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# ON INDUCED REPRESENTATIONS

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Let $G$ be a locally compact topological group and $K$ a compact subgroup of $G$. For any irreducible unitary representation $\sigma$ of $K$, we denote by $U(\sigma)$ the induced representation generated by $\sigma$ (see §1). In general, $U(\sigma)$ is not irreducible.

The purpose of this paper is to give a method of extracting the irreducible components of $U(\sigma)$ when $G$ is one of the special types of Lie groups.

**1.** Let $G$ be a connected non-compact semisimple Lie group with a finite dimensional faithful representation and $K$ a maximal compact subgroup of $G$. We assume that rank $G$=rank $K$. For any given irreducible unitary representation $\sigma$ of $K$ on a representation space $V$, we can construct a unitary representation $U(\sigma)$ of $G$ as follows. Let $\mathfrak{H}(\sigma)$ be the set of all "Haar-measurable" $V$-valued functions $f$ which satisfy the following conditions;

$$f(kx) = \sigma(k)f(x) \qquad (k\in K,\ x\in G)$$

and

$$\|f\|^2 = \int_G \|f(x)\|_V^2\, dx < \infty$$

where $\| \ \|_V$ denotes the norm in $V$.

Then $\mathfrak{H}(\sigma)$ is a Hilbert space if we identify functions which differ only on subsets of $G$ of Haar measure zero. The inner product $(\ ,\ )$ in $\mathfrak{H}(\sigma)$ is given by

$$(f_1, f_2) = \int_G (f_1(x), f_2(x))_V dx \qquad (f_1, f_2\in \mathfrak{H}(\sigma))$$

where $(\ ,\ )_V$ denotes the inner product in $V$. Finally for any $g\in G$, $U_g(\sigma)$ is defined by

$$(U_g(\sigma)f)(x) = f(xg) \qquad (f\in \mathfrak{H}(\sigma),\ x\in G)\,.$$

Thus we obtained the induced representation $U(\sigma)$ generated by $\sigma$ (cf. [7] ($d$)).

Our aim is to find out an irreducible closed subspace of $\mathfrak{H}(\sigma)$.

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**2.** Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $G$ (resp. $K$) and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Let $T$ be a maximal torus in $K$ and $\mathfrak{h}$ the Lie algebra of $T$. Then since $G$ has a finite dimensional faithful representation and rank $G$=rank $K$, $T$ is a Cartan subgroup of $G$; i.e.

$$T=\{g\in G;\ \mathrm{Ad}(g)H=H \quad \text{for all} \quad H\in\mathfrak{h}\}$$

where Ad denotes the adjoint representation of $G$. Let $\mathfrak{g}^C$ denote the complexification of $\mathfrak{g}$. Let $\Sigma$ be the set of all non-zero roots of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h}^C$ ($\mathfrak{h}^C$ is the subspace of $\mathfrak{g}^C$ spanned by $\mathfrak{h}$). We denote by $\Sigma_k$ the set of all compact roots (see for definition [8]). Let $\mathcal{F}$ be the vector space over $\boldsymbol{R}$ (the field of real numbers) consisting of all purely imaginary complex valued linear forms on $\mathfrak{h}$. Then $\Sigma\subset\mathcal{F}$. Introduce a linear order in $\mathcal{F}$ and let $P$ (resp. $P_k$) be the set of all positive roots in $\Sigma$ (resp. $\Sigma_k$). We denote by $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}^C$. As usual we regard elements of $\mathfrak{G}$ as left-invariant differential operators on $G$. Since $V$ is finite dimensional, it has the canonical structure of an analytic manifold. We denote by $C^0(G, V)$ the vector space of all continuous mappings from $G$ to $V$. For any $f\in C^0(G, V)$ and $v\in V$, we put

$$f_v(x)=(f(x), v)_V \qquad (x\in G)\,.$$

Let $C^\infty(G)$ be the vector space of all infinitely differentiable complex valued functions on $G$. We denote by $C^\infty(G, V)$ the set of all $f\in C^0(G, V)$ such that

$$f_v\in C^\infty(G) \quad \text{for all} \quad v\in V\,.$$

We often call $f\in C^\infty(G, V)$ a $V$-valued $C^\infty$-function. Define

$$(U_X(\sigma)f)(g)=\lim_{t\to 0}\frac{1}{t}\,(f(g\exp tX)-f(g)) \qquad (g\in G)$$

for $X\in\mathfrak{g}$ and $f\in C^\infty(G, V)$. Then we have a representation $X\to U_X(\sigma)$ of $\mathfrak{g}$ on $C^\infty(G, V)$. This extends uniquely to a representation of $\mathfrak{G}$. It is obvious that

$$(U_u(\sigma)f)_v=uf_v \quad \text{for all} \quad u\in\mathfrak{G}\,.$$

In the following, we shall simply write $uf$ instead of $U_u(\sigma)f$. Let $\mathfrak{Z}$ be the center of $\mathfrak{G}$ and $\Omega$ the Casimir operator of $\mathfrak{g}$. Then $\Omega\in\mathfrak{Z}$. For any $g\in G$, we define

$$(R(g)f)(x)=f(xg) \qquad (x\in G, f\in C^\infty(G))\,.$$

Then an element $u$ of $\mathfrak{G}$ belongs to $\mathfrak{Z}$ if and only if $R(g)\circ u=u\circ R(g)$ for all $g\in G$. Fix a subalgebra $\mathfrak{A}$ of $\mathfrak{Z}$ such that $\Omega\in\mathfrak{A}$. We denote by Hom $(\mathfrak{A}, \boldsymbol{C})$ the set of all homomorphisms of $\mathfrak{A}$ into $\boldsymbol{C}$. For any $\chi\in$ Hom $(\mathfrak{A}, \boldsymbol{C})$ we put

$$\mathfrak{H}(\sigma, \chi) = \{f \in \mathfrak{H}(\sigma) \cap C^\infty(G, V);\ zf = \chi(z)f \quad \text{for all} \quad z \in \mathfrak{A}\}.$$

Then we have

**Proposition 1.** *$\mathfrak{H}(\sigma, \chi)$ is a closed invariant subspace of $\mathfrak{H}(\sigma)$. Moreover, every element of $\mathfrak{H}(\sigma, \chi)$ is an analytic mapping from $G$ into $V$.*

Proof. Let $B$ be the Killing form of $\mathfrak{g}^C$ and put

$$\langle X, Y\rangle = -B(X, \theta(Y)) \qquad (X, Y \in \mathfrak{g}^C)$$

where $\theta$ denotes the conjugation of $\mathfrak{g}^C$ with respect to the compact real form $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then $\langle\ ,\ \rangle$ is an inner product in $\mathfrak{g}^C$. Select orthonormal bases $(X_1, \cdots, X_m)$ and $(Y_1, \cdots, Y_n)$ for $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Then, it follows from the definition of the Casimir operator $\Omega$ that

$$\Omega = -(X_1^2 + \cdots + X_m^2) + Y_1^2 + \cdots + Y_n^2.$$

We put

$$\Omega_{\mathfrak{k}} = X_1^2 + \cdots + X_m^2, \quad \Omega_{\mathfrak{p}} = Y_1^2 + \cdots + Y_n^2.$$

For any $X \in \mathfrak{g}$, let $X'$ denote the right invariant vector field on $G$ given by

$$(X'f)(x) = \left[\frac{d}{dt} f(\exp(tX)x)\right]_{t=0} \quad (x \in G, f \in C^\infty(G)).$$

Then the mapping $X \to X'$ $(X \in \mathfrak{g})$ can be extended uniquely to an anti-homomorphism of $\mathfrak{G}$ onto the algebra of right-invariant differential operators on $G$. It is easy to see that $\Omega' = \Omega$ as differential operators on $G$. For any $\lambda \in \mathcal{F}$, we shall denote as usual by $H_\lambda$ an element of $\mathfrak{h}^C$ such that $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{h}$; the inner product $\langle\lambda, \mu\rangle$ of two linear forms $\lambda, \mu \in \mathcal{F}$ means the value $\langle H_\lambda, H_\mu\rangle$. We denote by the same notation the infinitesimal representation of $\sigma$. Let $\Lambda \in \mathcal{F}$ be the highest weight of $\sigma$. Then it is well known that

$$\sigma(\Omega_k) = -\langle\Lambda + 2\rho_k, \Lambda\rangle I$$

where $\rho_k = \frac{1}{2}\sum_{\alpha \in P_k} \alpha$ and $I$ denotes the identity operator on $V$. Fix any $f \in \mathfrak{H}(\sigma, \chi)$ and $v \in V$. Then

$$\Omega'_{\mathfrak{k}} f_v(x) = (\sigma(\Omega_{\mathfrak{k}})f(x).\ v)_V = -\langle\Lambda + 2\rho_k, \Lambda\rangle f_v(x)$$

where $f_v(x) = (f(x), v)_V$. It follows that

$$\Omega' f_v(x) = 2\langle\Lambda + 2\rho_k, \Lambda\rangle f_v(x) + (\Omega'_{\mathfrak{k}} + \Omega'_{\mathfrak{p}}) f_v(x).$$

On the other hand,

$$\Omega f_v(x) = (\Omega f(x), v)_V = \chi(\Omega) f_v(x)$$

Therefore, we have

$$(*) \qquad (\Omega'_{\mathfrak{k}}+\Omega'_{\mathfrak{p}})f_v = (\chi(\Omega)-2\langle\Lambda+2\rho_{\mathfrak{k}},\ \Lambda\rangle)f_v\,.$$

Since $\Omega'_{\mathfrak{k}}+\Omega'_{\mathfrak{p}}$ is obviously an elliptic differential operators on $G$, we conclude that $f_v$ is an analytic function. This shows that $f$ is an analytic mapping from $G$ to $V$. Moreover, owing to the ellipticity of $\Omega'_{\mathfrak{k}}+\Omega'_{\mathfrak{p}}$, all solutions of the above equation (*) in the distribution sense are analytic. It is an immediate consequence of this fact that $\mathfrak{H}(\sigma, \chi)$ is closed in $\mathfrak{H}(\sigma)$. Since $R(g)\circ z=z\circ R(g)$ for all $g\in G$, we see that $U_g(\sigma)U_z(\sigma)=U_z(\sigma)U_g(\sigma)$. It follows immediately that $\mathfrak{H}(\sigma, \chi)$ is an invariant subspace of $\mathfrak{H}(\sigma)$. This completes the proof of the proposition.

We denote by $U(\sigma, \chi)$ the subrepresentation of $U(\sigma)$ obtained by restricting $U(\sigma)$ on $\mathfrak{H}(\sigma, \chi)$. In the following, we shall discuss when $U(\sigma, \chi)$ is non-trivial and irreducible.

**3.** Let $\mathcal{E}$ (resp. $\mathcal{E}_K$) be the set of all equivalence classes of irreducible unitary representations of $G$ (resp. $K$). For any irreducible unitary representation $\pi$ of $G$, let $\pi|K$ denote the restriction of the representation $\pi$ to the subgroup $K$. For any $\mathfrak{d}\in\mathcal{E}_K$, we denote by $(\pi|K\colon \mathfrak{d})$ the multiplicity with which the representation $\mathfrak{d}$ occurs in $\pi|K$. $(\pi|K\colon \mathfrak{d})$ depends only on the equivalence class $\omega$ which contains $\pi$. In this case, we also write $(\omega|K\colon \mathfrak{d})$ instead of $(\pi|K\colon \mathfrak{d})$. Let $\xi_\sigma$ be the character of $\sigma$. We define a projection operator $E_\sigma$ by

$$E_\sigma = d(\sigma)\int_K \overline{\xi_\sigma(k)}U_k(\sigma, \chi)dk$$

where $d(\sigma)$ denotes the degree of $\sigma$ and $dk$ is the normalized Haar measure of $K$. We denote by $[\sigma]$ the class in $\mathcal{E}_K$ to which $\sigma$ belongs.

**Proposition 2.** *If* $(U(\sigma, \chi)|K\colon [\sigma])=1$, *then* $U(\sigma, \chi)$ *is irreducible.*

Proof. It is sufficient to prove that every non-zero closed invariant subspace of $\mathfrak{H}(\sigma, \chi)$ contains $E_\sigma\mathfrak{H}(\sigma, \chi)$. Let $\mathfrak{H}$ be an arbitrary non-zero closed invariant subspace of $\mathfrak{H}(\sigma, \chi)$. Fix a non-zero element $f\in\mathfrak{H}$. Then from Proposition 1, $f$ is analytic. Hence there exists a $g_0\in G$ such that $f(g_0)\neq 0$. Put $f_0=U_{g_0}f$. Then it is obvious that $f_0(1)=f(g_0)\neq 0$ (1 is the identity element of $G$) and that $f_0$ is analytic on $G$. Notice that

$$\begin{aligned}(E_\sigma f_0)(1) &= d(\sigma)\int_K \overline{\xi_\sigma(k)}U_k(\sigma, \chi)f_0(1)dk \\ &= d(\sigma)\int_K \overline{\xi_\sigma(k)}f_0(k)dk \\ &= d(\sigma)\int_K \overline{\xi_\sigma(k)}\sigma(k)dkf_0(1) \\ &= f_0(1)\neq 0\,.\end{aligned}$$

Then since $E_\sigma f_0$ is again analytic, we can conclude that $E_\sigma f_0 \neq 0$. Moreover, since $\mathfrak{H}$ is closed invariant subspace, we have $E_\sigma f_0 \in \mathfrak{H}$. It follows from the assumption $(U(\sigma, \chi)|K: [\sigma])=1$ that $E_\sigma \mathfrak{H}(\sigma, \chi) \subset \mathfrak{H}$. This proves the proposition.

We denote by End $(V)$ the algebra of all linear endomorphisms of $V$. An End $(V)$-valued $C^\infty$-function $\varphi$ on $G$ is called a zornal spherical functions of type $(\sigma, \chi)$ if it satisfies the conditions

(1) $$\varphi(k_1 g k_2) = \sigma(k_1)\varphi(g)\sigma(k_2) \qquad (k_1, k_2 \in K, g \in G)$$

(2) $$z\varphi = \chi(z)\varphi \qquad \text{for all} \qquad z \in \mathfrak{A}.$$

Let $\varphi$ be a zornal spherical function of type $(\sigma, \chi)$. We call $\varphi$ square-integrable if

$$\int_G \|\varphi(g)\|_V^2 dg < +\infty$$

where $\|\ \|_V$ is the Hilbert-Schmidt norm of End$(V)$. Here we mean by the Hilbert-Schmidt norm of an element of $A \in \text{End}(V)$ the square root of the trace of the operator $A^*A$, where $A^*$ denotes the adjoint operator of $A$.

**Proposition 3.** *If there exists a non-zero square-integrable zornal spherical function of type* $(\sigma, \chi)$, *then* $U(\sigma, \chi)$ *is not trivial* (*i.e.* $\mathfrak{H}(\sigma, \chi) \neq (0)$).

Proof. Let $\varphi$ be a non-zero square-integrable zornal spherical function of type $(\sigma, \chi)$. Then there exists $v \in V$ such that $\varphi_v \neq 0$ where $\varphi_v(g) = \varphi(g)v$. It is easy to see that $\varphi_v \in \mathfrak{H}(\sigma, \chi)$. This completes the proof of the proposition.

**4.** Now we need some results of F. I. Mautner. For any unitary representation $\pi$ of $G$ or $K$, we denote by the $[\pi]$ equivalence class to which $\pi$ belongs. Then it is easy to see that $[U(\sigma_1)] = [U(\sigma_2)]$ if $[\sigma_1] = [\sigma_2] \in \mathcal{E}_K$. In case $\sigma \in \mathfrak{d}$, we shall write $U(\mathfrak{d})$ instead of $[U(\sigma)]$.

**Lemma 1.** *Put* $\mathcal{E}(\sigma) = \{\omega \in \mathcal{E}; (\omega|K: [\sigma]) \neq 0\}$. *Then*

$$[U(\sigma)] = \int_{\mathcal{E}(\sigma)} (\omega|K: [\sigma])\omega d\mu(\omega) \qquad (\textit{direct integral})$$

*where* $\mu$ *is the Plancherel measure for* $G$. *This means that the multiplicity with which* $\omega$ *occurs in* $U(\sigma)$ *coincides with the multiplicity with which* $[\sigma]$ *occurs in* $\omega|K$.

For a proof, see [7] (*c*), and notice the following. Let $R$ (resp. $r$) be the right-regular representation of $G$ (resp. $K$). Then owing to the Peter-Weyl theorem, one knows that

$$[r] = \sum_{\mathfrak{d} \in \mathcal{E}_K} m(\mathfrak{d})\mathfrak{d} \qquad \text{(direct sum)}$$

where $m(\mathfrak{d})$ is the multiplicity with which $\mathfrak{d}$ occurs in $r$ ($m(\mathfrak{d})=\deg \mathfrak{d}$). It follows from the theorem on inducing a representation "in stages" (see [7] (*d*)) that

$$[R] = \sum_{\mathfrak{d}\in\mathcal{E}_K} m(\mathfrak{d})U(\mathfrak{d}) \qquad \text{(direct sum).}$$

This shows that $[U(\sigma)]$ is a subrepresentation of the regular representation of $G$.

Now we shall need another lemma due to F.I. Mautner.

Consider the decomposition in Lemma 1. Then there exists a choice of representatives $\tilde{\pi}_\omega\in\omega$ ($\omega\in\mathcal{E}(\sigma)$) with the following property. Let $\tilde{\mathfrak{H}}_\omega$ denote the representation space of $\tilde{\pi}_\omega$. We denote by $\pi_\omega$ the $(\omega|K: [\sigma])$-times direct sum of $\tilde{\pi}_\omega$ and let $\mathfrak{H}_\omega$ be the representation space of $\pi_\omega$. Then we have

$$\mathfrak{H}_\omega = \tilde{\mathfrak{H}}_\omega\oplus\cdots\oplus\tilde{\mathfrak{H}}_\omega \qquad ((\omega|K: [\sigma])\text{-times direct sum).}$$

Then we have

$$\mathfrak{H}(\sigma) = \int_{\mathcal{E}(\sigma)} \mathfrak{H}_\omega d\mu(\omega) \qquad (\textit{direct integral}).$$

For any $f\in\mathfrak{H}(\sigma)$, let $f_\omega$ denote the "component" of $f$ in $\mathfrak{H}_\omega$. We denote by the same notations the infinitesimal representations of $\mathfrak{G}$ for $U(\sigma)$ (resp. $\pi_\omega$) on the Gårding subspaces $\mathfrak{H}^0(\sigma)$ (resp. $\mathfrak{H}^0_\omega$) where $\omega\in\mathcal{E}(\sigma)$ (cf. [7] (*a*))

**Lemma 2.** *For any $f\in\mathfrak{H}^0(\sigma)$ and $u\in\mathfrak{G}$, we have*

$$(U_u(\sigma)f)_\omega = \pi_\omega(u)f_\omega$$

*for almost every $\omega\in\mathcal{E}(\sigma)$.*

For a proof, see [7] (*a*), (*b*).

Let $\chi_\omega$ be the infinitesimal character of $\omega\in\mathcal{E}$. For any $\chi\in\mathrm{Hom}(\mathfrak{Z}, \boldsymbol{C})$, we denote by $\chi|\mathfrak{A}$ the restriction of $\chi$ on $\mathfrak{A}$. Then $\chi|\mathfrak{A}\in\mathrm{Hom}(\mathfrak{A}, \boldsymbol{C})$. For any $\chi\in\mathrm{Hom}(\mathfrak{A}, \boldsymbol{C})$, we put

$$\mathcal{E}(\chi) = \{\omega\in\mathcal{E};\ \chi_\omega|\mathfrak{A} = \chi\}\,.$$

Let $\mathcal{E}_d$ be the set of all discrete classes in $\mathcal{E}$ (see [4] (*d*)). We denote by $L$ the set of all $\lambda\in\mathcal{F}$ such that

$$\frac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}\in\boldsymbol{Z} \qquad \text{for all} \quad \alpha\in\Sigma\,,$$

where $\boldsymbol{Z}$ is the set of all integers. Let $L'$ be the set of all $\lambda\in L$ such that $\langle\lambda, \alpha\rangle\neq0$ for all $\alpha\in\Sigma$. Then owing to the profound result of Harish-Chandra ([4] (*d*) Theorem 16, p. 96), one has that for any $\lambda\in L'$, there corresponds an element $\omega(\lambda)\in\mathcal{E}_d$ such that

$$\chi_{\omega(\lambda)}(\Omega) = |\lambda|^2-|\rho|^2$$

where $|\ |^2=\langle\ ,\ \rangle$ and $\rho=\frac{1}{2}\sum_{\alpha\in P}\alpha$. As is easily seen, $\lambda\to|\lambda|^2-|\rho|^2$ $(\lambda\in\mathcal{F})$ is a polynomial of degree 2 and its homogeneous part of degree 2 is a positive definite quardatic form. It follows that $\mathcal{E}(\chi)\cap\mathcal{E}_d$ is finite set.

**Theorem 1.** *Let $\mathfrak{A}$ be an arbitrary subalgebra of $\mathfrak{Z}$ such that $\Omega\in\mathfrak{A}$ and let $\chi$ be a homomorphism of $\mathfrak{A}$ into $\boldsymbol{C}$. Then $\mathcal{E}(\chi)\cap\mathcal{E}_d$ is a finite set. Moreover, let $\sigma$ be an irreducible unitary representation of $K$ such that*

$$\text{(A)}\qquad \mathcal{E}(\sigma)\cap\mathcal{E}(\chi)-\mathcal{E}_d$$

*is of measure zero with respect to the Plancherel measure for $G$. Then we have*

$$[U(\sigma,\chi)]=\sum_{\omega}(\omega|K\colon[\sigma])\omega\quad(\omega\in\mathcal{E}(\sigma)\cap\mathcal{E}(\chi)\cap\mathcal{E}_d)\,.$$

Proof. We have already proved the first assertion. We consider the decompositions in Lemma 1 and 2 and use the notations in Lemma 2. Fix any $f\in\mathfrak{H}(\sigma,\chi)\cap\mathfrak{H}^0(\sigma)$. Then we know that

$$U_z(\sigma)f=\chi(z)f\quad\text{and}\quad\pi_\omega(z)f_\omega=\chi_\omega(z)f_\omega\qquad\text{for all}\quad z\in\mathfrak{A}\,.$$

On the other hand, from Lemma 2 we have

$$(U_z(\sigma)f)_\omega=\pi_\omega(z)f_\omega$$

for almost every $\omega\in\mathcal{E}(\sigma)$. Hence, there exists a subset $\mathcal{N}\subset\mathcal{E}(\sigma)$ of measure zero such that

$$(\chi(z)-\chi_\omega(z))f_\omega=0\quad\text{for all}\quad\omega\in\mathcal{E}(\sigma)-\mathcal{N}\,.$$

In general, $\mathcal{N}$ depends on $z$ as well as $f$. But one knows that $\mathfrak{A}$ is finitely generated. Therefore, every $\chi\in\mathrm{Hom}\,(\mathfrak{A},\boldsymbol{C})$ is uniquely determined by its values at a finite number of elements of $\mathfrak{A}$. Hence, we can assume that $\mathcal{N}$ does not depend on $z$. It follows immediately from the assumption (A) in the theorem that

$$f=\sum_{\omega}f_\omega\qquad(\omega\in\mathcal{E}(\sigma)\cap\mathcal{E}(\chi)\cap\mathcal{E}_d)\,.$$

This completes the proof of the theorem.

REMARK 1. For any real number $c$, define

$$\mathcal{E}_c=\{\omega\in\mathcal{E};\ \chi_\omega(\Omega)=c\}\,.$$

Then in case rank $G/K=1$, we can show that $\mathcal{E}_c-\mathcal{E}_d$ is of measure zero with respect to the Plancherel measure for $G$, using the explicit form of the Plancherel measure given in [4] (*c*), [8]. We have a conjecture that it holds in general. If this is true, then the condition (A) in Theorem 1 is always satisfied for all $\sigma$.

Now we have assumed that $G$ has a compact Cartan subgroup $T$. Owing to Harish-Chandra [4] ($d$), one sees that $\mathcal{E}_d \neq \emptyset$. Fix an $\omega \in \mathcal{E}_d$ and put $\chi = \chi_\omega | \mathfrak{A}$. Then it is obvious that there exists a $[\sigma] \in \mathcal{E}_K$ such that $\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}(\chi) \cap \mathcal{E}_d$. It follows from Theorem 1 that $\omega$ is a subrepresentation of $U(\sigma, \chi)$. If $\pi \in \omega$, we say that "$\pi$ is a realization of $\omega$" or that "$\omega$ is realized by $\pi$."

**Corollary.** *Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{Z}$ such that $\Omega \in \mathfrak{A}$. Fix an $\omega \in \mathcal{E}_d$ and put $\chi = \chi_\omega | \mathfrak{A}$. Assume that there exists an irreducible unitary representation $\sigma$ of $K$ which satisfies the following conditions* ($A$.1)~($A$.3).

($A$.1) $\mathcal{E}(\sigma) \cap \mathcal{E}(\chi) \cap \mathcal{E}_d = \{\omega\}$.

($A$.2) $(\omega | K: \sigma) = 1$.

($A$.3) $\mathcal{E}(\sigma) \cap \mathcal{E}(\chi) - \mathcal{E}_d$ *is of measure zero with respect to the Plancherel measure for $G$.*

*Then $\omega$ is realized by $U(\sigma, \chi)$.*

**5.** Consider the special case that $\mathfrak{A} = \mathfrak{Z}$. Then it is known (see [4] (a)) that $\mathcal{E}(\sigma) \cap \mathcal{E}(\chi)$ is always a finite set. Hence, in case $\mathfrak{A} = \mathfrak{Z}$, the assumption (A) in Theorem 1 and the assumption (A.3) in the corollary to Theorem 1 are always satisfied.

**Theorem 2.** *Fix any $[\sigma] \in \mathcal{E}_K$ and $\chi \in \mathrm{Hom}(\mathfrak{Z}, \boldsymbol{C})$. Then $U(\sigma, \chi)$ is non-trivial and irreducible if and only if $\sigma$ and $\chi$ satisfy the following condition* (C).

(C) $\mathcal{E}(\sigma) \cap \mathcal{E}(\chi) \cap \mathcal{E}_d$ *consists of only one element $\omega$ such that* $(\omega | K: \sigma) = 1$.

*Moreover, the condition (C) implies that $U(\sigma, \chi)$ is a realization of $\omega$.*

Remark 2. Since $K \backslash G$ is simply connected, $\mathfrak{H}(\sigma)$ can be realized as $V$-valued square-integrable functions on a certain submanifold of $G$ with respect to a certain measure. If the rank of the symmetric space $K \backslash G$ is equal to be one, the radial components of $U_z(\sigma)$ ($z \in \mathfrak{Z}$) coincide with ordinary differential equations (see [9] and cf. [4] (b)). It is very cumbersome to calculate the radial components of $U_z(\sigma)$ ($z \in \mathfrak{Z}$) even if $G$ is the lower dimensional Lie group such as the universal covering group of De Sitter group. However, R. Takahashi [9] computed the radial component of $U_\Omega(\sigma)$ in a very ingenious manner, making use of the quaternion field. Thus he proved that $U(\sigma, \chi)$ is non-trivial and irreducible for a certain $[\sigma] \in \mathcal{E}_K$ and $\chi \in \mathrm{Hom}(\mathfrak{A}, \boldsymbol{C})$ in case $\mathfrak{A} = \boldsymbol{C}[\Omega]$ (the algebra of all polynomials of $\Omega$).

Now we shall give here an another proof of this fact, making use of the corollary to Theorem 1 and the result of J. Dixmier [1] (b). In the following, we use the notations of [1] ($b$) and [9]. Let $G$ be the universal covering group of De Sitter group. We consider the irreducible unitary representation $\rho_K^{n,0}$ of the maximal compact subgroup $K$ of $G$ (where $2n\in \boldsymbol{Z}$ and $n\geqslant 1$). Put $\sigma_n = \rho_K^{n,0}$. Then it follows immediately from Fig. 2–3–4–5 ([1] ($b$) p. 24) that

$$\mathcal{E}(\sigma_n) = \left\{\pi^+_{n,q};\ q = n,\ n-1,\ \cdots,\ 1 \quad \text{or} \quad \frac{1}{2}\right\} \cup \{\nu_{n,s};\ s>0\}\,.$$

On the other hand, from (12) (in [1] ($b$) p. 12) and (53), (55) in ([1] (b) p. 27) one gets that

$$\chi_{\pi^+_{n,q}}(\Omega) = n^2+n+q^2-q-2\,,$$
$$\chi_{\nu_{n,s}}(\Omega) = n^2+n-s-2\,.$$

We denote by $\chi_{n,p}$ the unique element of Hom $(\mathfrak{A}, \boldsymbol{C})$ such that $\chi_{n,p}(\Omega)=n^2+n+p^2-p-2$. Then it is clear that $\mathcal{E}(\sigma_n)\cap\mathcal{E}(\chi_{n,p})=\{\pi^+_{n,p}\}$ for any $p$ such that $2p$, $n-p\in\boldsymbol{Z}$ and $n\geqslant p\geqslant 1$. Since every $\mathfrak{d}\in\mathcal{E}_K$ is contained at most once in each $\omega\,|\,K$ $(\omega\in\mathcal{E})$, it follows from the Corollary to Theorem 1 that $[U(\sigma_n;\chi_{n,p})]=\pi^+_{n,p}$. This shows that $U(\sigma_n, \chi_{n,p})$ is non-trivial and irreducible. If we take $\sigma_n=\rho_K^{0,n}$, similarly we have $[U(\sigma_n, \chi_{n,p})]=\pi^-_{n,p}$ These facts together with Theorem 1 and 2 in [1] (b) prove the following.

**Proposition 4.** (R. Takahashi) *Let $G$ be the universal covering group of De Sitter group. Then every irreducible unitary representation of discrete class $\omega\in\mathcal{E}_d$ can be realized by $U(\sigma, \chi)$ for some $[\sigma]\in\mathcal{E}_K$ and $\chi\in$* Hom $(\mathfrak{A}, \boldsymbol{C})$ *where $\mathfrak{A}=\boldsymbol{C}[\Omega]$ (the algebra of all polynomials of $\Omega$). More precisely, $\omega$ is realized by*

$$U(\rho_K^{n,0}, \chi_{n,p}) \text{ (resp. } U(\rho_K^{0,n}, \chi_{n,p}))$$
$$\textit{if} \quad \omega = \pi^+_{n,p} \text{ (resp. } \omega = \pi^-_{n,p})$$

*where $\chi_{n,p}$ is the unique element of* Hom $(\mathfrak{A}, \boldsymbol{C})$ *such that*

$$\chi_{n,p}(\Omega) = n^2+n+p^2-p-2\,.$$

REMARK 3. It is interesting to observe the fact that the theory of unitary representations has an application to the theory of partially differential equations; i.e. the differential equation (31) on page 399 in [9] has non trivial solutions in $H_0^{\rho,p}$ (for the notations, see [9]).

**6.** Finally, we shall apply the above theory to the group $SU(m, 1)$ and the universal covering group of $SO_0(2m, 1)$ where $m$ is an arbitrary positive integer (for the notations, see [5]). Let $G$ be any one of these groups. Then it is known that every $\mathfrak{d}\in\mathcal{E}_K$ is contained at most once in each $\omega\,|\,K$ $(\omega\in\mathcal{E})$ (cf. [1] (b), [2], [3]). We fix an element $\chi$ of Hom $(\mathfrak{Z}, \boldsymbol{C})$. If $\omega_1, \omega_2\in\mathcal{E}(\chi)\cap\mathcal{E}_d$, then

$\omega_1|K$ and $\omega_2|K$ are disjoint, that is, $\omega_1|K$ and $\omega_2|K$ have no irreducible components in common (see for proof, [2], [6]). Therefore, if $\omega\in\mathcal{E}(\sigma)\cap\mathcal{E}_d$, then we have $\mathcal{E}(\sigma)\cap\mathcal{E}(\chi_\omega)\cap\mathcal{E}_d=\{\omega\}$. Making use of Theorem 2 we obtain the following proposition.

**Proposition 5.** *Let G be either $SU(m, 1)$ or the universal covering group of $SO_0(2m, 1)$ where m is an arbitrary positive integer. Then every $\omega\in\mathcal{E}_d$ is realized by $U(\sigma, \chi_\omega)$ for any $[\sigma]\in\mathcal{E}_K$ such that $(\omega|K: [\sigma])\neq 0$.*

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