Watabe, T.
Osaka J. Math.
4 (1967), 133-140

$SO(r)$-COBORDISM AND EMBEDDING OF 4-MANIFOLDS*

TSUYOSHI WATABE

(Received December 15, 1966)

Introduction. Let M be a C^∞-manifold, which we assume to be compact and orientable. We shall call M an $SO(r)$-manifold if the structure group of the stable normal bundle of M, which is the stable special orthogonal group SO, is reducible to a subgroup $SO(r)$. Two n-dimensional closed $SO(r)$-manifolds M_1 and M_2 are called to be $SO(r)$-cobordant if there exists an $(n+1)$-dimensional $SO(r)$-manifold W whose boundary is union of M_1 and $-M_2$ ($-M_2$ denotes M_2 with the reversed orientation) and the restriction of the $SO(r)$-structure of W to boundary induces the given structure of M_1 and M_2. We can define the $SO(r)$-cobordism group, which we denote by $\Omega_n(SO(r))$. In his paper [5], Liulevicius has calculated the group $\Omega_n(SO(r))$ for $r=2$ and $n\leqq 8$.

In this note, we shall apply Liulevicius'result to the embedding of 4-manifold in Euclidean space. Our main result is the following:

Theorem (5.1) *Let M be an orientable 4-manifold which is oriented cobordant to zero. If M is immersible in R^6, then M is embeddable in R^7.*

As a corollary to this theorem, we have

Theorem (5.2) *Any simply connected 4-manifold which is oriented cobordant to zero is embeddable in R^7.*

Throughout this note, we assume that a manifold is compact, orientable and of the class C^∞.

The author wishes to express his hearty thanks to Professors M. Adachi and Y. Shikata for their kind discussions and valuable suggestions.

1. $SO(r)$-cobordism group

Let M be an n-manifold and embedded in a Euclidean $(n+N)$-space with normal bundle ν. If the structure group of ν is reducible to a subgroup $SO(r)$, then M is called an $SO(r)$-manifold. More precisely, by the theorem 9.4 in [6], the structure group of ν is reducible to $SO(r)$ if and only if there exists a map $f: M\to E$ so that the diagram

---

* This paper was partially supported by Yukawa Fellowship.

$$\begin{tikzcd} & E \arrow[d] \\ M \arrow[ur, "f"] \arrow[r, "f_\nu"'] & BSO(N) \end{tikzcd}$$

is commutative, where E is the fibre bundle associated to the universal N-plane bundle with $SO(N)/SO(r)$ as fibre and f_ν is the classifying map for ν. Then we call the pair (M, f) an $SO(r)$-manifold and f an $SO(r)$-structure. We shall identify an $SO(r)$-structure with those induced from it by suspension. A homotopy class of f determines uniquely an $SO(r)$-structure on M.

The following lemma is well known.

Lemma (1.1) (1) *If a manifold W with boundary bW admits an $SO(r)$-structure f, then its boundary bW also admits an $SO(r)$-structure, i.e. f/bW.*
(2) *If two manifolds W_1 and W_2 admit $SO(r)$-structures f_1 and f_2 respectively which induce the same structure on the common boundary $bW_1 = bW_2$, then the manifold W obtained from the union of W_1 and W_2 by identifying the boundary also admits an $SO(r)$-structure, i.e. $f_1 \cup f_2$.*

An n-dimensional $SO(r)$-manifold (M, f) without boundary is called to be $SO(r)$-cobordant to zero if there is an $(n+1)$-dimensional $SO(r)$-manifold (W, F) such that $bW = M$ and $F/bW = f$. Two $SO(r)$-manifolds (M_1, f_1) and (M_2, f_2) are called to be $SO(r)$-cobordant if the disjoint union $(M_2 \cup -M_1, f_1 \cup f_2)$ is $SO(r)$-cobordant to zero.

As a corollary to lemma (1.1), we have

Corollary. *The $SO(r)$-cobordism is an equivalence relation.*

Let $[M, f]$ denote the $SO(r)$-cobordism calss of (M, f). We define an addition of two classes by $[M_1, f_1] + [M_2, f_2] = [M_1 \cup M_2, f_1 \cup f_2]$, where $M_1 \cup M_2$ denotes the disjoint union of M_1 and M_2. By this addition, the set of all cobordism classes admits an abelian group structure. We denote this group by $\Omega_n(SO(r))$.

2. Homotopy interpretation of the group $\Omega_n(SO(r))$

In this section, we shall prove the following

Proposition (2.1) *We have an isomorphism*

$$\Omega_n(SO(r)) \cong \pi_{N+n}(S^{N-r}MSO(r)) \qquad (N \geqq n+2)$$

where $MSO(r)$ is the Thom space of the universal r-plane bundle and $S^{N-r}MSO(r)$ the $(N$-$r)$-fold suspension of $MSO(r)$.

We shall first prove the stability of homotopy group of $S^{N-r}MSO(r)$; the suspension homomorphism

$$S\colon \pi_{n+N}(S^{N-r}MSO(r)) \to \pi_{n+1+N}(S^{N+1-r}MSO(r))$$

is an isomorphism for $n+1 \leqq N$. In fact, since $MSO(r)$ is $(r-1)$-connected, the suspension homomorphism $S\colon \pi_j(MSO(r)) \to \pi_{j+1}(SMSO(r))$ is isomorphic for $j \leqq 2r-1$, by a theorem of Blaker-Massey. Thus $SMSO(r)$ is r-connected. Similarly it is known that $S^{N-r}(MSO(r))$ is $(N-1)$-connected. By the theorem of Blaker-Massey,

$$S\colon \pi_j(S^{N-r}MSO(r)) \to \pi_{j+1}(S^{N-r+1}MSO(r))$$

is an isomorphism for $j \leqq 2N-1$. This implies that

$$S\colon \pi_{n+N}(S^{N-r}MSO(r)) \to \pi_{n+N+1}(S^{N-r+1}MSO(r))$$

is isomorphic for $n+1 \leqq N$.

Now we shall construct an isomorphism $\Omega_n(SO(r)) \cong \pi_{n+N}(S^{N-r}MSO(r))$. Suppose that M be a closed n-dimensional $SO(r)$-manifold. Embed M in S^{n+N} with normal bundle (we consider normal disk bundle) $\nu\colon A \to M$, which we assume to admit an $SO(r)$-structure f. There is a bundle map

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}_\nu} & E(\eta_N) \\ {\scriptstyle \nu}\downarrow & & \downarrow \\ M & \xrightarrow[f_\nu]{} & BSO(N) \end{array}$$

where η_N denotes the universal N-disk bundle. Since ν admits an $SO(r)$-structure, there exists a map $g\colon M \to BSO(r)$ such that $f_\nu = i \circ g$, where i: $BSO(r) \to BSO(N)$ is the inclusion map. Thus we have a commutative diagram of bundle maps

$$\begin{array}{ccccc} A & \xrightarrow{\bar{g}} & E(\eta_r \oplus \varepsilon^{N-r}) & \xrightarrow{\bar{i}} & E(\eta_N) \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow[g]{} & BSO(r) & \xrightarrow[i]{} & BSO(N) \end{array}$$

and an induced map $A/\mathring{A} \to S^{N-r}MSO(r)$, where $\mathring{A}$ denotes the associated sphere bundle of A. Then there exists the composite map F given by

$$F\colon S^{n+N} \to S^{n+N}/S^{n+N} - \operatorname{int} A \cong A/\mathring{A} \to S^{N-r}MSO(r)\,.$$

Thus to an $SO(r)$-manifold corresponds a map $F\colon S^{n+N} \to S^{N-r}MSO(r)$. Conversely if a map $F\colon S^{n+N} \to S^{N-r}MSO(r)$ is given, we can find a map $F'\colon S^{n+N} \to S^{N-r}MSO(r)$ so that

(1) F' is a smooth map homotopic to F

and

(2) F' is t-regular on $BSO(r)$.

Then F'^{-1} $(BSO(r))$ is an n-dimensional smooth submanifold of S^{n+N} with normal bundle $\nu\oplus\varepsilon^{N-r}$. These correspondence induces an isomorphism between $\Omega_n(SO(r))$ and π_{n+N} $(S^{N-r}MSO(r))$. (For the details, see [7])

Next we shall prove the following

Proposition (2.2) *The natural homomorphism $\Omega_n(SO(r))\to\Omega_n$ is isomorphic for $n+1\leqq r$.*

Proof. To prove the proposition, we need the following lemmas.

Lemma (2.1) *The homomorphism $\pi_j(SMSO(r))\to\pi_j(MSO(r+1))$ induced by the inclusion $SMSO(r)\to MSO(r+1)$ is isomorphic for $j\leqq 2r$.*

For the proof, see [1].

Lemma (2.2) *The iterated suspension homomorphism S^{N-r-1}: $\pi_j(SMSO(r))\to\pi_{j+N-r-1}(S^{N-r}MSO(r))$ is isomorphic for $j\leqq 2r+1$.*

Proof. This is a straightforward application of the theorem of Blaker-Massey.

Now we consider the diagram

$$\begin{array}{ccc}\pi_j(SMSO(r)) & \longrightarrow & \pi_j(MSO(r+1)) \\ \big\downarrow S^{N-r-1} & & \big\downarrow S^{N-r-1} \\ \pi_{j+N-r-1}(S^{N-r}MSO(r)) & \longrightarrow & \pi_{j+N-r-1}(S^{N-r-1}MSO(r+1))\end{array}$$

Since the vertical and top horizontal homomorphism are isomorphic for $j\leqq 2r$, we have

$$\pi_{j+N-r-1}(S^{N-r}MSO(r)) \cong \pi_{j+N-r-1}(S^{N-r-1}MSO(r+1)) \quad (j\leqq 2r)$$

Thus we have

$$\pi_{n+N}(S^{N-r}MSO(r)) \cong \pi_{n+N}(S^{N-r-1}MSO(r+1))$$

for $n+1\leqq r$. In other words, if $n+1\leqq r$, $\Omega_n(SO(r))\cong\Omega_n(SO(r+1))$.

3. The group $\Omega_n(SO(2))$

In section 2, we have shown that $\Omega_n(SO(2))\cong\pi_{n+N}(S^{N-2}MSO(2))$ (N is sufficiently large integer). R. Thom has shown that $MSO(2)$ can be identified with the infinite dimensional complex projective space CP^∞. For a space X, let $\pi_m^s(X)$ denote the m-th stable homotopy group of X.

In his paper [5], Liulevicius has obtained the following results;

m	1	2	3	4	5	6	7	8
$\pi_m^s(CP^\infty)$	0	Z	0	Z	Z_2	Z	Z_2	$Z+Z_2$

By definition, $\pi_{m+2}^s(CP^\infty)$ is nothing else than $\Omega_m(SO(2))$.

4. The characteristic numbers

Let M be a closed $SO(r)$-manifold with dimension n and f the classifying map for the normal bundle; $f\colon M\to BSO(r))$. f induces a homomorphism $f^*\colon H^n(BSO(r)); Q)\to H^n(M; Q)$. For any element $x\in H^n(BSO(r); Q)$, we call $f^*(x)\,[M]\in Q$ the normal characteristic number corresponding to x, where $[M]$ is the fundamental class of M.

We have the commutative diagram

$$\begin{array}{ccccc} H^{n+r+N}(S^NMSO(r); Q) & \xrightarrow{T(f)^*} & H^{n+r+N}(T(M); Q) & \longrightarrow & H^{n+r+N}(S^{n+r+N}; Q) \\ \uparrow \cong & & \uparrow \cong & & \\ H^n(BSO(r); Q) & \xrightarrow[f^*]{} & H^n(M; Q) & & \end{array}$$

where $T(M)$ denotes the Thom space of the normal bundle of M in S^{n+r+N}. The isomorphism $\pi_{n+r+N}(S^NMSO(r))\otimes Q\cong H_{n+r+N}(S^NMSO(r); Q)$ implies that if $\bar{f}\colon S^{n+r+N}\to S^NMSO(r)$ represents a class in the free part of $\pi_{n+r+N}\times(S^NMSO(r))$, $\bar{f}^*\colon H^{n+r+N}(S^NMSO(r); Q)\to H^{n+r+N}(S^{n+r+N}; Q)$ is a zero homomorphism only if $\bar{f}$ is homotopic to zero. Hence f^* is a zero homomorphism only if $\bar{f}$ is homotopic to zero. In other words, if all normal characteristic numbers of M are zero, then the cobordism class of M is zero in $\Omega_n(SO(r))/$ torsion subgroup. It is easy to see that if two $SO(r)$-manifolds are $SO(r)$-cobordant, then they have the same characteristic number. Thus we have proved.

Theorem (4.1) *The class of $\Omega_n(SO(r))/$torsion subgroup is completely determined by the normal Euler and Pontrjagin numbers.*

Corollary. *The natural homomorphism $\Omega_4(SO(2))\to\Omega_4$ is monomorphic.*

Proof. Since $\Omega_4(SO(2))$ has no torsion, for 4-manifold, the normal Euler number completely determines $SO(2)$-cobordism class. Suppose an $SO(2)$-manifold M be oriented cobordant to zero. Then the Pontrjagin number $p_1[M]$ is zero, and hence $\bar{p}_1[M]=0$. Since $\bar{p}_1=\bar{X}_2^2$, this implies that the normal Euler number $\bar{X}_2^2[M]=0$. Therefore M is $SO(2)$-cobordant to zero. This completes the proof.

5. Embedding of 4-manifolds

In this section, we shall consider the embeddability of closed 4-manifold M in Euclidean 7-space R^7.

We know the following results about the embeddability of M.

(1) Any closed 4-manifold is embeddable in R^8.

(2) Any closed 4-manifold M is embeddable in R^7 almost differentaibly (i.e. M-x is embeddable in R^7, where x is a point of M) [2].

(3) Any orientable closed 4-manifold is embeddable in R^7 piecewise linearly [3].

In the following, we shall prove

Theorem (5.1) *Let M be an orientable, closed 4-manifold which is oriented cobordant to zero. If M is immersible in R^6, then M is embeddable in R^7.*

Theorem (5.2) *Let M be closed, simply connected and oriented cobordant to zero. Then M is embeddable in R^7.*

In his paper [9], the author has proved the following result.

(4) Let M be a closed, simply connected and s-parallelizable 4-manifold. Then M is embeddable in R^6.

The proof of theorem (5.1).

Let (M, f) be an n-dimensional $SO(2)$-manifold with or without boundary. We shall prove the following lemma.

Lemma (5.1) *Suppose $n \geqq 2$. Then there exists an n-dimensional $SO(2)$-manifold (M', f') such that*

(1) *M' is simply connected,*

and

(2) *If M has boundary, then $bM=bM'$ and $f|bM=f'|bM'$.*

Proof. Let α be a non-zero element of $\pi_1(M)$ represented by a map $\bar{g}: S^1 \to M$. Since the dimension of M is greater than 1 and M is orientable, $\bar{g}$ is homotopic to an embedding $g: S^1 \to M$ with a trivial normal bundle. g can be considered as a map

$$g\colon\ S^1 \times D^{n-1} \longrightarrow M .$$

If $bM \neq \phi$, we may assume that $g(S^1 \times D^{n-1}) \subset \operatorname{int} M$. Define an $(n+1)$-manifold W as follows;

$$W = (M \times [0, 1]) \cup D^2 \times D^{n-1} ,$$

where $D^2 \times D^{n-1}$ is attached to $M \times 1$ by the embedding $g\colon S^1 \times D^{n-1} \to M$.

By smoothing the corners, W admits a smooth structure (Compare Kervaire-Milnor [4], p. 519). Clearly W has the homotopy type of $M \cup D^2$. Then

$$H^i(W, M; Z) = \begin{cases} Z & \text{if } i=2 \\ 0 & \text{otherwise.} \end{cases}$$

We shall extend the given $SO(2)$-structure on M over W. Consider the following diagram

$$\begin{tikzcd} & E \arrow[d] \\ M \arrow[ur, "f"] \arrow[r, "f_0"'] & BSO(N) \end{tikzcd}$$

where E is the bundle associated to the universal N-plane bundle with fibre $SO(N)/SO(2)$. Obstructions to the extending f over W are in $H^i(W, M; \pi_{i-1}(V_{N,N-2}))$. Since $H^i(W, M)=0$ for $i \neq 2$, there is only one obstruction in $H^2(W, M; \pi_1(V_{N,N-2}))$, which is zero. Hence f is extendable over W.

Let M'' be the component of bW different from M. Since W is an $SO(2)$-manifold, M'' is also an $SO(2)$-manifold. As well known,

$$\pi_1(M'') = \pi_1(M)/\{\alpha\},$$

where $\{\alpha\}$ is some subgroup of $\pi_1(M)$ containing α. After a finite number of steps, we can obtain a manifold M' with the desired properties.

Now let M be a 4-manifold which is oriented cobordant to zero. By corollary of theorem (4.1), M is $SO(2)$-cobordant to zero. In other words, there exists an $SO(2)$-manifold W with dimension 5 whose boundary is M. By lemma (5.1), we may suppose that W is simply connected. $\tilde{W}$ is the double of W. By lemma (1.1), $\tilde{W}$ is also an $SO(2)$-manifold, and by van Kampen theorem, $\tilde{W}$ is simply connected. The following theorem which is due to Hirsch [3] implies that $\tilde{W}-x$ is embeddable in R^7 and hence M is embeddable in R^7.

Theorem of Hirsch. *Let M be a closed $(m-1)$-connected n-manifold, $2 \leqq 2m \leqq n$, with a smooth triangulation; let x be a point of M. If a neighbourhood of the $(n$-$m)$-skelton can be immersed in R^q for $q \geqq 2n-2m+1$, then M-x can be embedded in R^q.*

A 4-manifold is immersible in R^6 if and only if it is an $SO(2)$-manifold. This completes the proof of theorem (5.1).

The proof of theorem (5.2).

To obtain the theorem, it is sufficient to show that any closed simply connected M which is oriented cobordant to zero is immersible in R^6. We can find a 5-manifold W whose boundary is M. By a theorem of Wall [8], we may suppose that W has the homotopy type of a bouquet of 2-spheres. Such a 5-manifold is immersible in R^7. In fact, let W be embedded in R^{5+N}, where N is sufficiently large, with normal bundle ν and $\tilde{\nu}$ the bundle associated to ν with $V_{N,N-2}$ as fibre. Then $\tilde{\nu}$ admits a cross section if and only if W can be immersed in R^7. Since $H^i(W)=0$ for $i\geqslant 3$, there is no obstruction to the existence of cross section of $\tilde{\nu}$. Since W is immersible in R^7, its boundary is immersible in R^6. The proof of theorem is concluded.

REMARK. The result of theorem (5.2) is best possible. In fact, the total space of non-trivial 2-sphere bundle over 2-sphere can not be embedded in R^6, because it has non-vanishing 2nd Stiefel-Whitney class.

NIIGATA UNIVERSITY

References

[1] P.E. Conner and E.E. Floyd: Differentiable periodic maps, Springer-Verlag, 1964.

[2] M.W. Hirsch: *On imbedding differentiable manifold in Euclidean space,* Ann. of Math. **73** (1961), 566–571.

[3] ————: *On embedding* 4-*manifold in* R^7, Proc. Cambridge Philos. Soc. **61** (1965), 657–658.

[4] M. Kervaire and J. Milnor: *Groups of homotopy spheres* I, Ann. of Math. **77** (1963), 504–537.

[5] A. Liulevicius: *A theorem in homological algebras and stable homotopy group of projective spaces,* Trans. Amer. Math. Soc. **109** (1963), 540–552.

[6] N.E. Steenrod: The topology of fibre bundles, Princeton, 1951.

[7] R. Thom: *Quelques propriétés globales des variétés difféntiables,* Comm. Math. Helv. **28** (1954), 17–86.

[8] C.T.C. Wall: *On simply connected* 4-*manifold,* J. London Math. Soc. **39** (1964), 141–149.

[9] T. Watabe: *On imbedding closed* 4-*manifolds in Euclidean space*, Sci. Rep. Niigata Univ. Ser. A **3** (1966), 9–13.