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K_O-GROUPS OF PROJECTIVE SPACES

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Introduction. The purpose of this note is to calculate $\tilde{K}_O^i$-groups of the real projective m-space $RP(m)$ and the complex projective n-space $CP(n)$. Consider the operations: complexification ε: $K_O(X)\to K_U(X)$, real restriction ρ: $K_U(X)\to K_O(X)$, and conjugation $*$: $K_U(X)\to K_U(X)$. The following formulas

$$\rho\varepsilon = 2 \quad : K_O(X) \to K_O(X)\ ,$$
$$\varepsilon\rho = 1+*: K_U(X) \to K_U(X)\,,$$

are well known (c.f. [4]). Let ξ be the canonical real line bundle over $PR(m)$, and let η be the canonical complex line bundle over $CP(n)$. Then generators for our groups are defined as follows:

$$\lambda = \xi - 1 \in \tilde{K}_O(RP(m))\,,$$
$$\nu = \varepsilon\lambda \ \in \tilde{K}_U(PR(m))\,,$$
$$\mu = \eta - 1 \in \tilde{K}_U(CP(n))\ ,$$
$$\mu_0 = \rho\mu \ \in \tilde{K}_O^0(CP(n))\ ,$$
$$\mu_i = \rho g^i\mu \in \tilde{K}_O^{-2i}(CP(n)) \qquad (i=1,\,2,\,3)\,,$$

where g is the generator of $\tilde{K}_U^0(S^2)$ given by the reduced Hopf bundle.

Our theorems are as follows.

Theorem 1. 1) *The groups $\tilde{K}_O^{-i}(RP(m))$ are isomorphic to the following groups*:

	i \ m	$8r$	$8r+1$	$8r+2$	$8r+3$	$8r+4$	$8r+5$	$8r+6$	$8r+7$
0)	0	(2^{4r})	(2^{4r+1})	(2^{4r+2})	(2^{4r+2})	(2^{4r+3})	(2^{4r+3})	(2^{4r+3})	(2^{4r+3})
i)	1	$r\neq 0$ (2)	(2)	(2)	$(\infty)+(2)$	(2)	(2)	(2)	$(\infty)+(2)$
ii)	2	$r\neq 0$ (2)+(2)	(2)	(2)	(2)	(2)	(2)	(2)+(2)	(2)+(2)+(2)
iii)	3	$r\neq 0$ (2)	(∞)	0	0	0	(∞)	(2)	(2)+(2)
iv)	4	(2^{4r})	(2^{4r})	(2^{4r})	(2^{4r})	(2^{4r+1})	(2^{4r+2})	(2^{4r+3})	(2^{4r+3})
v)	5	0	0	0	(∞)	0	0	0	(∞)
vi)	6	0	0	(2)	(2)+(2)	(2)	0	0	0
vii)	7	0	(∞)	(2)	(2)+(2)	(2)	(∞)	0	0

where (t) *means the cyclic group of order* t.

2) $\tilde{K}_O^0(RP(m))$ *is generated by* λ *with two relations* $\lambda^2=-2\lambda$, $\lambda^{f+1}=0$, *where* $f=\varphi(m)$ *is the number of integers* s *such that* $0<s\leqslant m$ *and* $s\equiv 0, 1, 2, 4 \bmod 8$, *and* $\tilde{K}_O^{-4}(RP(m))$ *is additively generated by* $g_2\lambda$ $(g_2=\rho g^2)$.

Theorem 2. 0) $K_O^0(CP(n))$ *is the trancated polynomial ring (over the integers) with one generator* μ_0 *and the following relations*:

(a) *if* $n=2t$, *then* $\mu_0^{t+1}=0$,

(b) *if* $n=4t+1$, *then* $2\mu_0^{2t+1}=0$ *and* $\mu_0^{2t+2}=0$,

(c) *if* $n=4t+3$, *then* $\mu_0^{2t+2}=0$.

i) $\tilde{K}_O^{-1}(CP(n))=0$.

ii) $\tilde{K}_O^{-2}(CP(n))$ *is the free module with basis* $\mu_1, \mu_1\mu_0, \cdots, \mu_1\mu_0^{t-1}$, *and also, in case* n *is odd*, $\mu_1\mu_0^t$ (*if* $n\equiv 1 \bmod 4$) *or* σ (*if* $n\equiv 3 \bmod 4$), *where* $2\sigma=\mu_1\mu_0^t$ *and* $t=\left[\frac{n}{2}\right]$ ([ ] is the Gauss notation).

iii) $\tilde{K}_O^{-3}(CP(n))=\begin{cases} Z_2 & \text{if } n=4t+3, \\ 0 & \text{otherwise.}\end{cases}$

iv) $\tilde{K}_O^{-4}(CP(n))$ *is the free module with basis* $\mu_2, \mu_2\mu_0, \cdots, \mu_2\mu_0^{t-1}$, *and also, in case* $n\equiv 3 \bmod 4$, $\mu_2\mu_0^t$ *with relation* $2\mu_2\mu_0^t=0$, *where* $t=\left[\frac{n}{2}\right]$.

v) $\tilde{K}_O^{-5}(CP(n))=0$.

vi) $\tilde{K}_O^{-6}(CP(n))$ *is the free module with basis* $\mu_3, \mu_3\mu_0, \cdots, \mu_3\mu_0^{t-1}$, *and also, in case* n *is odd*, $\mu_3\mu_0^t$ (*if* $n\equiv 3 \bmod 4$) *or* τ (*if* $n\equiv 1 \bmod 4$), *where* $2\tau=\mu_3\mu_0^t$ *and* $t=\left[\frac{n}{2}\right]$.

vii) $\tilde{K}_O^{-7}(CP(n))=\begin{cases} Z_2 & \text{if } n=4t+1, \\ 0 & \text{otherwise.}\end{cases}$

Theorem 3. *The ring structures of* $\tilde{K}_O^{\text{even}}(CP(n))=\sum\tilde{K}_O^{-2i}(CP(n))$ *are given by the followings*:

i) $\mu_1^2=4\mu_2+\mu_2\mu_0$,	ii) $\mu_2^2=\mu_0^2$,	iii) $\mu_3^2=4\mu_2+\mu_2\mu_0$,
iv) $\mu_2\mu_1=\mu_3\mu_0$,	v) $\mu_3\mu_2=\mu_1\mu_0$,	vi) $\mu_1\mu_3=4\mu_0+\mu_0^2$.

REMARK. Theorem 2 is an unpublished result of S. Araki, who computed the result directly from the spectral sequence.

1. Preliminaries

First we recall from [1] that

$$\begin{array}{lcccccccccl} q & \equiv & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \bmod 8 \\ K_U^{-q}(*) = \tilde{K}_U(S^q) & = & Z & 0 & Z & 0 & Z & 0 & Z & 0 & \\ K_O^{-q}(*) = \tilde{K}_O(S^q) & = & Z & Z_2 & Z_2 & 0 & Z & 0 & 0 & 0 & \end{array}$$

and if q is even

(1.1) $$\varepsilon\colon Z = \tilde{K}_O(S^{2q}) \to \tilde{K}_U(S^{2q}) = Z$$

is monomorphic, in fact, Im $\varepsilon = Z$ if $q \equiv 0 \bmod 4$, while Im $\varepsilon = 2Z$ if $q \equiv 2 \bmod 4$. Then we can easily obtain the next lemma.

Lemma (1.2). *The Conjugation*

$$* : \tilde{K}_U(S^{2q}) \to \tilde{K}_U(S^{2q})$$

is given by

$$* = 1 \quad \textit{if } q \textit{ is even} \quad \textit{and} \quad * = -1 \quad \textit{if } q \textit{ is odd.}$$

Next we recall from [3] and [6] that the E_2 and E_∞ terms of the spectral sequence of $\tilde{K}_O$-theory are given by

$$E_2^{p,q} \cong \tilde{H}^p(X, K_O^q(*)),$$
$$E_\infty^{p,q} \cong G_p\tilde{K}_O^{p+q}(X) = \tilde{K}_p^{p+q}(X)/\tilde{K}_{p+1}^{p+q}(X),$$

where $\tilde{K}_p^n(X) = \operatorname{Ker}[\tilde{K}_O^n(X) \to \tilde{K}_O^n(X^{p-1})]$. The Ω-spectrum $\boldsymbol{Y} = \{Y_q, h_q\}$ in $\tilde{K}_O$-theory is given by $Y_{8k-i} = \Omega^i B_O$ $(i = 7, \cdots, 1, 0)$, where B_O is a classifying space for the orthogonal group O, ΩB_O is the space of loops on B_O and $\Omega^p B_O$ is the space $\Omega(\Omega^{p-1}B_O)$. As for differentials $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ we have $d_r^{p,q} = \Omega d_r^{p+1,q}$ and $d_r^{p,q} = d_r^{p,q+8}$. On the other hand, Theorem 3.4 of [6] asserts that $d_2^{8t,0}$, $d_2^{8t,-1}$ and $d_3^{8t,-2}$ are induced by the cohomology operations defined by the k-invariants $k^{8t,8t} \in H^{8t+2}(Z, 8t, Z_2)$, $k^{8t,8t} \in H^{8t+2}(Z_2, 8t, Z_2)$ and $k^{8t,8t+1} \in H^{8t+3}(Z_2, 8t, Z)$, respectively. Therefore we have (c.f. §2 and Theorem 4.2 of [8])

(1.3) $$\begin{cases} d_2^{p,-8t} = Sq^2\colon H^p(X, Z) \to H^{p+2}(X, Z_2), \\ d_2^{p,-8t-1} = Sq^2\colon H^p(X, Z_2) \to H^{p+2}(X, Z_2), \\ d_3^{p,-8t-2} = \delta_2 Sq^2\colon H^p(X, Z_2) \to H^{p+3}(X, Z), \end{cases}$$

where δ_2 is the Bockstein operator associated with the exact coefficient sequence

$$0 \to Z \to Z \to Z_2 \to 0.$$

2. Proof of Theorem 1

0) was proved by J.F. Adams [1].

Proof of iv). We begin by applying for $PR(m)$ the spectral sequence of $\tilde{K}_O$-theory. Let $\psi(m)$ be the number of integers s such that $0 < s \leqslant m$ and $s \equiv 0, 4, 5, 6 \bmod 8$. Since we find (apart from zero terms) just $\psi(m)$ copies of Z_2 in E_2-terms which have total degree -4, there are at most $2^{\psi(m)}$ elements in $\tilde{K}_O^{-4}(RP(m))$.

On the other hand we show that $\tilde{K}_O^{-4}(RP(m))$ contains at least 2^{f-1} elements, where $f = \left[\dfrac{m}{2}\right]$. Consider

$$I^{-2}\varepsilon: \tilde{K}_O^{-4}(RP(m)) \to \tilde{K}_U^0(RP(m)) = Z_{2^f},$$

where I is the Bott isomorphism. By (1.1) we have $I^{-2}\varepsilon\,(kg_2\lambda)=2kv$, where g_2 is the generator of $\tilde{K}_O^0(S^4)$. Therefore in $\tilde{K}_O^{-4}(RP(m))$ we find 2^{f-1} elements $kg_2\lambda$ ($k=1, 2, \cdots, 2^{f-1}$). If $m\equiv 2$, 3 or 4 mod 8, then $\psi(m)=f-1$, so that $\tilde{K}_O^{-4}(RP(m))=Z_{2^{\psi(m)}}$, and $g_2\lambda$ generates the group.

The proof for the cases $m=0$, 1, 5, 6, 7 mod 8 is similar to that in the case 0) (c.f. [1]).

Proof of i). Consider the spectral sequence, if $m\neq 4r+3$ the term $E_2^{p+7,-p}$ is Z_2 for $p\equiv 1$ or 2 mod 8 such that $-7<p\leqslant m-7$, otherwise zero. However, if $m=4r+3$ we find an extra term $E_2^{4r+3,-4(r-1)}=Z$ in addition to the above.

By (1.3) the differentials

$$(2.1)\qquad d_2: E_2^{8t+6,-8t} \to E_2^{8t+8,-8t-1}$$

$$(2.2)\qquad d_2: E_2^{8t+7,-8t-1} \to E_2^{8t+9,-8t-2}$$

are isomorphisms except $d_2^{-1,7}=0$, therefore $E_3^{p+7,-p}=0$ except $E_3^{1,6}=Z_2$ and $E_3^{4r+3,-4(r-1)}=Z$ for $m=4r+3$. Since $d_k: E_k^{p+7,-p}\to E_k^{p+k+7,-p-k+1}$ (total degree 8) is a zero map for $k\geqslant 2$ (c.f. 0)), $E_3^{1,6}=Z_2$ survives to E_∞. Also $E_3^{4r+3,-4(r-1)}=Z$ survives to E_∞. Hence, we have

$$\tilde{K}_O^{-1}(RP(m)) = \begin{cases} Z_2 & \text{if } m\neq 4r+3\,, \\ Z+Z_2 \text{ or } Z & \text{if } m=4r+3\,. \end{cases}$$

Lemma (2.3). *In* $\varepsilon: \tilde{K}_O^{-1}(RP(4r+3))\to\tilde{K}_U^{-1}(RP(4r+3))=Z$, *we have*

$$\operatorname{Im}\varepsilon = \begin{cases} Z & \textit{if } r \textit{ is odd,} \\ 2Z & \textit{if } r \textit{ is even.} \end{cases}$$

Proof. By Theorem (3.3) of [5] we have $\tilde{K}_U^{-1}(RP(4r+3))=Z$. Considering the commutative diagram

$$\begin{array}{ccccccc} \tilde{K}_O^{-1}(RP(4r+3)) & \to & \tilde{K}_O^0(S^{4r+4}) & \to & \tilde{K}_O^0(RP(4r+4)) & \xrightarrow{i^!} & \tilde{K}_O^0(RP(4r+3)) \\ \varepsilon\downarrow & & \varepsilon\downarrow & & \varepsilon\downarrow & & \varepsilon\downarrow \\ \tilde{K}_U^{-1}(RP(4r+3)) & \to & \tilde{K}_U^0(S^{4r+4}) & \to & \tilde{K}_U^0(RP(4r+4)) & \xrightarrow{i^!} & \tilde{K}_U^0(RP(4r+3))\,, \end{array}$$

we can easily obtain the result by (1.1) and Ker $i^!=Z_2$.

Now, considering the commutative diagram

$$\begin{array}{ccccccc} Z=\tilde{K}_O^{-1}(S^{4r+3}) & \to & \tilde{K}_O^{-1}(RP(4r+3)) & \to & \tilde{K}_O^{-1}(RP(4r+2)) & \to & 0 \\ \varepsilon\downarrow & & \varepsilon\downarrow & & & & \\ Z=\tilde{K}_U^{-1}(S^{4r+3}) & \xrightarrow{\cong} & \tilde{K}_U^{-1}(RP(4r+3))\,, & & & & \end{array}$$

we obtain $\tilde{K}_O^{-1}(RP(4r+3))=Z+Z_2$. Finishing the proof of i).

Proof of v). We can easily obtain the results in the same way as the proof of i).

Proof of iii). If $m\neq 4r+1$ the term $E_2^{p+5,-p}$ is Z_2 for $p\equiv 1$ or $2 \bmod 8$ such that $-5<p\leqslant m-5$, otherwise zero. However, if $m=4r+1$ we find an extra term $E_2^{4r+1,-4(r-1)}=Z$ in addition to the above.

By (1.3) the differential

$$d_2 : E_2^{8t+6,-8t-1} \to E_2^{8t+8,-8t-2} \tag{2.4}$$

is an isomorphism except $d_2^{8r+6,-8r-1}=0$ for $m=8r+6$ or $8r+7$, therefore $E_3^{8t+6,-8t-1}=0$ except $E_3^{8r+6,-8r-1}=Z_2$ for $m=8r+6$ or $8r+7$.

By $d_k^{p+4,-p}=0$ $(k\geqslant 2)$ (c.f. iv)) and $d_2^{p,-8t-2}=0$, we have $E_2^{8t+7,-8t-2}=E_3^{8t+7,-8t-2}$. By (1.3) the differential

$$d_3 : E_3^{8t+7,-8t-2} \to E_3^{8t+10,-8t-4} \tag{2.5}$$

is an isomorphism except $d_3^{8r+7,-8r-2}=0$ for $m=8r+7$, $8r+8$ or $8r+9$, therefore $E_3^{8t+7,-8t-2}=0$ except $E_4^{8r+7,-8r-2}=Z_2$ for $m=8r+7$, $8r+8$ or $8r+9$.

By $d_k^{p+4,-p}=0$ $(k\geqslant 2)$

$$\begin{aligned} E_2^{4r+1,-4(r-1)} &= Z && \text{for } m=4r+1\,, \\ E_3^{8r+6,-8r-1} &= Z_2 && \text{for } m=8r+6 \text{ or } 8r+7\,, \\ E_4^{8r+7,-8r-2} &= Z_2 && \text{for } m=8r+7,\ 8r+8 \text{ or } 8r+9\,, \end{aligned}$$

all survive to E_∞. Hence, we have the following posibilities

$$\tilde{K}_O^{-3}(RP(m)) = \begin{cases} Z \text{ or } Z+Z_2 & \text{if } m=8r+1\,, \\ Z_2+Z_2 \text{ or } Z_4 & \text{if } m=8r+7\,, \end{cases}$$

and $\tilde{K}_O^{-3}(RP(m))$ is as stated in Theorem 1 for otherwise.

Now, considering the exact sequence

$$0=\tilde{K}_O^{-3}(RP(8r+2)) \to \tilde{K}_O^{-3}(RP(8r+1)) \to \tilde{K}_O^{-2}(S^{8r+2})=Z\,,$$

we obtain $\tilde{K}_O^{-3}(RP(8r+1))=Z$.

Next, by $RP(8r+7)/RP(8r+5)\approx S^{8r+6}\vee S^{8r+7}$ we have $\tilde{K}_O^{-3}(RP(8r+7)/RP(8r+5))=Z_2+Z_2$. Thus, considering the exact sequence

$$\tilde{K}_O^{-3}(RP(8r+7)/RP(8r+5)) \to \tilde{K}_O^{-3}(RP(8r+7)) \to \tilde{K}_O^{-3}(RP(8r+5))=Z\,,$$

we obtain $\tilde{K}_O^{-3}(RP(8r+7))=Z_2+Z_2$. Finishing the proof of iii).

Proof of vii). Similar to the proof of iii).

Proof of ii). The term $E_2^{p+6,-p}$ is Z_2 for $p=0$, 1, 2 or 4 mod 8 such that $-6<p\leqslant m-6$, otherwise zero. By (2.1), (2.2) and (2.4) we have $E_3^{8t+6,-8t}=E_3^{8t+7,-8t-1}=E_3^{8t+8,-8t-2}=0$ except $E_3^{8r+6,-8r}=Z_2$ for $m=8r+6$ or $8r+7$ and $E_3^{8r+7,-8r-1}=Z_2$ for $m=8r+7$ or $8r+8$. Also, by (2.5) we have $E_4^{8t+10,-8t-4}=0$ except $E_4^{2,4}=Z_2$.

Obviously $E_3^{8r+6,-8r}=E_4^{8r+6,-8r}$ and $E_3^{8r+7,-8r-1}=E_4^{8r+7,-8r-1}$, and since we have $d_k^{p+5,-p}=0$ for $k\geqslant 4$ (c.f. iii)), $E_4^{8r+6,-8r}=Z_2$ (for $m=8r+6$ or $8r+7$) and $E_4^{8r+7,-8r-1}=Z_2$ (for $m=8r+7$ or $8r+8$) survive to E_∞. Also, since d_k: $E_k^{p+6,-p}\to E_k^{p+k+6,-p-k+1}$ (total degree 7) is a zero mape for $k\geqslant 3$ (c.f. i)), $E_4^{2,4}=Z_2$ survives to E_∞. Hence, we have the following posibilities

$$\tilde{K}\tilde{O}^{-2}(RP(m))=\begin{cases} Z_2 & \text{if } m=8r+1,\ 8r+2,\ 8r+3,\ 8r+4 \text{ or } 8r+5\,, \\ Z_2+Z_2 \text{ or } Z_4 & \text{if } m=8r(r\neq 0) \text{ or } 8r+6\,, \\ Z_2+Z_2+Z_2,\ Z_4+Z_2 \text{ or } Z_8 & \text{if } m=8r+7\,. \end{cases}$$

Now, in order to complete the proof we show the next lemma.

Lemma (2.6). $2\tilde{K}\tilde{O}^{-2}(RP(m))=0$.

Proof. It is sufficient to ensure that it is true for $m=8r+6$, $8r+7$ or $8r+8$ ($r=0, 1, \cdots$). First we show $4\tilde{K}\tilde{O}^{-2}(RP(m))=0$. Considering the exact sequence

$$\tilde{K}\tilde{O}^{-3}(RP(8r+7))\to\tilde{K}\tilde{O}^{-3}(RP(8r+5))\to\tilde{K}\tilde{O}^{-2}(RP(8r+7)/RP(8r+5))$$
$$\to\tilde{K}\tilde{O}^{-2}(RP(8r+7))\to\tilde{K}\tilde{O}^{-2}(RP(8r+5))\to\tilde{K}\tilde{O}^{-1}(RP(8r+7)/RP(8r+5))\,,$$

we have $\tilde{K}\tilde{O}^{-2}(RP(8r+7))\neq Z_8$. That is $4\tilde{K}\tilde{O}^{-2}(RP(m))=0$.

We have the following exact sequence (2.7) for the fibering $U\to U/O$, $B_O\times Z=\Omega(U/O)$ (c.f. p. 314 of [10]).

$$\cdots\to\tilde{K}\tilde{O}^{n}(X)\xrightarrow{\varepsilon}\tilde{K}\tilde{U}^{n}(X)\xrightarrow{p_*}\tilde{K}\tilde{O}^{n+2}(X)\xrightarrow{\partial}\tilde{K}\tilde{O}^{n+1}(X)\to\cdots. \tag{2.7}$$

Applying the exact sequence (2.7) for $RP(m)$ and $n=-2$, we obtain the exact sequence

$$\to\tilde{K}\tilde{O}^{-2}(RP(m))\xrightarrow{\varepsilon}\tilde{K}\tilde{U}^{-2}(RP(m))\xrightarrow{p_*}\tilde{K}\tilde{O}^{0}(RP(m))\xrightarrow{\partial}\tilde{K}\tilde{O}^{-1}(RP(m))\to\tilde{K}\tilde{U}^{-1}(RP(m))\,.$$

If $m-8r=6$, 7 or 8, then $f=\left[\dfrac{m}{2}\right]=\varphi(m)$, so that we have $\tilde{K}\tilde{U}^{-2}(RP(m))=Z_{2^f}$ and $\tilde{K}\tilde{O}^{0}(RP(m))=Z_{2^f}$. Since $\tilde{K}\tilde{O}^{-1}(RP(m))=Z_2$ or $Z+Z_2$ and $\tilde{K}\tilde{U}^{-1}(RP(m))=0$ or Z (c.f. Theorem (3.3) of [5]), we have $\operatorname{Im}\partial=Z_2$. Therefore $\operatorname{Im}p_*=\operatorname{Ker}\partial=Z_{2^{f-1}}$. Hence $\operatorname{Im}\varepsilon=\operatorname{Ker}p_*=Z_2$, that is $2\operatorname{Im}\varepsilon=0$ and $\operatorname{Im}\varepsilon\subset 2^{f-1}\times\tilde{K}\tilde{U}^{-2}(RP(m))$.

Now, considering

$$\tilde{K}\tilde{O}^{-2}(RP(m))\xrightarrow{\varepsilon}\tilde{K}\tilde{U}^{-2}(RP(m))\xrightarrow{\rho}\tilde{K}\tilde{O}^{-2}(RP(m))\,,$$

we have $2\tilde{K}\tilde{O}^{-2}(RP(m))=\operatorname{Im}\rho\varepsilon\subset 2^{f-1}\times\tilde{K}\tilde{O}^{-2}(RP(m))=2^{f-3}\times 4\tilde{K}\tilde{O}^{-2}(RP(m))=0$. This shows the lemma. Finishing the proof of ii).

Proof of vi). We can easily obtain the results in the same way as the proof of ii).

This completes the proof of Theorem 1.

3. Proof of Theorem 2

0) was proved by B.J. Sanderson [7].

Proof of vii). The term $E_2^{p+1,-p}$ is Z_2 for $p\equiv 1 \bmod 8$ such that $-1<p\leqslant 2n-1$, otherwise zero. By (1.3) the differential

$$d_2 : E_2^{8t+2,-8t-1} \to E_2^{8t+4,-8t-2}$$

is an isomorphism except $d_2^{8r+2,-8r-1}=0$ for $n=4r+1$. Therefore $E_3^{p+1,-p}=0$ except $E_3^{8r+2,-8r-1}=Z_2$ for $n=4r+1$. Hence, we have the following posibilities

$$\tilde{K}_O^{-7}(CP(n)) = \begin{cases} 0 & \text{if } n\neq 4r+1\,, \\ 0 \text{ or } Z_2 & \text{if } n=4r+1\,. \end{cases}$$

Now, considering the exact sequence

$$\tilde{K}_O^0(CP(4r+1)) \to \tilde{K}_O^0(CP(4r)) \to \tilde{K}_O^1(S^{8r+2}) \to \tilde{K}_O^1(CP(4r+1)) \to 0\,,$$

we obtain $\tilde{K}_O^{-7}(CP(4r+1))=Z_2$. Finishing the proof of vii).

Proof of v) and i). We can easily obtain the results in the same way as the proof of vii).

Proof of vi). The proof is given by induction on n. For $n=0$ our assertion is trivial. Suppose that $\tilde{K}_O^{-6}(CP(n))$ is as stated for $n<4t+1$. Considering the exact sequence

$$0 \to \tilde{K}_O^{-6}(S^{8t+2}) \xrightarrow{j^!} \tilde{K}_O^{-6}(CP(4t+1)) \to \tilde{K}_O^{-6}(CP(4t)) \to 0\,,$$

we have

$$\tilde{K}_O^{-6}(CP(4t+1)) \cong \tilde{K}_O^{-6}(CP(4t))+Z\,.$$

Let α is a generator of $\tilde{K}_O^{-6}(S^{8t+2})=\tilde{K}_O^0(S^{8t+8})$, then we have $j_c^!\varepsilon\alpha=g^3\mu^{4t+1}$. On the other hand we have $\varepsilon\mu_3\mu_0^{2t}=g^3(\mu-\bar{\mu})(\mu+\bar{\mu})^{2t}=2g^3\mu^{4t+1}$, because $\bar{\mu}=-\mu+\mu^2-\cdots-\mu^{4t+1}$ from Theorem (7.2) of [1]. Therefore, putting $\tau=j^!\alpha$, we have $2\tau=\mu_3\mu_0^{2t}$. Thus, μ_3, $\mu_3\mu_0$, $\cdots$, $\mu_3\mu_0^{2t-1}$, τ additively generate $\tilde{K}_O^{-6}(CP(4t+1))$.

Next, considering the exact sequence

$$\begin{aligned} 0 \to \tilde{K}_O^{-7}(CP(4t+1)) \to \tilde{K}_O^{-6}(S^{8t+4}) \to \tilde{K}_O^{-6}(CP(4t+2)) \\ \to \tilde{K}_O^{-6}(CP(4t+1)) \to \tilde{K}_O^{-5}(S^{8t+4}) \to 0\,, \end{aligned}$$

we have

$$\tilde{K}_O^{-6}(CP(4t+2)) = \overbrace{Z+\cdots+Z}^{2t+1}$$

and μ_3, $\mu_3\mu_0$, $\cdots$, $\mu_3\mu_0^{2t}$ additively generate the group.

Next, considering the exact sequence

$$0 \to \tilde{K}_O^{-6}(S^{8t+6}) \xrightarrow{j^!} \tilde{K}_O^{-6}(CP(4t+3)) \to \tilde{K}_O^{-6}(CP(4t+2)) \to 0\,,$$

we have

$$\tilde{K}_O^{-6}(CP(4t+3)) \cong \tilde{K}_O^{-6}(CP(4t+2))+Z$$

and μ_3, $\mu_3\mu_0$, $\cdots$, $\mu_3\mu_0^{2t+1}$ additively generate the group, because $j^!\alpha=\mu_3\mu_0^{2t+1}$ for a generator α of $\tilde{K}_O^{-6}(S^{8t+6})$.

Moreover, considering the exact sequence

$$0 \to \tilde{K}_O^{-6}(CP(4t+4)) \to \tilde{K}_O^{-6}(CP(4t+3)) \to 0\,,$$

we have

$$\tilde{K}_O^{-6}(CP(4t+4)) \cong \tilde{K}_O^{-6}(CP(4t+3))\,.$$

This completes the induction.

Proof of iv) can be treated in the same way as that of vi).

Proof of iii) and ii) can be treated in the same way as that of vii) and vi) respectively.

4. Proof of Theorem 3

We apply the Chern characters for $\tilde{K}_O^{-2i}(CP(n))$. By Lemma (1.2) we have

$$\text{ch}\,\varepsilon\mu_i = \begin{cases} e^y+e^{-y}-2 & \text{if } i \text{ is even,} \\ e^y-e^{-y} & \text{if } i \text{ is odd,} \end{cases} \tag{4.1}$$

where y is a generator of the cohomology group $H^2(CP(n); Z)$. Therefore we have

$$\begin{aligned} &\text{(4.2)} \quad \text{ch}\,\varepsilon\mu_i\mu_j = (e^y-e^{-y})^2 \\ &\qquad = 4(e^y+e^{-y}-2)+(e^y+e^{-y}-2)^2 \quad \text{if } i, j \text{ odd,} \\ &\text{(4.3)} \quad \text{ch}\,\varepsilon\mu_i\mu_j = (e^y+e^y-2)^2 \quad \text{if } i, j \text{ even,} \\ &\text{(4.4)} \quad \text{ch}\,\varepsilon\mu_i\mu_j = (e^y-e^{-y})(e^y+e^{-y}-2) \quad \text{if } i \text{ odd, } j \text{ even.} \end{aligned}$$

If n is even $\text{ch}\,\varepsilon$ is a monomorphism (c.f. Theorem 2). Hence, (4.1) and (4.2) imply i), iii) and vi); (4.1) and (4.3) imply ii); and (4.1) and (4.4) imply iv) and v).

In case of $n=2t-1$, the results of Theorem 3 are induced from that in case of $n=2t$ by the inclusion map $CP(2t-1)\subset CP(2t)$. This completes the proof.

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