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CONSTRUCTION AND CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF SPECIAL LINEAR GROUP OF THE SECOND ORDER OVER A FINITE FIELD

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1. Introduction

The purpose of the present paper is to construct all irreducible representations of $SL(2, \boldsymbol{F})$, where $\boldsymbol{F}$ is a finite field, closely following the study of H.D. Kloosterman [9] on the representation of the modular congruence group. Most of the results described here are known, however the argument which depends on theta series and was the starting point for H.D. Kloosterman's work is replaced by the recent results of A. Weil [11] on a certain projective representation of symplectic groups. By this construction we hope to give some insight to the structure of the remarkable irreducible representations of SL $(2, \boldsymbol{F})$ discovered by I.M. Gel'fand and M.I. Graev [7] as an analogue of the discrete series of SL (2) over a non-discrete locally compact field.

We consider $\boldsymbol{F}^2$, the two dimensional vector space over $\boldsymbol{F}$ and identify its dual with $\boldsymbol{F}^2$ (as an additive group) in two ways. The one is connected with indefinite quadratic form in two variables and the other is connected with definite quadratic form (or rather with quadratic extension of $\boldsymbol{F}$). On the other hand, a natural projective unitary representation of the symplectic group Sp(G) associated with a locally compact abelian group G was constructed on $L^2(G)$ and studied by A. Weil [11]. $SL(2, \boldsymbol{F})$ is imbedded into $Sp(\boldsymbol{F}^2)$, so we have two projective representations of $SL(2, \boldsymbol{F})$, which turn out to be ordinary representations. We call the one associated with indefinite quadratic form the representation of the first kind and the other of the second kind.

To decompose these representations, we consider the hyperbolic rotation group H and the rotation group C operating on $\boldsymbol{F}^2$. The operators induced by hyperbolic rotations commute with the operators of the representation of the first kind, and those induced by rotations with the operators of the representation of the second kind. So by Fourier transformation with respect to H or C, these representations are decomposed into direct sums of representations. Each operator of the component representations is expressed by a sum analogous to

Kloosterman's sum (or to the integral representation of the ordinary Bessel function) and representations discovered by I.M. Gel'fand and M.I. Graev are thus reconstructed. We call the sum Bessel function of the first or the second kind over a finite field. For a representation of the second kind, the operator corresponding to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is essentially the Fourier transform in two variables, so the above construction is an analogue of classical construction of Fourier-Bessel transform. Bessel functions of the first kind are also obtainable as kernels of the so-called $\mathcal{X}$-realization [7, P216] of the representations, obtained by decomposing the quasi-regular representation which is naturally defined on the two dimensional affine space over $\boldsymbol{F}$.

The author was informed by Professor H. Yoshizawa that since 1963 J.A. Shalika had undertaken the problem of determining all irreducible representations of the modular congruence group with considerable progress. After the author had finished the present work, he was also informed by Professor M. Kuga about the general outline of J.A. Shalika's lecture on the similar problem as considered in this paper at 1965-A.M.S.-Summer Institute on algebraic groups.

The representation theory of $SL(2, \boldsymbol{F})$ was originated by G.F. Frobenius [3] who calculated the traces of all irreducible representations of $SL(2, \boldsymbol{F})$. Later E. Hecke [8] gave the construction to a half of irreducible representations (obtained from quasi-regular representation) of this group. In connection with his study of the general theory of modular functions, E. Hecke raised the problem of determining all irreducible representations and their traces of the modular congruence group mod p^{λ}. The problem was attacked by H.D. Kloosterman [9] by means of the transformation formulas under modular substitutions of certain theta series. He constructed the greater part (in fact, for the case $\lambda=1$, all) of irreducible representations. So all irreducible representations of $SL(2, \boldsymbol{F})$ with $\boldsymbol{F}=\boldsymbol{Z}/(p)$ have been constructed by H.D. Kloosterman.

In recent years, I.M. Gel'fand and M.I. Graev have undertaken the study of representation theory of Dickson-Chevalley groups over an arbitrary field [4], [5], [6], [7]. In particular, they gave the formulas of representations of the discrete series of $SL(2, \boldsymbol{K})$, where $\boldsymbol{K}$ is a non-discrete locally compact field, and analogous ones for $SL(2, \boldsymbol{F})$ [7]. But the author of the present paper feels that the meaning of these formulas remains rather implicit.

The present paper is closely connected with our previous paper [10]. In Part I of [10], the discrete series (of $SL(2, \boldsymbol{K})$) of I.M. Gel'fand and M.I. Graev were reconstructed by the method described before. We hope our construction described in Part I of [10] and in this paper make the structure of those representations clear. In Part II of [10], the representations of the modular congruence group mod p^{λ} constructed by H.D. Kloosterman were reconstructed. Modifying the construction, we obtained a new representation which may give some of irreducible representations absent in H.D. Kloosterman's work. For the special case of $\lambda=2$, all irreducible representations absent in H.D. Kloosterman's work were thus obtained.

We shall proceed to outline the contents of this paper. §2, §3, §4 of this paper are preliminaries for later constructions. In §5, the representations of the first and second kind are constructed depending on results of A. Weil. Decomposition of the constructed representation into invariant subspaces is descirbed in §6 and §7. There are exceptional invariant subspaces which can be decomposed further. Description of them are given in §8. In §9, irreducibility of the constructed representations are proved and it is shown that all irreducible representations of $SL(2, \boldsymbol{F})$ are thus obtained.

2. Properties of a finite field

For general facts about a finite field, see [2]. Let $\boldsymbol{F}$ be a finite field with $q=p^n$ (p a prime number) elements. Let $\boldsymbol{F}^+$ be the additive group of $\boldsymbol{F}$, $\boldsymbol{F}^*$ its multiplicative group. Let ε be a generator of $\boldsymbol{F}^*$.

Let $\boldsymbol{L}$ be the quadratic extension $\boldsymbol{F}(\sqrt{\varepsilon})$ of $\boldsymbol{F}$. For $z=x+\sqrt{\varepsilon}y$ $(x, y\in F)$, define $\bar{z}=x-\sqrt{\varepsilon}y$, $S(z)=z+\bar{z}$ and $N(z)=z\bar{z}$. The set of elements t in $\boldsymbol{L}$ which satisfy $N(t)=c$ $(c\in F^*)$ is called a circle in $\mathbf{L}$. Each circle consists of $q+1$ elements. Let us denote by C the circle $N(t)=1$.

Let $\tilde{\boldsymbol{F}}^*$ and $\tilde{C}$ be the character group of $\boldsymbol{F}^*$ and C respectively and π_0 (same symbol for both case) be the identities of $\tilde{\boldsymbol{F}}^*$ and C. There are characters of $\boldsymbol{F}^*$ or $\tilde{C}$ with real values. Apart from π_0 there is only one such characters in each case, which we denote by π_1 and π_2 respectively. Let t_0 be a generator of C. π_1 and π_2 are characterized by $\pi_1(\varepsilon)=-1$ and $\pi_2(t_0)=-1$. Put $\boldsymbol{F}^*_\pm=\{x\in\boldsymbol{F}^*;\ \pi_1(x)=\pm1\}$, $\boldsymbol{F}_\pm=\boldsymbol{F}^*_\pm\cup\{0\}$ and $C_\pm=\{t\in C;\ \pi_2(t)=\pm1\}$.

Lemma 1. $\pi_1(-1)=-1$ *if and only if* $\pi_2(-1)=1$.

Proof. Let $-1=t^2$, $t\in C$. Then $t+\bar{t}=0$, so t is written as $\sqrt{\varepsilon}b$, $b\in F$. Therefore $-1=t^2=\varepsilon b^2$, so $\pi_1(-1)=\pi_1(\varepsilon)=-1$. Conversely, let $-1=\varepsilon b^2$, $b\in\boldsymbol{F}$. Putting $t=\sqrt{\varepsilon}b$, we have $t\bar{t}=-b^2\varepsilon=1$ and $t^2=-1$.

Lemma 2. *For any* $t\in C_-$, *there exists an element* $e\in\boldsymbol{L}$ *such that* $N(e)=\varepsilon$ *and* $\bar{e}/e=t$.

Proof. Let e' be an element of $\boldsymbol{L}$ which satisfies $N(e')=\varepsilon$. If $\bar{e}'/e'$ is expressible as t'^2, $t'\in C$, then we have $\overline{(e't')}=e't'$ and $\varepsilon=N(e')=(e't')^2$, which contradict the definition of ε. So $\bar{e}'/e'=t_0^{2m+1}$ for some m. If $t=t_0^{2n+1}$, put $e=t_0^{-n}t_0^{m}e'$.

Let us fix e_0 which satisfies $N(e_0)=\varepsilon$ and $\bar{e}_0/e_0=t_0$.

Let us fix a non-trivial character χ of $\tilde{\boldsymbol{F}}^+$. For $\pi\in\boldsymbol{F}^*$, $\pi\neq\pi_0$, define

$$\tau(\pi)=\sum_{x\in F^*}\chi(x)\pi(x)$$

and put $\tau=\tau(\pi_1)$. It is known that $|\tau(\pi)|^2=q$ (see for instance [1, P27–29]) and in particular $\tau^2=q\pi_1(-1)$.

Let $\delta(x)$ be δ-function on $\boldsymbol{F}$ i.e. $\delta(x)=0$ if $x\in\boldsymbol{F}^*$ and $\delta(0)=1$. We also introduce function $\delta(\pi)$ on $\tilde{\boldsymbol{F}}^*$ and $\tilde{C}$ defined by $\delta(\pi)=0$ if $\pi\neq\pi_0$ and $\delta(\pi_0)=1$. Given $\pi\in\tilde{\boldsymbol{F}}^*$, we define $\pi(0)=0$ and consider π as a function on $\boldsymbol{F}$.

3. Quasi-regular representation

Let $\mathfrak{H}$ be the finite dimensional Hilbert space consisting of all functions on $\boldsymbol{F}\times\boldsymbol{F}-\{(0,0)\}$ with the inner product $(f_1,f_2)=\sum f_1(x_1,x_2)\overline{f_2(x_1,x_2)}$. For $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,\boldsymbol{F})$, define the operator $T(g)$ on $\mathfrak{H}$ by the formula

$$T(g)f(x_1,x_2)=f(\alpha x_1+\gamma x_2,\ \beta x_1+\delta x_2)\,.$$

Then $T(g)$ is a unitary representation of $SL(2,\boldsymbol{F})$. For $\pi\in\tilde{\boldsymbol{F}}^*$, we call $f\in\mathfrak{H}$ which satisfies $f(\lambda x_1,\lambda x_2)=\pi(\lambda)f(x_1,x_2)$, $\lambda\in\boldsymbol{F}^*$, a homogeneous function of degree π. Let $\mathfrak{H}_\pi$ be the subpsace of $\mathfrak{H}$ which consists of all homogeneous functions of degree π. $\mathfrak{H}_\pi$ is the invariant subspace of the representation $\{T(g),\mathfrak{H}\}$ and let us write $T_\pi(g)=T(g)|\mathfrak{H}_\pi$.

For $f\in\mathfrak{H}_\pi$, define $\psi(x)=f(1,x)$, $x\in\boldsymbol{F}$, and $\psi(\infty)=f(0,1)$. Induced action of $T_\pi(g)$ on ψ is written as follows. If $\gamma\neq 0$,

$$T_\pi(g)\psi(x)=\psi\Big(\frac{\delta x+\beta}{\gamma x+\alpha}\Big)\pi(\gamma x+\alpha)\,,\qquad \Big(x\neq-\frac{\alpha}{\gamma}\Big)$$

$$T_\pi(g)\psi\Big(-\frac{\alpha}{\gamma}\Big)=\pi\Big(-\frac{1}{\gamma}\Big)\psi(\infty)\,,$$

$$T_\pi(g)\psi(\infty)=\pi(\gamma)\psi\Big(\frac{\delta}{\gamma}\Big);$$

if $\gamma=0$,

$$T_\pi(g)\psi(x)=\psi\Big(\frac{\delta x+\beta}{\alpha}\Big)\pi(\alpha)\,,$$

$$T_\pi(g)\psi(\infty)=\pi(\delta)\psi(\infty)\,.$$

Let $\pi\neq\pi_0$ and put $\varphi(y)=\sum_{x\in F}\psi(x)\chi(-yx)$, $\varphi(\infty)=\psi(\infty)$. Realization of $T_\pi(g)$ expressed on functions $\varphi(y)$, $\varphi(\infty)$ is called χ-realization and written by the formula in §7. which we shall obtain by another method.

$T_{\pi_0}(g)$ is equivalent to direct sum of $T_{\pi_0}^{(2)}(g)$ (which will be introduced in §9.) and an identity representation.

4. Summary of results of A. Weil

This section contains summary of definitions and results in Chapter I. of [11] which we need later.

Let G be a commutative locally compact group and G^* be its dual group. For $u\in G$ and $u^*\in G^*$, put $\langle u, u^*\rangle=u^*(u)$. For notational convenience, we assume G is a finite group with n elements. Let $\mathfrak{H}$ be the finite dimensional Hilbert space consisting of all functions on G, with inner product $(\Phi_1, \Phi_2)=n^{-1/2}\sum_{u\in G}\Phi_1(u)\overline{\Phi_2(u)}$. For $\Phi\in\mathfrak{H}$, define its Fourier transform Φ^* by

$$\Phi^*(u^*) = n^{-1/2}\sum_{u\in G}\Phi(u)\langle u, u^*\rangle$$

Let G and H be commutative finite groups. If $u\to u\alpha$ is a homomorphism of G into H, there exists a homomorphism α^* of H^* into G^* such that $\langle u\alpha, v^*\rangle=\langle u, v^*\alpha^*\rangle$ for any $u\in G$ and $v^*\in H^*$. If $H^*=G$ and $\alpha=\alpha^*$, α is called a symmetric homomorphism.

Let $w\to w\sigma$ be an automorphism of $G\times G^*$. Putting $w=(u, u^*)$, σ can be represented by matrix:

$$(u, u^*)\to(u, u^*)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = (u\alpha+u^*\gamma,\ u\beta+u^*\delta)$$

where α, β, γ and δ are homomorphism of G into G, of G into G^*, of G^* into G and of G^* into G^* respectively. An automorphism of $G\times G^*$ is called *symplectic* if $\sigma\sigma^I=1$ where

$$\sigma^I = \begin{pmatrix}\delta^* & -\beta^*\\ -\gamma^* & \alpha^*\end{pmatrix}$$

The group of all symplectic automorphism is denoted by $Sp(G)$.

Let T be the multiplicative group of the complex numbers of modulus 1. Put $F(w_1, w_2)=\langle u_1, u_2^*\rangle$ for $w_i=(u_i, u_i^*)\in G\times G^*$ $(i=1, 2)$. Let $A(G)$ be the group whose underlying space is $G\times G^*\times T$, with the multiplication rule defined by $(w_1, t_1)(w_2, t_2)=(w_1+w_2, F(w_1, w_2)t_1t_2)$. We call the group $A(G)$ the *Heisenberg group* associated with G. It's center is $\{(0, t), t\in T\}$. Define the unitary operator $U(w, t)$ in $\mathfrak{H}$ by the formula

$$U(w, t)\Phi(v) = t\Phi(v+u)\langle v, u^*\rangle \qquad (\Phi\in\mathfrak{H})$$

U(w, t) is an irreducible unitary representation of the Heisenberg group $A(G)$.

Let $B_0(G)$ be the group of all automorphisms of Heisenberg group $A(G)$ which fix the elements of the center of $A(G)$. For $s\in B_0(G)$, define the representation U^s of Heisenberg group by $U^s(w, t)=U((w, t)^s)$. It can be shown that U^s and U are equivalent, so there exist unitary operators, unique up to the constant factors, which define the equivalence. We fix one of them and denote it by $V(s)$. The mapping $s\to V(s)$ is a projective unitary representation of $B_0(G)$. If $u\to 2u$ is an automorphism of G, a natural injective homomorphism $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\to(\sigma, f_\sigma)$ of $S_p(G)$ into $B_0(G)$ exists, where

$$f_\sigma(u, u^*) = \langle u, 2^{-1}u\alpha\beta^*\rangle\langle 2^{-1}u^*\gamma\delta^*, u^*\rangle\langle u^*\gamma, u\beta\rangle.$$

(σ, f_σ) is simply denoted by σ.

Let α, γ and ρ be an automorphism of G, an isomorphism of G^* on G and a symmetric homomorphism of G into G^* respectively. Define unitary operators $\boldsymbol{d}_0(\alpha)$, $\boldsymbol{d}_0'(\gamma)$ and $\boldsymbol{t}_0(\rho)$ by formulas

$$\begin{aligned}\boldsymbol{d}_0(\alpha)\Phi(u) &= \Phi(u\alpha)\\ \boldsymbol{d}_0'(\gamma)\Phi(u) &= \Phi^*(-u\gamma^{*-1}) \qquad (\Phi\in\mathfrak{H})\\ \boldsymbol{t}_0(\rho)\Phi(u) &= \Phi(u)\langle u, 2^{-1}u\rho\rangle\end{aligned}$$

Then $V\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$, $V\begin{pmatrix}0 & -\gamma^{*-1}\\ \gamma & 0\end{pmatrix}$ and $V\begin{pmatrix}1 & \rho\\ 0 & 1\end{pmatrix}$ are equal to $\boldsymbol{d}_0(\alpha)$, $\boldsymbol{d}_0'(\gamma)$ and $\boldsymbol{t}_0(\rho)$ up to constant factors.

5. The representations of the first and second kind

Let us now proceed to construction of the reprerentations of the first and second kind of $SL(2, \boldsymbol{F})$. Assume that the characteristic p of $\boldsymbol{F}$ be odd. Let G be the additive group of the two-dimensional vector spcae $\boldsymbol{F}^2$ over $\boldsymbol{F}$. Now let us define two functions on $G\times G$ which define self-duality of G. Put

$$\langle u, v\rangle_1 = \chi(u_1v_2+u_2v_1)$$

and

$$\langle u, v\rangle_2 = \chi(2(u_1v_1+\varepsilon u_2v_2))\,.$$

The self-dualities of G defined by $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are called of the first kind and of the second kind respectively. Let $\boldsymbol{L}$ be the quadratic extension $\boldsymbol{F}(\sqrt{\varepsilon})$ of $\boldsymbol{F}$. The self-duality of the second kind is canonically associated with $\boldsymbol{L}$. We identify G and $\boldsymbol{L}$ by $G\ni(u_1, u_2)u\to u_1+\sqrt{\varepsilon}u_2\in\boldsymbol{L}$ and then $\langle u, v\rangle_2=\chi(S(u\bar{v}))$.

First we consider both cases simultaneously, so we omit the suffices and write $\langle\ ,\ \rangle$. Let $\alpha\in\boldsymbol{F}$ and the homomorphism of G defined by $u\to u\alpha=(\alpha u_1, \alpha u_2)$ be denoted also by α. Then each element of $SL(2, \boldsymbol{F})$ can be considerd as an element of $Sp(G)$, so $SL(2, \boldsymbol{F})$ can be imbedded homomorphically into $B_0(G)$.

If $\gamma\neq 0$, $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is expressed uniquely as

$$g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}1 & \alpha\gamma^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -\gamma^{-1}\\ \gamma & 0\end{pmatrix}\begin{pmatrix}1 & \delta\gamma^{-1}\\ 0 & 1\end{pmatrix}$$

Define the operator $\boldsymbol{r}_0(g)$ by the formula

$$\boldsymbol{r}_0(g) = \boldsymbol{t}_0(\alpha\gamma^{-1})\boldsymbol{d}'_0(\gamma)\boldsymbol{t}_0(\delta\gamma^{-1}) .$$

or, in an explicit form, by

$$\boldsymbol{r}_0(g)\Phi(u) = \sum_{v\in G} k(g\,|\,u,\ v)\Phi(v) ,$$

where

$$k(g\,|\,u,\ v) = \sum_{v\in G} \chi\left(\frac{\alpha 2^{-1}\langle u,\ u\rangle + \delta 2^{-1}\langle v,\ v\rangle - \langle u,\ v\rangle}{\gamma}\right)$$

Let $g_1 g_2 = g_3$, $g_i = \begin{pmatrix}\alpha_i & \beta_i \\ \gamma_i & \delta_i\end{pmatrix}$ and $\gamma_i \neq 0$ $(i=1,\ 2,\ 3)$. Then $\boldsymbol{r}_0(g_1)\boldsymbol{r}_0(g_2) = c(g_1,\ g_2)\boldsymbol{r}_0(g_3)$, i.e.

$$\sum_{v\in G} k(g_1|u,\ v)k(g_2|v,\ w) = c(g_1,\ g_2)k(g_3|u,\ w) .$$

Putting $u=w=0$, we have

$$\begin{aligned} c = c(g_1,\ g_2) &= \frac{1}{q}\sum_{a\in G} \chi(\gamma_1^{-1}\delta_1 2^{-1}\langle v,\ v\rangle)\chi(\gamma_2^{-1}\alpha_2 2^{-1}\langle v,\ v\rangle) \\ &= \frac{1}{q}\sum_{v\in G} (\gamma_3\gamma_1^{-1}\gamma_2^{-1}2^{-1}\langle v,\ v\rangle) . \end{aligned}$$

Now let us consider the two cases separately. We have

$$\begin{aligned} &\frac{1}{q}\sum_{v\in G} \chi(\gamma_3\gamma_1^{-1}\gamma_2^{-1}2^{-1}\langle v,\ v\rangle_1) \\ &= \frac{1}{q}\sum_{v\in G} \chi(\gamma_3\gamma_1^{-1}\gamma_2^{-1}v_1 v_2) \\ &= \frac{2q-1}{q} + \frac{1}{q}\sum_{v\in G,\ v_1 v_2 \neq 0} \chi(\gamma_3\gamma_1^{-1}\gamma_2^{-1}v_2 v_1) \\ &= \frac{2q-1}{q} + \frac{q-1}{q}\sum_{x\in F^*} \chi(\gamma_3\gamma_1^{-1}\gamma_2^{-1}x) \\ &= \frac{2q-1}{q} + \frac{q-1}{q}(-1) = 1 . \end{aligned}$$

We also have

$$\begin{aligned} &\frac{1}{q}\sum_{v\in G} \chi(\gamma_3\gamma_1^{-1}\gamma_2^{-1}2^{-1}\langle v,\ v\rangle_2) \\ &= \frac{1}{q} + \frac{1}{q}\sum_{v\in L^*} \chi(\gamma_1^{-1}\gamma_1^{-1}\gamma_2^{-1}N(v)) \\ &= \frac{1}{q} + \frac{q+1}{q}\sum_{x\in F^*} \chi(\gamma_3\gamma_1^{-1}\gamma_1^{-1}x) \\ &= \frac{1}{q} + \frac{q+1}{q}(-1) = -1 . \end{aligned}$$

If we multiply $r_0(g)$ by $(-1)^{i+1}$ and write it by $T^{(i)}(g)$, we have $T^{(i)}(g_1)T^{(i)}(g_2)=T^{(i)}(g_3)$. Taking into consideration the operators corresponding to elements of the type $g=\begin{pmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{pmatrix}$, we obtain following representations.

1. The representation of the first kind. Operators of the representation are defined by formulas

$$T^{(1)}(g)\Phi(u)=\sum_{v\in G}K^{(1)}(g\,|\,u,\,v)\Phi(v)\,,$$

where

$$K^{(1)}(g\,|\,u,\,v)=\frac{1}{q}\chi\Big(\frac{\alpha u_1u_2+\delta v_1v_2-(u_1v_2+u_2v_1)}{\gamma}\Big)$$

if $\gamma\neq 0$;

$$K^{(1)}(g\,|\,u,\,v)=\chi(\alpha\beta u_1u_2)\delta(v-u\alpha)$$

if $\gamma=0$.

2. The representation of the second kind. Operators of the representation are defined by formulas

$$T^{(2)}(g)\Phi(u)=\sum_{v\in L}K^{(2)}(g\,|\,u,\,v)\Phi(v)\,,$$

where

$$K^{(2)}(g\,|\,u,\,v)=\frac{-1}{q}\chi\Big(\frac{\alpha N(u)+\delta N(v)-S(u\bar{v})}{\gamma}\Big)\,,$$

if $\gamma\neq 0$;

$$K^{(2)}(g\,|\,u,\,v)=\chi(\alpha\beta N(u))\delta(v-u\alpha)$$

if $\gamma=0$.

Now let us calculate the traces of these representations, which will be used in §9.

$$Tr\,T^{(i)}(g)=\sum_{u\in G}K^{(i)}(g\,|\,u,\,u)\,,$$

so

$$Tr\,T^{(i)}(g)=\frac{(-1)^{i+1}}{q}\sum_{u\in G}\chi\Big(\frac{\alpha+\delta-2}{\gamma}\cdot 2^{-1}\langle u,\,u\rangle\Big)\,,$$

if $\gamma\neq 0$; and

$$\begin{aligned}Tr\,T^{(i)}(g)&=\sum_{u\in G}\chi(\alpha\beta 2^{-1}\langle u,\,u\rangle)\delta(u-u\alpha)\\&=1+\delta(\alpha-1)\sum_{u\in G-\{(0,0)\}}\chi(\beta 2^{-1}\langle u,\,u\rangle)\,,\end{aligned}$$

if $\gamma=0$.

So they are computed analogously as the calculation of $c(g_1,\,g_2)$. We omit the procedure and state the final results.

$$Tr\, T^{(1)}(g) = (q-1)\delta(\alpha+\delta-2)+1\,,$$

if $\gamma \neq 0$;

$$Tr\, T^{(1)}(g) = 1+(q-1)\delta(\alpha-1)\{q\delta(\beta)+1\}\,,$$

if $\gamma = 0$.

$$Tr\, T^{(2)}(g) = -(q+1)\delta(\alpha+\delta-2)+1$$

if $\gamma \neq 0$;

$$Tr\, T^{(2)}(g) = 1+(q+1)\delta(\alpha-1)\{q\delta(\beta)-1\}\,,$$

if $\gamma = 0$.

6. Decomposition of $T^{(2)}(g)$ into invariant subspaces

Let us now decompose the representation $T^{(2)}(g)$ constructed in §5 into invariant subspaces. For $t \in C$, define the operator R_t on $\mathfrak{H}$ by

$$R_t \Phi(u) = \Phi(tu) \qquad (\Phi \in \mathfrak{H})\,.$$

Then R_t commute with $T^{(2)}(g)$. Let π be an element of $\tilde{C}$, the character group of C, and $\mathfrak{H}_\pi$ be the subspace of $\mathfrak{H}$ consisting of elements which satisfy, for all $t \in C$,

$$R_t \Phi = \pi(t)\Phi$$

If $\Phi \in \mathfrak{H}_\pi$ and $\pi \neq \pi_0$, $\Phi(0)=0$.

By above mentioned commutativity, $\mathfrak{H}_\pi$ is the invariant subspace of the representation $\{T^{(2)}(g), \mathfrak{H}\}$. Put $T^{(2)}_\pi(g) = T^{(2)}(g) \,|\, \mathfrak{H}_\pi$.

Now for $\Phi \in \mathfrak{H}$, Φ_π be the function defined by

$$\Phi_\pi = \sum_{t \in C} R_t \Phi \cdot \overline{\pi(t)}\,.$$

Clearly $\Phi_\pi \in \mathfrak{H}_\pi$ and $(T^{(2)}(g)\Phi)_\pi = T^{(2)}_\pi(g)\Phi_\pi$. Moreover inversion formula

$$\Phi(u) = \frac{1}{q+1} \sum_{\pi \in \tilde{C}} \Phi_\pi(u)$$

and Plancherel formula

$$(\Phi, \Phi) = \frac{1}{q+1} \sum_{\pi \in \tilde{C}} (\Phi_\pi, \Phi_\pi)$$

are obvious. So $\{T^{(2)}(g), \mathfrak{H}\}$ is decomposed into direct sum of $\{T^{(1)}_\pi(g), \mathfrak{H}_\pi\}$, $\pi \in \tilde{C}$.

Let θ be a system of representatives of the C-transitive part (each consisting of $q+1$ elements) of $\boldsymbol{L}^*$. For $\Phi \in \mathfrak{H}_\pi$,

$$\begin{aligned} T^{(2)}(g)\Phi(u) &= \sum_{v\in F} K^{(2)}(g\,|\,u,\, v)\Phi(v) \\ &= K^{(2)}(g\,|\,u,\, 0)\Phi(0)+\sum_{v\in\theta}\sum_{t\in C} K^{(2)}(g\,|\,u,\, tv)\Phi(tv) \\ &= K^{(2)}(g\,|\,u,\, 0)\Phi(0)+\sum_{v\in\theta}\,[\sum_{t\in C} K^{(2)}(g\,|\,u,\, tv)\pi(t)]\Phi(v)\,. \end{aligned}$$

This is the general formula of action of the representation $T_\pi^{(2)}(g)$.

Now assume that $\pi\neq\pi_0$. Extend π to a character of $\boldsymbol{L}^*$ and define $\Psi'(u)=\Psi(u)\overline{\pi(u)}$. Then $\Psi'(tu)=\Psi'(u)$ for all $t\in C$, so $\Psi'(u)=\varphi(N(u))$, where φ is a function on $\boldsymbol{F}^*$. By

$$\sum_{t\in C}\chi\Big(-\frac{S(u\,\overline{vt})}{\gamma}\Big)\,\pi(t)=\sum_{t\bar{t}=N(v)N(u)^{-1}}\chi\Big(-\frac{N(u)t+N(v)t^{-1}}{\gamma}\Big)\pi(t)\pi\Big(\frac{u}{v}\Big)\,,$$

we have, if $\gamma\neq 0$,

$$\begin{aligned} (T^{(2)}(g)\Psi)'(u) = -\frac{1}{q}\sum_{v\in\theta}\chi\Big(\frac{N(u)\alpha+N(v)\delta}{\gamma}\Big) \\ \times\Big[\sum_{t\bar{t}=N(v)N(u)^{-1}}\chi\Big(-\frac{N(u)t+N(v)t^{-1}}{\gamma}\Big)\pi(t)\Big]\Psi'(u)\,. \end{aligned}$$

So the induced action of $T_\pi^{(2)}(g)$ on $\varphi(x)$ is written as

$$T_\pi^{(2)}(g)\varphi(x)=\sum_{y\in F^*}K_\pi^{(2)}(g\,|\,x,\, y)\varphi(y)\,,$$

where

$$K_\pi^{(2)}(g\,|\,x,\, y)=-\frac{1}{q}\chi\Big(\frac{\alpha x+\delta y}{\gamma}\Big)\sum_{t\bar{t}=yx^{-1}}\chi\Big(-\frac{xt+yt^{-1}}{\gamma}\Big)\pi(t)\,,$$

if $\gamma\neq 0$;

$$K_\pi^{(2)}(g\,|\,x,\, y)=\pi(\alpha)\chi(\alpha\beta x)\delta(y-\alpha^2x)\,,$$

if $\gamma=0$. These representations are analogous, in form, to discrete series of $SL(2,\, \boldsymbol{K})$ and discovered by I.M. Gel'fand and M.I. Graev.

Now let $\pi=\pi_0$, then $\Psi\in\mathfrak{H}_\pi$ can be written as $\Psi(u)=\varphi(N(u))$, where φ is a function on $\boldsymbol{F}$. So arguing as above, we have, for $\gamma\neq 0$,

$$\begin{aligned} T_\pi(g)\varphi(x) &= -\frac{1}{q}\sum_{y\in F^*}\chi\Big(\frac{\alpha x+\delta y}{\gamma}\Big)\sum_{t\bar{t}=yx^{-1}}\chi\Big(-\frac{xt+yt^{-1}}{\gamma}\Big)\varphi(y) \\ &\quad -\frac{1}{q}\chi\Big(\frac{\alpha x}{\gamma}\Big)\varphi(0) \qquad (x\in\boldsymbol{F}^*) \\ T_\pi(g)\varphi(0) &= -\frac{q+1}{q}\sum_{y\in F^*}\chi\Big(\frac{\delta y}{\gamma}\Big)\varphi(y)-\frac{1}{q}\varphi(0)\,. \end{aligned}$$

For $\gamma=0$,

$$\begin{aligned} T_\pi(g)\varphi(x) &= \chi(\alpha\beta x)\varphi(\alpha^2x) \qquad (x\in\boldsymbol{F}^*) \\ T_\pi(g)\varphi(0) &= \varphi(0) \end{aligned}$$

Now let us compute the traces of the representation $T_\pi^{(2)}(g)$. $Tr\,T_\pi^{(2)}(g)= K^{(2)}(g\,|\,0,\,0)\delta(\pi)+\sum_{u\in\theta}[\sum_{t\in C}K^{(2)}(g\,|\,u,\,tu)\pi(t)]$.

If $\gamma\neq 0$,

$$\begin{aligned} Tr\,T_\pi^{(2)}(g) &= -\frac{1}{q}\delta(\pi)-\frac{1}{q}\sum_{x\in F^*}\sum_{t\in C}\chi\left(\frac{\alpha+\delta-t-t^{-1}}{\gamma}x\right)\pi(t) \\ &= -\frac{1}{q}\delta(\pi)-\frac{1}{q}\sum_{t\in C}\{q\delta(\alpha+\delta-t-t^{-1})-1\}\pi(t) \\ &= \delta(\pi)-\sum_{t\in C}\delta(\alpha+\delta-t-t^{-1})\pi(t)\,. \end{aligned}$$

So we have

$$\begin{aligned} Tr\,T_\pi^{(2)}(g) &= \delta(\pi)-\pi(\lambda_g)-\pi(\lambda_g^{-1}) && \text{if } \lambda_g\in C \text{ and } \lambda_g\neq\pm 1; \\ &= \delta(\pi)-\pi(\lambda_g) && \text{if } \lambda_g=\pm 1; \\ &= \delta(\pi) && \text{if } \lambda_g\notin C\,, \end{aligned}$$

where λ_g is an eigenvalue of $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

If $\gamma=0$,

$$\begin{aligned} Tr\,T_\pi^{(2)}(g) &= \delta(\pi)+\delta(\alpha^2-1)\sum_{x\in F^*}\pi(\alpha)\chi(\alpha\beta x) \\ &= \delta(\pi)+\delta(\alpha^2-1)\pi(\alpha)\{q\delta(\beta)-1\}\,. \end{aligned}$$

Therefore $T_\pi^{(2)}(g)$ and $T_{\pi'}^{(2)}(g)$ are equivalent if and only if $\pi=\pi'$ or $\pi=\pi'^{-1}$. It is proved in §9 that $T_\pi^{(2)}(g)$ $(\pi\neq\pi_2)$ are irreducible. Traces of $T_\pi^{(2)}(g)$ $(\pi\neq\pi_2)$ are collected in Table 2.

7. Decomposition of $T^{(1)}(g)$ into invariant subspaces

Let us now decompose the representation $T^{(1)}(g)$ into invariant subspaces. For $t\in \boldsymbol{F}^*$, define the transformation $u\to u^t$ of G by $u^t=(tu_1,\,t^{-1}u_2)$. Operators R_t on $\mathfrak{H}$ defined by $R_t\Phi(u)=\Phi(u^t)$ commute with the operators of the representation. Let π be an element of $\tilde{\boldsymbol{F}}^*$, the character group of $\boldsymbol{F}^*$, and $\mathfrak{H}_\pi$ be the subspace of $\mathfrak{H}_t$ consisting of elements which satisfy $R_t\Phi=\pi(t)\Phi$ for all $t\in\boldsymbol{F}^*$. If $\Phi\in\mathfrak{H}_\pi$ and $\pi\neq\pi_0$, then $\Phi(0)=0$. By the above mentioned commutativity, $\mathfrak{H}_\pi$ is the invariant subspace of the representation $\{T^{(1)}(g),\,\mathfrak{H}\}$. Let $T_\pi^{(1)}(g)=T^{(1)}(g)\,|\,\mathfrak{H}_\pi$. Formula of decomposition into invariant subspaces is analogous as in §6 and it is omitted.

Let θ be a system of representatives of $\boldsymbol{F}^*$-transitive part (each consisting of $q-1$ elements) of $G-\{(0,\,0)\}$. For $\Phi\in\mathfrak{H}_\pi$,

$$\begin{aligned} T^{(1)}(g)\Phi(u) &= \sum_{v\in G}K^{(1)}(g\,|\,u,\,v)\Phi(v) \\ &= K^{(1)}(g\,|\,u,\,0)\Phi(0)+\sum_{v\in\theta}\sum_{t\in F^*}K^{(1)}(g\,|\,u,\,v^t)\Phi(v^t) \\ &= K^{(1)}(g\,|\,u,\,0)\Phi(0)+\sum_{v\in\theta}[\sum_{t\in F^*}K^{(1)}(g\,|\,u,\,v^t)\pi(t)]\Phi(v)\,. \end{aligned}$$

This is the general formula of action of the representation $T_\pi^{(1)}(g)$.

Let us write more explicitly the formula of the representation $T_\pi^{(1)}(g)$, fixing $\pi \neq \pi_0$. Then $\Phi \in \mathfrak{H}_\pi$ implies $\Phi(0)=0$ and elements of $\mathfrak{H}_\pi$ is determined by their values on θ. We take $\{(1, x), x \in \boldsymbol{F}; (0, 1)\}$ as θ and put $\varphi(x)=\Phi(1, x)$, $\varphi(\infty)=\frac{1}{q}\tau(\pi^{-1})\Phi(0, 1)$. If $\gamma \neq 0$,

$$T_\pi^{(1)}(g)\varphi(x) = \frac{1}{q}\sum_{y \in \boldsymbol{F}^*} \chi\Big(\frac{\alpha x+\delta y}{\gamma}\Big)\Big[\sum_{t \in \boldsymbol{F}^*}\chi\Big(-\frac{xt+yt^{-1}}{\gamma}\Big)\pi(t)\Big]\varphi(y)$$
$$+\chi\Big(\frac{\alpha x}{\gamma}\Big)\pi\Big(-\frac{1}{\gamma}\Big)\varphi(\infty)$$

and

$$T_\pi^{(1)}(g)\varphi(\infty) = \frac{\pi(\gamma)}{q}\sum_y \varphi(y)\chi\Big(\frac{\delta y}{\gamma}\Big)$$

If $\gamma=0$,

$$T_\pi^{(1)}(g)\varphi(x) = \pi(\alpha)\chi(\alpha\beta x)\varphi(\alpha^2 x)$$

and

$$T_\pi^{(1)}(g)\varphi(\infty) = \pi^{-1}(\alpha)\varphi(\infty)$$

These representations are χ-realization of T_π introduced in §3.

Let us compute traces of constructed representations.

$$Tr\, T_\pi^{(1)}(g) = K^{(1)}(g\,|\,0, 0)\delta(\pi)+\sum_{u \in \theta}\big[\sum_{t \in \boldsymbol{F}^*}K^{(1)}(g\,|\,u, u^t)\pi(t)\big]$$

If $\gamma \neq 0$,

$$\begin{aligned}
Tr\, T_\pi^{(1)}(g) &= \frac{1}{q}\delta(\pi)+\frac{1}{q}\sum_{u \in \theta}\sum_{t \in \boldsymbol{F}^*}\chi\Big(\frac{\alpha+\delta-t-t^{-1}}{\gamma}u_1u_2\Big)\pi(t)\\
&= \frac{1}{q}\delta(\pi)+\frac{2}{q}\sum_{t \in \boldsymbol{F}^*}\pi(t)+\frac{1}{q}\sum_{t, x \in \boldsymbol{F}^*}\chi\Big(\frac{\alpha+\delta-t-t^{-1}}{\gamma}x\Big)\pi(t)\\
&= \frac{1}{q}\delta(\pi)+\frac{2}{q}\sum_{t \in \boldsymbol{F}^*}\pi(t)+\frac{1}{q}\sum_{t \in \boldsymbol{F}^*}\{q\delta(\alpha+\delta-t-t^{-1})-1\}\pi(t)\\
&= \delta(\pi)+\sum_{t \in \boldsymbol{F}^*}\delta(\alpha+\delta-t-t^{-1})\pi(t)
\end{aligned}$$

So we have

$$\begin{aligned}
Tr\, T_\pi^{(1)}(g) &= \delta(\pi)+\pi(\lambda_g)+\pi(\lambda_g^{-1}) && \text{if } \lambda_g \in \boldsymbol{F} \text{ and } \lambda_g \neq \pm 1;\\
&= \delta(\pi)+\pi(\lambda_g) && \text{if } \lambda_g \neq \pm 1;\\
&= \delta(\pi) && \text{if } \lambda_g \notin \boldsymbol{F}
\end{aligned}$$

If $\gamma=0$,

$$\begin{aligned}
Tr\, T_\pi^{(1)}(g) &= \delta(\pi)+\sum_{u \in \theta}\sum_{t \in \boldsymbol{F}^*}\chi(\alpha\beta u_1u_2)\delta(u^t-u\alpha)\pi(t)\\
&= \delta(\pi)+\pi(\alpha)+\pi(\alpha^{-1})+\delta(\alpha^2-1)\pi(\alpha)\sum_{x \in \boldsymbol{F}^*}\chi(\alpha\beta x)\\
&= \delta(\pi)+\pi(\alpha)+\pi(\alpha^{-1})+\delta(\alpha^2-1)\pi(\alpha)\{q\delta(\beta)-1\}
\end{aligned}$$

So we have

$$\begin{aligned} Tr\,T_{\pi}^{(1)}(g) &= \delta(\pi)+\pi(\alpha)+\pi(\alpha^{-1}) && \text{if } \alpha\neq\pm 1; \\ &= \delta(\pi)+\pi(\alpha)\{1+q\delta(\beta)\} && \text{if } \alpha=\pm 1 . \end{aligned}$$

Therefore $T_{\pi}^{(1)}$ and $T_{\pi'}^{(1)}$ are equivalent if and only if $\pi=\pi'$ or $\pi=\pi'^{-1}$. $T_{\pi_0}^{(1)}$ is equivalent to direct sum of $T_{\pi_2}^{(2)}$ and twice the identity representation. It will be proved in § 9. that $T_{\pi}^{(1)}(g)$ $(\pi\neq\pi_1,\ \pi_0)$ are irreducible. Traces of $T_{\pi}^{(1)}(g)$ $(\pi\neq\pi_1,\ \pi_0)$ are collected in Table 2.

8. Decomposition of $T_{\pi_1}^{(1)}(g)$ and $T_{\pi_2}^{(2)}(g)$ into invariant subspaces

8.1. Algebraic lemmas.

Lemma 3. *Let us consider the mapping $\boldsymbol{F}^*\ni t\to\alpha=t+t^{-1}\in\boldsymbol{F}$. Then*

(i) *The range consists of α which satisfy $\alpha^2-4\in\boldsymbol{F}_+$.*

(ii) *The image of the set $\boldsymbol{F}_+^*$ is the set of α which satisfy $\alpha+2,\ \alpha-2\in\boldsymbol{F}_+$.*

Proof. (i) If $\alpha=t+t^{-1}$, then $\alpha^2-4=(t-t^{-1})^2\in\boldsymbol{F}_+$. Conversely let $\alpha^2-4=a^2$, $a\in\boldsymbol{F}$. If we put $t=2^{-1}(\alpha+a)$, then $t=2(\alpha-a)^{-1}$. So $\alpha=t+t^{-1}$.
(ii) If $\alpha=s^2+s^{-2}$, then $\alpha\pm 2=(s\pm s^{-1})^2\in\boldsymbol{F}_+$. Conversely let $\alpha+2=a^2$ and $\alpha-2=b^2$, $a,\ b\in\boldsymbol{F}$. Put $s=2^{-1}(b+a)$. Then $s=4^{-1}(b^2+2ab+a^2)=2^{-1}(\alpha+ab)=2(\alpha-ab)^{-1}$. So $\alpha=s^2+s^{-2}$.

Lemma 4. *Let us consider the mapping $C\ni t\to\alpha=t+t^{-1}\in\boldsymbol{F}$. Then*

(i) *The range consists of α which satisfy $\alpha^2-4\in\boldsymbol{F}_-$.*

(ii) *The image of $C_+-\{-1\}$ is the set of α which satisfy $\alpha+2\in\boldsymbol{F}_+^*$ and $\alpha^2-4\in\boldsymbol{F}_-$.*

Proof. (i) $\alpha=t+t^{-1}$, $t\in C$, then $\alpha^2-4=(t-\bar{t})^2\in\boldsymbol{F}_-$. Conversely let $\alpha^2-4=a^2\varepsilon$, $a\in\boldsymbol{F}$. Put $t=2^{-1}(\alpha+a\sqrt{\varepsilon})$, then $t\in C$ and $\alpha=t+t^{-1}$.
(ii) If $\alpha=s^2+s^{-2}$, $s\in C$, then $\alpha+2=(s+\bar{s})^2$ and $s=\neq-1$ implies that $s+\bar{s}\neq 0$.

Conversely, let $\alpha^2-4\in\boldsymbol{F}_-$ and $\alpha+2=a^2$, $a\in\boldsymbol{F}^*$. Then $\alpha=t+t^{-1}$, $t\in C$, by (i). Suppose $t\in C_-$, then it is written as $t=\bar{e}/e$, where $N(e)=\varepsilon$. So $\boldsymbol{F}_+\ni(e+\bar{e}^2)=e\bar{e}(\bar{e}/e+2+e/\bar{e})=a^2\varepsilon\in\boldsymbol{F}_-^*$. This is a contradiction and we have $t\in C_+$. $t\neq-1$ is obvious.

8.2. Decomposition of $T_{\pi_2}^{(2)}(g)$ into invariant subspaces. Representation space of $T_{\pi_2}(g)$ consist of all functions defined on $\boldsymbol{F}^*$. Let us prove that spaces of functions which vanish on $\boldsymbol{F}_+^*$ and $\boldsymbol{F}_-^*$ are invariant subspaces. In fact, let $yx^{-1}=a^2\varepsilon$, $a\in\boldsymbol{F}^*$. Then

$$\sum_{t\bar{t}=yx^{-1}}\chi\left(-\frac{xt+yt^{-1}}{\gamma}\right)\pi_2(t)=\pi_2(ae_0)\sum_{t\in C}\chi\left(-\frac{xa(e_0t+\overline{e_0t})}{\gamma}\right)\pi_2(t)\,.$$

Put

$$A=\sum_{t\in C}\chi(a(e_0t+\overline{e_0t}))\pi_2(t)$$

Replacing t by $(\bar{e}_0/e_0)s^{-1}$, we have

$$A=\sum_{s\in C}\chi(a(\bar{e}_0s^{-1}+e_0s))\pi_2(\bar{e}_0/e_0)\pi_2(s^{-1})=-A\,.$$

So it is proved that if $yx^{-1}\in\boldsymbol{F}^*_{\pm}$, then $K^{(2)}_{\pi_2}(g\,|\,x,y)=0$. Restriction of $T^{(2)}_{\pi_2}(g)$ on the space of functions vanishing on $\boldsymbol{F}^*_{\pm}$ and $\boldsymbol{F}^*_{\pm}$ are denoted by $T^+_{\pi_2}(g)$ and $T^-_{\pi_2}(g)$. It will be proved in §9. that $T^{\pm}_{\pi_2}(g)$ are irreducible.

8.3. Calculation of traces of $T^{\pm}_{\pi_2}(g)$. $Tr\,T^{(2)}_{\pi_2}(g)=Tr\,T^-_{\pi_2}(g)+Tr\,T^-_{\pi_2}(g)$ is already known, so it is sufficient to calculate $Tr\,T^+_{\pi_2}(g)-Tr\,T^-_{\pi_2}(g)$. Put, for $a\in\boldsymbol{F}$,

$$S=\sum_{t\in C}\chi(aS(t))\pi_2(t)\,.$$

Then by Lemma 4.,

$$\begin{aligned}
S&=\sum_{u\in F,\,u\neq\pm2}\chi(au)\pi_1(u+2)\{1-\pi_1(u^2-4)\}+\chi(2a)+\chi(-2a)\pi_2(-1)\\
&=\sum_{u\in F,\,u\neq\pm2}\chi(au)\{\pi_1(u+2)-\pi_1(u-2)\}+\chi(2a)+\chi(-2a)\pi_2(-1)\\
&=\chi(-2a)\pi_1(a)\sum_{u\in F^*}\chi(u)\pi_1(u)-\chi(2a)\pi_1(a)\sum_{u\in F^*}\chi(u)\pi_1(u)\\
&\quad+\chi(-2a)\pi_1(-4)+\chi(-2a)\pi_2(-1)\,.\\
&=\tau\{\chi(-2a)-\chi(2a)\}\pi_1(a)\,.
\end{aligned}$$

If $\gamma\neq0$,

$$\begin{aligned}
&T_r\,T^+_{\pi_2}(g)-T_r\,T^-_{\pi_2}(g)\\
&=-\frac{1}{q}\sum_{x\in F^*}\pi_1(x)\chi\Big(\frac{\alpha+\delta}{\gamma}x\Big)\sum_{t\in C}\chi\Big(-\frac{t+t^{-1}}{\gamma}x\Big)\pi_2(t)\\
&=-\frac{1}{q}\sum_{x\in F^*}\pi_1(x)\chi\Big(\frac{\alpha+\delta}{\gamma}x\Big)\tau\Big\{\chi\Big(2\frac{x}{\gamma}\Big)-\chi\Big(-2\frac{x}{\gamma}\Big)\Big\}\pi_1\Big(-\frac{x}{\gamma}\Big)\\
&=-\frac{\tau}{q}\pi_1\Big(-\frac{1}{\gamma}\Big)\sum_{x\in F^*}\Big\{\chi\Big(\frac{\alpha+\delta+2}{\gamma}x\Big)-\chi\Big(\frac{\alpha+\delta-2}{\gamma}x\Big)\Big\}\\
&=-\tau\pi_1\Big(-\frac{1}{\gamma}\Big)\{\delta(\alpha+\delta+2)-\delta(\alpha+\delta-2)\}\,.
\end{aligned}$$

If $\gamma=0$,

$$\begin{aligned}
Tr\,T^+_{\pi_2}(g)-Tr\,T^-_{\pi_2}(g)&=\delta(\alpha^2-1)\pi_2(\alpha)\sum_{x\in F^*}\chi(\alpha\beta x)\pi_1(x)\\
&=\delta(\alpha^2-1)\pi_2(\alpha)\pi_1(\alpha\beta)\tau\,.
\end{aligned}$$

The results are collected in Table 2.

8.4. Decomposition of $T_{\pi_1}(g)$ into invariant subspaces. The representation space of $T_{\pi_1}(g)$ consist of all functions defined on $\boldsymbol{F}\cup\{\infty\}$.

Proposition. *The sapce of functions vanishing on $\boldsymbol{F}^*_-$ and satisfying*

$$(1) \qquad -\frac{\tau}{q}\varphi(0)+\varphi(\infty)=0$$

is an invariant subspace of the representation $T_{\pi_1}(g)$.

Proof. Put

$$A=\sum_{t\in \boldsymbol{F}^*}\chi(a^2\varepsilon t+b^2t^{-1})\pi_1(t) \qquad (a,\ b\in \boldsymbol{F}^*)$$

Replacing t by $(b^2/a^2\varepsilon)\ s^{-1}$, we have

$$A=\sum_{s\in \boldsymbol{F}^*}\chi(b^2s^{-1}+a^2\varepsilon s)\pi_1(s)\pi_1(\varepsilon)=-A\,.$$

So $A=0$.

Let φ vanishes on $\boldsymbol{F}^*_-$ and satisfies (1). It is obvious that $T_{\pi_1}(g)\varphi$ vanishes on $\boldsymbol{F}^*_-$ and satisfies (1), if $\gamma=0$. We have only to consider the case $g=s=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$$\begin{aligned}(T_{\pi_1}(s)\varphi)(a^2\varepsilon)&=\frac{1}{q}\sum_{y\in \boldsymbol{F}_+}\{\sum_{t\in \boldsymbol{F}^*}\chi(-(a^2\varepsilon t+yt^{-1}))\pi_1(t)\}\varphi(y)\\&\quad+\pi_1(-1)\varphi(\infty) \qquad (a\in \boldsymbol{F}^*)\,.\end{aligned}$$

By $A=0$, we have

$$\begin{aligned}(T_{\pi_1}(s)\varphi)(a^2\varepsilon)&=\frac{1}{q}\{\sum_{t\in \boldsymbol{F}^*}\chi(-a^2\varepsilon t)\pi_1(t)\}\varphi(0)+\pi_1(-1)\varphi(\infty)\\&=-\frac{\pi_1(-1)\tau}{q}\varphi(0)+\pi_1(-1)\varphi(\infty)=0\,.\end{aligned}$$

We also have

$$\begin{aligned}&-\frac{\tau}{q}(T_{\pi_1}(s)\varphi)(0)+(T_{\pi_1}(s)\varphi)(\infty)\\&=-\frac{\tau}{q}\left\{\frac{\pi_1(-1)\tau}{q}\sum_{y\in \boldsymbol{F}^*_+}\varphi(y)+\pi_1(-1)\varphi(\infty)\right\}+\frac{1}{q}\sum_{y\in \boldsymbol{F}_+}\varphi(y)\\&=q^{-2}\{-\tau^2\pi_1(-1)+q\}\sum_{y\in \boldsymbol{F}^*_+}\varphi(y)+q^{-1}\{-\tau\pi_1(-1)\varphi(\infty)+\varphi(0)\}\,.\end{aligned}$$

Making use of the fact $\tau^2=q\pi_1(-1)$ and (1), we have

$$-\frac{\tau}{q}(T_{\pi_1}(s)\varphi)(0)+(T_{\pi_1}(s)\varphi)(\infty)=0\,.$$

So the proposition is proved.

Let us denote by $T^+_{\pi_1}(g)$ the restriction of $T_{\pi_1}(g)$ to the subspace described in the statement of the proposition. By the same way, we can show that the space of functions vanishing on $\boldsymbol{F}^*_+$ and satisfying

$$\frac{q}{\tau}\varphi(0)+\varphi(\infty)=0$$

is an invariant subspace of $T_{\pi_1}(g)$. Let us denote by $T^-_{\pi_1}(g)$ the restriction of $T_{\pi_1}(g)$ to this subspace. It will be proved in §9. that $T^{\pm}_{\pi_1}(g)$ are irreducible.

Now let us write the formulas of the action of $T^+_{\pi_1}(g)$ on $\varphi(x)$, $x\in \boldsymbol{F}_+$. If $\gamma\neq 0$,

$$T^+_{\pi_1}(g)\varphi(x)=\frac{1}{q}\sum_{y\in \boldsymbol{F}^*_+}\chi\left(\frac{\alpha x+\delta y}{\gamma}\right)\left[\sum_{t\in \boldsymbol{F}^*}\chi\left(-\frac{xt+yt^{-1}}{\gamma}\right)\pi_1(t)\right]\varphi(y)$$
$$+\frac{2\tau}{q}\chi\left(\frac{\alpha x}{\gamma}\right)\pi_1(-\gamma)\varphi(0) \qquad (x\in \boldsymbol{F}^*_+)$$

$$T^+_{\pi_1}(g)\varphi(0)=\frac{\pi_1(\gamma)}{\tau}\sum_{y\in \boldsymbol{F}_+}\varphi(y)\chi\left(\frac{\delta y}{\gamma}\right)$$

If $\gamma=0$,

$$T^+_{\pi_1}(g)\varphi(x)=\pi(\alpha)\chi(\alpha\beta x)\varphi(\alpha^2 x) \qquad (x\in \boldsymbol{F}_+)\,.$$

The action of $T^-_{\pi_1}$ on $\varphi(x)$, $x\in \boldsymbol{F}$, are written as follows. If $\gamma\neq 0$,

$$T^-_{\pi_1}(g)\varphi(x)=\frac{1}{q}\sum_{y\in \boldsymbol{F}^*_-}\chi\left(\frac{\alpha x+\delta y}{\gamma}\right)\left[\sum_{t\in \boldsymbol{F}^*}\chi\left(-\frac{xt+yt^{-1}}{\gamma}\right)\pi_1(t)\right]\varphi(y)$$
$$-\frac{2\tau}{q}\chi\left(\frac{\alpha x}{\gamma}\right)\pi_1(-\gamma)\varphi(0) \qquad (x\in \boldsymbol{F}^*)$$

$$T^-_{\pi_1}(g)\varphi(0)=-\frac{\pi_1(\gamma)}{\tau}\sum_{y\in \boldsymbol{F}_-}\varphi(y)\chi\left(\frac{\delta y}{\gamma}\right)$$

If $\gamma=0$,

$$T^-_{\pi_1}(g)\varphi(x)=\pi(\alpha)\chi(\alpha\beta x)\varphi(\alpha^2 x) \qquad (x\in \boldsymbol{F}_-)\,.$$

8.5. Calculation of traces of $T^{\pm}_{\pi_1}(g)$. It is only necessary to calculate $Tr\,T^+_{\pi_1}(g)-Tr\,T^-_{\pi_1}(g)$. Put

$$S=\sum_{t\in \boldsymbol{F}^*}\chi(a(t+t^{-1}))\pi_1(t)\,.$$

By Lemma 4., we have

$$\begin{aligned} S&=\sum_{u\neq\pm 2}\chi(au)\{1+\pi_1(u^2-4)\}\pi_1(u-2)+\chi(2a)+\chi(-2a)\pi_1(-1)\\ &=\sum_{u\in \boldsymbol{F}}\chi(au)\pi_1(u-2)+\sum_{u\in \boldsymbol{F}}\chi(au)\pi_1(u+2)\\ &=\{\chi(2a)+\chi(-2a)\}\pi_1(a)\tau\,. \end{aligned}$$

If $\gamma\neq 0$,

$$\begin{aligned} Tr\,T^+_{\pi_1}(g)&=\frac{1}{q}\sum_{x\in \boldsymbol{F}^*_+}\chi\left(\frac{\alpha+\delta}{\gamma}x\right)\sum_{t\in \boldsymbol{F}^*}\chi\left(-\frac{t+t^{-1}}{\gamma}x\right)\pi_1(t)+\pi_1\left(-\frac{1}{\gamma}\right)\frac{\tau}{q}\\ &=\frac{\tau}{q}\sum_{x\in \boldsymbol{F}^*_+}\chi\left(\frac{\alpha+\delta}{\gamma}x\right)\left\{\chi\left(-\frac{2x}{\gamma}\right)+\chi\left(\frac{2x}{\gamma}\right)\right\}\pi_1\left(-\frac{x}{\gamma}\right)+\pi_1\left(-\frac{1}{\gamma}\right)\frac{\tau}{q} \end{aligned}$$

Analogously,

$$Tr\,T^{-}_{\pi_1}(g)=\frac{\tau}{q}\sum_{x\in F^*_-}\chi\Big(\frac{\alpha+\delta}{\gamma}x\Big)\Big\{\chi\Big(-\frac{2x}{\gamma}\Big)+\chi\Big(\frac{2x}{\gamma}\Big)\Big\}\pi_1\Big(-\frac{x}{\gamma}\Big)-\pi_1\Big(-\frac{1}{\gamma}\Big)\frac{\tau}{q}$$

Therefore, we have

$$\begin{aligned}Tr\,T^{+}_{\pi_1}(g)-Tr\,T^{-}_{\pi_1}(g)&=\frac{\tau}{q}\sum_{x\in F}\Big\{\chi\Big(\frac{\alpha+\delta-2}{\gamma}x\Big)+\chi\Big(\frac{\alpha+\delta+2}{\gamma}x\Big)\Big\}\pi_1\Big(-\frac{1}{\gamma}\Big)\\&=\frac{\tau}{q}\{q\delta(\alpha+\delta-2)+q\delta(\alpha+\delta+2)\}\pi_1\Big(-\frac{1}{\gamma}\Big)\\&=\tau\delta((\alpha+\delta)^2-4)\pi_1\Big(-\frac{1}{\gamma}\Big),\end{aligned}$$

If $\gamma=0$,

$$Tr\,T^{\pm}_{\pi_1}(g)=\delta(\alpha^2-1)\sum_{x\in F^*_\pm}\pi_1(\alpha)\chi(\alpha\beta x)+\pi_1(\alpha)\,.$$

Therefore, we have

$$\begin{aligned}Tr\,T^{+}_{\pi_1}(g)-Tr\,T^{-}_{\pi_1}(g)&=\delta\,(\alpha^2-1)\sum_{x\in F^*}\pi_1(\alpha)\chi\Big(\frac{\beta}{\alpha}x\Big)\pi_1(x)\\&=\delta(\alpha^2-1)\pi_1(\beta)\tau\,.\end{aligned}$$

The results are collected in Table 2.

9. Description of all irreducible representation

We have proved that the representation of the first kind splits up into

1. $T^{(1)}_{\pi}(g)$ $(\pi\neq\pi_0,\ \pi_1)$ $\frac{1}{2}(q-3)$ inequivalent representations of degree $q+1$, each with multiplicity 2;
2. $T^{\pm}_{\pi_1}(g)$ two inequivalent representations of degree $\frac{1}{2}(q+1)$;
3. A representation of degree q equivalent to $T^{(2)}_{\pi_0}(g)$;
4. twice the identical representation.

The representation of the second kind splits up into

1. $T^{(2)}_{\pi}(g)$ $(\pi\neq\pi_0,\pi_1)$ $\frac{1}{2}(q-1)$ inequivalent representations of degree $q-1$, each with multiplicity 2;
2. $T^{\pm}_{\pi_2}(g)$ two inequivalent representations of degree $\frac{1}{2}(q-1)$;
3. $T^{(2)}_{\pi_0}(g)$

It will next be shown that we have thus obtained the complete decomposition of the representations of the first and second kind into irreducible representations. The proof of this statement rests on the following

Lemma 5. Let $R_1, R_2,\cdots, R_k$ be (reducible or irreducible) representations of a finite group G of order g. Suppose that no two of these representations are equivalent. Let $n_1, n_2, \cdots, n_k$ be positive integers, let R be the representation

$$R = n_1R_1 + n_2R_2 + \cdots + n_kR_k$$

and let T be its trace function. Then we have

$$\sum_{U \in G} |T(U)|^2 \geqslant (n_1^2 + n_2^2 + \cdots + n_k^2)g ,$$

where the equality sign holds if and only if all representations $R_1, R_2, \cdots, R_k$ are irreducible.

For the proof of this lemma, see [9, p. 402].

Now let us recall some properties of $SL(2, \boldsymbol{F})$. Order of $SL(2, \boldsymbol{F})$ is $q(q^2-1)$. Put $g_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $\lambda \in \boldsymbol{F}^*$ and $\lambda \neq \pm 1$. For $t \in C$, $t \neq \pm 1$, let g_t denote an (arbitralily fixed) element of $SL(2, \boldsymbol{F})$ with eigenvalues t and t^{-1}. Then the conjugate class decomposition of $SL(2, \boldsymbol{F})$ is described as in following table.

Table 1

Representative	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	g_λ	g_t	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$
Number of sets of conjugates	1	1	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-1)$	1	1	1	1
Number of elements in each set	1	1	$q(q+1)$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$

The sum of the square of the multiplicities with which the enumerated representations occur in the representation of the first kind is

$$\frac{1}{2}(q-3) \cdot 2^2 + 2 \cdot 1 + 1 + 2^2 = 2q + 1 .$$

On the other hand it is shown that

$$\sum_g |Tr\, T^{(1)}(g)|^2 = q(q-1)(2q+1)$$

Therefore, by Lemma 5, $T_\pi^{(1)}(g)$ $(\pi \neq \pi_0, \pi_2)$, $T_{\pi_1}^{\pm}(g)$, $T_{\pi_0}^{(2)}(g)$ are irreducible. In the same way, it is proved that $T_\pi^{(2)}(g)$ $(\pi \neq \pi_0, \pi_2)$, $T_{\pi_2}^{\pm}(g)$, $T_{\pi_0}^{(2)}(g)$ are irreducible. Their traces are collected on Table 2.

By Table 2 it is shown that there are no more equivalence between constructed irreducible representations. Together with the trivial representation, there are $q+4$ irreducible representations. The number is equal to that of the conjugate classes in $SL(2, \boldsymbol{F})$. So all irreducible representations of $SL(2, \boldsymbol{F})$ are thus obtained.

Table 2

Representative	$TrT_{\pi}^{(1)}$	$TrT_{\pi_0}^{\pm}$	$TrT_{\pi_2}^{(2)}$	$TrT_{\pi}^{(2)}$	$TrT_{\pi_2}^{\pm}$
	$(\pi\neq\pi_0,\ \pi_1)$			$(\pi\neq\pi_0,\ \pi_2)$	
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$q+1$	$\frac{1}{2}(q+1)$	q	$q-1$	$\frac{1}{2}(q-1)$
$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\pi(-1)(q+1)$	$\frac{1}{2}\pi_1(-1)(q+1)$	q	$-\pi(-1)(q-1)$	$\frac{1}{2}\pi_1(-1)(q-1)$
$g_\lambda(\lambda\in \boldsymbol{F}^*,\ \lambda\neq\pm1)$	$\pi(\lambda)+\pi(\lambda^{-1})$	$\pi_1(\lambda)$	1	0	0
$g_t(t\in C,\ t\neq\pm1)$	0	0	-1	$-\pi(t)-\pi(t^{-1})$	$-\pi_2(t)$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	1	$\frac{1}{2}(1\pm\tau)$	0	-1	$\frac{1}{2}(-1\pm\tau)$
$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	1	$\frac{1}{2}(1\mp\tau)$	0	-1	$\frac{1}{2}(-1\mp\tau)$
$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$\pi(-1)$	$\frac{1}{2}\{\pi_1(-1)\pm\tau\}$	0	$-\pi(-1)$	$\frac{1}{2}\{\pi_1(-1)\mp\tau\}$
$\begin{pmatrix}-1&\varepsilon\\0&-1\end{pmatrix}$	$\pi(-1)$	$\frac{1}{2}\{\pi_1(-1)\mp\tau\}$	0	$-\pi(-1)$	$\frac{1}{2}\{\pi_1(-1)\pm\tau\}$

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