\Sigma\oplus\Gamma'')=\Gamma'$, where Γ'' runs through all h-orders of same type containing Λ'. Therefore, Γ' is a direct summand of $(\Sigma\oplus\Gamma'')^m$ for some m. Hence, Γ' is a left Λ'-projective[2)] by Lemma 1, which implies from Theorem that $\Lambda'=\Gamma'$. Furthermore, since $\Sigma\oplus\Gamma''$ is Λ-projective, Γ' is Λ-projective. Every h-order contains Z by [2], Proposition 2.2. Hence, we obtain $\Lambda'=\Lambda$ from Theorem.

Corollary 2. *Let R and Σ be as above and Λ an R-order. Λ is an h-order if one of the following conditions is satisfied.*

1) *Every maximal order containing Λ is left Λ-projective.*

2) *We have an h-order Γ containing Λ such that there exists a maximal chain $\Gamma=\Gamma_1\supset\Gamma_2\supset\cdots\supset\Gamma_n=\Lambda$ between Γ and Λ and furthermore, each Γ_i is left Λ-projective,* (cf. [2], Theorem 5.3 and [6], Theorem 5.1)

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2) This is an analogy to Silver's method in [6].

Proof. It is clear from Proposition 2 and Theorem.

Next we shall consider a condition that an h-order Γ is Λ-projective for an order $\Gamma \supseteqq \Lambda$.

Lemma 3. *Let $\Gamma \supseteqq \Lambda$ be orders. Then Γ is left Λ-projective if and only if $C\Gamma = \Gamma$, where $C = \mathrm{Hom}^l_\Lambda(\Gamma, \Lambda)$.*

Proof. It is clear from the standard map: $C \underset{\Lambda}{\otimes} \Gamma \rightarrow \mathrm{Hom}^l_\Lambda(\Gamma, \Gamma) = \Gamma$.

Proposition 3. *Let Λ be an order in Σ and C an idempotent two-sided ideal in Λ. If $\mathrm{End}^r_\Lambda(C) = \Gamma$ is an h-order, then Γ is left Λ-projective, C is left and right Λ-projective and $C = \mathrm{Hom}^r_\Lambda(\Gamma, \Lambda)$.*

Proof. Put $C^{-1} = \mathrm{Hom}^l_\Lambda(C, \Gamma)$. Since Γ is an h-order, $CC^{-1} = \Gamma$ by [4], p. 84. Hence, $C\Gamma = CCC^{-1} = CC^{-1} = \Gamma$. Since $C' = \mathrm{Hom}^l_\Lambda(\Gamma, \Lambda) \supseteqq C$, $C'\Gamma = \Gamma$. Therefore, Γ is left Λ-projective by Lemma 3 and C is left Λ-projective. Furthermore, $\mathrm{End}^l_\Lambda(C) = \Gamma'$ is an h-order by [4], Theorem 1.1. Exchanging left and right in the above, we obtain that C is right Λ-projective. Hence, $C' = C$ by [2], Proposition 1.6.

Corollary 3. *Let Λ be an order and Γ an h-order containing Λ. If $C = \mathrm{Hom}^l_\Lambda(\Gamma, \Lambda)$ is idempotent, every two-sided idempotent ideal contained in C is left and right Λ-projective and there exists an h-order Γ_0 ($\supseteqq \Lambda$) such that Γ_0 is Λ-projective. Especially if Γ is a maximal order, then Γ is left Λ-projective if and only if C is idempotent.*

Proof. Let D be an idempotent ideal in C. Then $\mathrm{End}^r_\Lambda(D) \supseteqq \mathrm{End}^r_\Lambda(C)$, since $CD = D$. Hence, $\mathrm{End}^r_\Lambda(D)$ is an h-order by [2], Corollary 1.4.

Finally, we shall give one more characterization of *h*-orders which is a converse of [2], Theorem 3.3.

We only consider a central simple K-algebra Σ and orders Λ over a discrete rank one, valuation ring R.

Let $\{\Omega_\alpha\}_{\alpha \in I}$ be the set of maximal orders containing Λ. If $\Lambda = \bigcap_{\alpha \in I} \Omega_\alpha$ and $\Lambda \subsetneqq \bigcap_{\beta \in I - \alpha} \Omega_\beta$, then we shall call Λ has a *unique irredundant representation* by Ω_α.

Proposition 4. *Let Σ and R be as above. The intersection of any two distinct maximal orders is contained in an h-order of rank two.*

Proof. Let Ω_1 and Ω_2 be maximal orders and $\Lambda = \Omega_1 \cap \Omega_2$. Then there exists an Ω_1-Ω_2 ideal A which is contained in Λ by [1]. Let N be the radical of Ω_1. If $A \subseteqq N^i$ and $A \nsubseteqq N^{i+1}$ for some i, $N^{-i}A \nsubseteqq N$. Hence, we may assume that $A \nsubseteqq N$. Then $B = A + N$ is not a two-sided ideal, since $A\Omega_1 = \Omega_1$. Let $\Omega' = \mathrm{End}^l_\Lambda(B)$. Since B is a right Λ-module, $\Omega' \subseteqq \Lambda$. Therefore, $\Omega_1 \cap \Omega'$

($\supseteqq \Lambda$) is an h-order of rank two by [3], Theorem 5.3.

Corollary 4. (Hijikata). *Let Σ and R be as above. Then an R-order has a unique irredundant representation by maximal orders if and only if Λ is an h-order.*

Proof. If Λ is an h-order then Λ has a unique representation by [2], Theorem 3.3. Conversely, we assume that Λ has a unique representation and $\{\Omega_\alpha\}_{\beta\in I}$ be the set of maximal orders containing Λ. Then $\Omega_1 \supsetneqq \Omega_1 \cap \Omega_2 \supsetneqq \cdots$ is a chain of orders containing Λ. Hence, I must be a finite set by Lemma 2. Since $\Omega_i \cap \Omega_j$ is contained in an h-order of rank two, which is written uniquely as $\Omega_i \cap \Omega_j'$. $\Omega_j = \Omega_j'$. Hence, Λ is an h-order by [3], Corollary 5.2.

EXAMPLES. 1. We give an example which shows $\Lambda \neq \Lambda Z$ in Lemma 1. Let R be the ring of integers and Q the field of rationals. $L = Q(\sqrt{5})$ and Z is the integrally closure of R in L. Put $Z_0 = R + R\sqrt{5}$, then $Z \supsetneqq Z_0$. Let A be an ideal of Z contained in Z_0. An maximal order $\Omega = \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix}$ is $\Lambda = \begin{pmatrix} Z & A \\ Z & Z_0 \end{pmatrix}$-projective. However, $\Lambda \not\supseteq Z$.

2. Let $R = R_p$, where p is a primeideal in R. $\Gamma_1 = \begin{pmatrix} R & R & P \\ R & R & P \\ R & R & R \end{pmatrix} \supseteqq \Gamma_2 = \begin{pmatrix} R & P & P \\ R & R & P \\ R & R & R \end{pmatrix}$ $\supseteqq \Lambda = \begin{pmatrix} R & P & P \\ P & R & P \\ R & R & R \end{pmatrix}$. Then $C_1 = \mathrm{Hom}_\Lambda(\Gamma_1, \Lambda) = \begin{pmatrix} P & P & P \\ P & P & P \\ R & R & R \end{pmatrix}$ and $C_2 = \mathrm{Hom}_\Lambda(\Lambda_2, \Lambda)$ $= \begin{pmatrix} P & P & P \\ P & R & P \\ R & R & R \end{pmatrix}$. C_1 and C_2 are idempotent and left and right Λ-projective. Furthermore, $\mathrm{End}^r_\Lambda(C_2) = \begin{pmatrix} R & P & P \\ R & R & P \\ P^{-1} & R & R \end{pmatrix} = \Gamma_3$.

This example shows that $\mathrm{Hom}_\Lambda(\Gamma_2, \Lambda)$ is idempotent for an h-order Γ_2, however $\mathrm{End}^r_\Lambda(C_2) \supsetneqq \Gamma_2$, (cf. [2], Proposition 1.6) and furthermore, h-order Γ_3 is Λ-projective, but Λ is not hereditary (cf. Corollary 2). Moreover, $\Lambda/N = R/p \oplus R/p \oplus R/p$ and $\Lambda \supset C_2 \supset C_1$ is a maximal chain of (projective) idempotent ideals, however Λ is not hereditary (cf. Corollary 2).

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References

[1] M. Deuring: Algebren, Berlin, 1953.
[2] M. Harada: *Hereditary orders,* Trans. Amer. Math. Soc. **107** (1963), 273–290.

[3] ――――――: *Structure of hereditary orders over local rings,* J. Math. Osaka City Univ. **14** (1963), 1–22.

[4] ――――――: *Multiplicative ideal theory in hereditary orders,* J. Math. Osaka City Univ. **14** (1963), 83–106.

[5] H. Hijikata: *Maximal compact subgroups in p-adic classical groups,* Sugaku no Ayumi **10** (1963), 66–97.

[6] L. Silver: *Noncommutative localizations and applications,* to appear.