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ON A DISTANCE FUNCTION ON THE SET OF DIFFERENTIABLE STRUCTURES

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1. Introduction

Let X be a given compact closed orientable topological manifold and let $\Sigma=\{\sigma_i\}$ denote the set of differentiable structures on X. In this paper one defines a pseudo distance ρ on Σ which is allowed to take ∞ as one of its values. It is proved that ρ is actually a distance function, namely that $\rho(\sigma_1, \sigma_2)=0$ implies $\sigma_1=\sigma_2$. More strongly, one proves the following

Theorem 1. *There exists a positive ε_1 depending on dimension of X, such that if $\rho(\sigma_1, \sigma_2)\leqq\varepsilon_1$ then σ_1 and σ_2 are differentiably equivalent.*

The proof is given in Part II. In Part I one investiates relations between the distance and the combinatorial equivalence.

It is seen that $\rho(\sigma_1, \sigma_2)<\infty$ if σ_1 and σ_2 are combinatorially equivalent, and the following theorem is proved:

Theorem 2. *There exists a positive ε_2, depending only on dimension of X, such that if $\rho(\sigma_1, \sigma_2)\leqq\varepsilon_2$ then σ_1 and σ_2 are combinatorially equivalent.*

In order to assure non triviality of the distance function the following remark might be sufficient. By J. Milnor and I. Tamura there is found a compact comibinatorial manifold which admits two smoothings having different integral Pontrjagin classes, therefore using the result of [S] which asserts that any two differentiable structures of distance less than $1/2\log 3/2$ have the same integral Pontrjagin classes, one sees that the distance between these structures is finite but not less than $1/2\log 3/2$.

Although the distance has been allowed to take ∞ as its value, it is possible to restrict Σ so that the distaece always gives finite value, by introducing a notion of Lipschitz manifold and compatible smoothing, as appears in a sequel.

PART I

2. Definition of a pseudo distance.

Let X_i $(i=1, 2)$ be metric spaces with metrics d_i $(i=1, 2)$. Then the *size* $\mathrm{I}(f)$ (rel. d_1, d_2) of a homeomorphism f of X_1 onto X_2 is defined to be

$$\mathrm{I}(f) = \begin{cases} \inf \{k \geqq 1 \mid \forall x, y \in X_1,\ d_1(x, y)/k \leqq d_2(f(x), f(y)) \leqq k d_1(x, y)\}\ , \\ \infty, \textit{ if the above set of } k \textit{ is empty.} \end{cases}$$

A homeomorphism f is said to be (regular) *Lipschitz* relative to d_1, d_2, if the size $\mathrm{I}(f)$ (rel. d_1, d_2) is bounded, and the size $\mathrm{I}(f)$ (rel. d_1, d_2) is sometimes called the *Lipschitz constant* of f.

The following properties 1)-3) of the size are elementary.

1) $\mathrm{I}(id.)$ (rel. d_1, d_1)$=1$,

2) $\mathrm{I}(f)$ (rel. d_1, d_2)$=\mathrm{I}(f^{-1})$ (rel. d_2, d_1),

3) Let X_3, d_3 be the third metric space and its metric, and let $f: X_1 \to X_2$, $g: X_2 \to X_3$ be homeomorphisms, then

$$\mathrm{I}(gf)\ (\text{rel. } d_1, d_3) \leqq \{\mathrm{I}(g)\ (\text{rel. } d_2, d_3)\}\{\mathrm{I}(f)\ (\text{rel. } d_1, d_2)\}\ .$$

Now let M_i $(i=1, 2)$ be compact differentiable manifolds and let f be a homeomorphism of M_1 onto M_2, then since M_i admit Riemannian metrics ρ_i $(i=1, 2)$, one can define the size $\mathrm{I}(f)$ (rel. ρ_1, ρ_2) of f. If h_i are diffeomorphisms on M_i $(i=1, 2)$, $h_i^{\#}\rho_i$ also are Riemannian metrics on M_i, and obviously

$$\mathrm{I}(h_2 f h_1)(\text{rel. } \rho_1, \rho_2) = \mathrm{I}(f)(\text{rel. } h_1^{\#}f_1, h_2^{\#}f_2)\,.$$

Thus denoting by $\mathrm{I}(f)$ the infimum of the sizes $\mathrm{I}(f)$ (rel. ρ_1, ρ_2) taken on all Riemannian metrics ρ_i on M_i $(i=1, 2)$ one sees that, for any diffeomorphisms h_i on M_i,

$$\mathrm{I}(h_2 f h_1) = \mathrm{I}(f)\,.$$

A differentiable manifold M which is homeomorphic to a topological manifold X is said to be a *smoothing* of X and two smoothings M_1, M_2 are said to be *equivalent* if there is a diffeomorphism between them. The equivalence classes are called *differentiable structures* on X.

Define a real valued function $\rho(M_1, M_2)$ of smoothings M_i $(i=1, 2)$ by

$$\rho(M_1, M_2) = \inf \{\log \mathrm{I}(h_2^{-1}h_1) \mid h_i : M_i \to X \text{ is homeomorphism}\}\ ,$$

where the infimum is taken over all the homeomorphisms h_i of M_i onto X $(i=1, 2)$.

Then obviously,

4) $\rho(M_1, M_2)$ depends only on the equivalence classes σ_i ($i=1, 2$) of smoothings M_i and therefore can be denoted by $\rho(\sigma_1, \sigma_2)$.

The function $\rho(\sigma_1, \sigma_2)$ defined on the set Σ of the differentiable structures on X has the following properties 5–7), which are easily deduced from 1–3):

5) $\rho(\sigma_1, \sigma_1)=0$.
6) $\rho(\sigma_1, \sigma_2)=\rho(\sigma_2, \sigma_1)$.
7) $\rho(\sigma_1, \sigma_3)\leqq\rho(\sigma_1, \sigma_2)+\rho(\sigma_2, \sigma_3)$, where σ_3 also is a differentiable structure on X.

In the other words,

Proposition 1. *ρ gives a pseudo distance on the set Σ of the equivalence classes of differentiable structures on a compact topological manifold X.*

The following non symmetric versions of the usual Lipschitz conditions are sometimes useful and referred simply as Lipschitz conditions.

A map f of a metric space X_1 into a metric space X_2 is said to satisfy *Lipschitz condition* with a positive λ, if

$$|d_2^2(f(x), f(y))-d_1^2(x, y)|\leqq\lambda d_1^2(x, y)$$

for all $x, y\in X_1$. Also f is said to satisfy *local Lipschitz condition* at $p\in X_1$, if there exists a neighbourhood $U(p)$ of p such that

$$|d_2^2(f(x), f(y))-d_1^2(x, y)|\leqq\lambda d_1^2(x, y)$$

for all $x, y\in U(p)$.

Obviously a (regular) Lipschitz map with Lipschitz constant $\mathrm{l}(f)$ satisfies both global and local Lipschitz conditions with $\lambda=\sqrt{\mathrm{l}^2(f)-1}$.

Conversely, a map satisfying the (global) Lipschitz condition with $\lambda<1$ is (regular) Lipschitz and has the Lipschitz constant less than $1/\sqrt{1-\lambda}$.

Proposition 2. *For any positive $\lambda<1$, there is a positive ρ such that $\log \mathrm{l}(f)<\rho$ implies the Lipschitz condition with λ for f. Conversely, for a given $\rho>0$, there is a positive λ such that the Lipschitz condition with λ for f yields $\log \mathrm{l}(f)<\rho$.*

3. Local properties of Lipschitz maps.

Throughout the section, R^n denotes the Euclidean n-space with the usual norm $|\ \ |$ and, unless otherwise stated, f is a map of R^n into R^N sending $0\in R^n$ to $0\in R^N$.

Lemma 1. *Assume that f satisfies the local Lipschitz condition with λ in an open neighbourhood $U(0)$ of 0, then*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \leqq 2\lambda(|x|^2 + |y|^2)$$

for all $x, y \in U(0)$.

Let s be an n-simplex in R^n having 0 as one of its vertices, then, the *s-approximation* f_s of a map f is the linear map which agrees to f on each vertices v_i of s, that is, f_s is the linear map characterized by $f_s(v_i) = f(v_i)$ for all verteces v_i of s.

Unless otherwise stated, a simplex s in $U(0)$ is understood to have 0 as one of its vertices.

Denote by $\delta(s)$ the *diameter* of a simplex s in R^n, and denote by $\theta(s)$ the *fullness* $\mathrm{vol}(s)/(\delta(s))^n$ of s.

Lemma 2. *For a positive* θ, *there is a positive* $\lambda(\theta, n)$ *depending only on* θ, n, *such that, if* f *satisfies the local Lipschitz condition with* $\lambda(\theta, n)$ *in* $U(0)$, *then for any* n*-simplex* s *in* $U(0)$ *of fullness* $\theta(s) \geqq \theta$, *it holds that*

$$\theta(f_s(s)) \geqq \theta/2 .$$

Proof. Choose a sufficiently large $A(n)$ such that, if $\max|x_{ij} - a_{ij}| \leqq \max|a_{ij}|$, then

$$|\det x_{ij} - \det a_{ij}| \leqq A(n) \max|x_{ij} - a_{ij}| \max|a_{ij}|^{n-1} .$$

Substitute x_{ij} by $\langle f(v_i), f(v_j)\rangle$ and a_{ij} by $\langle v_i, v_j\rangle$ where v_k $(k = 1, \cdots, n)$ are the vertices of s except the origin 0. Then it follows that

$$(n!)^2 |\mathrm{vol}^2 f_s(s) - \mathrm{vol}^2(s)| \leqq 4A(n)\lambda\delta^{2n}(s) .$$

Hence,

$$\theta^2(f_s(s))/(\sqrt{1-\lambda})^{2n} \geqq \theta^2(s) - 4\lambda A(n)/n!^2$$

and the conclusion follows easily.

Lemma 3. *Let* s, t *be* n*-simplexes in* $U(0)$ *such that*

$$\beta\delta \leq \delta(s),\ \delta(t) \leq \delta, \quad \theta \leq \theta(s),\ \theta(t)$$

then, if f *satisfies the local Lipschitz condition with* λ *in* $U(0)$,

$$|f_s(x) - f_t(x)|^2 \leq 8|x|^2 \lambda n^2 (1 + 1/\beta)/n!^4 \theta^4 .$$

for any $x \in U(0)$

Proof. Making use of the fullness $\theta(s)$, one gets [Wy, p. 126] that

(3.1) $\quad n!\theta(s)\delta(s) \leq |v_i| \leq \delta(s)$, *for all vertex* v_i *of* s,

and that,

(3.2) $\quad |x_i| \leq |\sum x_i v_i| / n!\theta(s)|v_i|$ *for any linear combination* $\sum x_i v_i$.

Therefore, one easily sees that

$$|f_s(x)-f_t(x)|^2 \leq 2\lambda n^2 |x|^2 \{2/(n!\theta(s))^4+2/(n!\theta(t))^4 \\ +2(\delta^2(s)+\delta^2(t))/(n!\theta(s))^2\delta(s)(n!\theta(t))^2\delta(t)\} .$$

Hence the evaluation in Lemma follows easily.

Proposition 3. *In case of $n=N$, there exists a positive $\mu(n, \beta, \theta)$ depending only upon n, β, θ, such that, if f of R^n into R^n satisfies the local Lipschitz condition with $\mu(n, \beta, \theta)$ in $U(0)$ and if $s=(0, v_1 \cdots v_{n-1}, p)$, $t=(0, v_1 \cdots v_{n-1}, q)$ are properly joined n-simplexes at the $(n-1)$ face $F=(0, v_1 \cdots v_{n-1})$ which satisfy*

$$s, t \subset U(0) \qquad \beta\delta \leqq \delta(s),\ \delta(t) \leqq \delta,\ \theta \leqq \theta(s),\ \theta(t) ,$$

then the simplexes $f_s(s)$ and $f_t(t)$ are non degenerate and properly joined, in the other words, non degenerate simplexes $f_s(s)$ and $f_t(t)$ are separated each other by the $(n-1)$-simplex $f_s(F)=f_t(F)$.

Proof. Assume that f satisfies the Lipschitz condition with the constant $\lambda_0=\lambda(n, \theta)$ in Lemma 1, then (see (3.1))

$$\text{any height of } f_s(s) \geqq n!\theta(f_s(s))\delta(f_s(s)) \\ \geqq n!\sqrt{1-\delta_0}\beta\delta\theta/2 ,$$

in particular

$$\text{dist}\,(f_s(p), \text{ plane of } f_s(F)) \geqq n!\sqrt{1-\lambda_0}\beta\delta\theta/2 ,$$

therefore if λ is closen so small that

$$4\sqrt{\lambda(1+1/\beta)}n/n!^2\theta^2 < n!\sqrt{1-\lambda_0}\beta\theta/2$$

then by Lemma 3, both $f_s(p)$ and $f_t(p)$ lie in the same side of the plane of $f_s(F)=f_t(F)$.

Lemma 4. *If f of R^n into R^N satisfies the Lipschitz condition with λ in $U(0)$, then for an n-simplex s in $U(0)$ and for $x \in U(0)$ such that*

$$\alpha\delta(s) \leqq |x| \leqq \delta(s) ,$$

it holds that

$$|f(x)-f_s(x)|^2 \leqq 2\lambda |x|^2 \{1+2n^2/(n!\theta(s))^4+2n/(n!\theta(s))^2+2n/\alpha n!\theta(s)\}$$

Proof. A calculation using (3, 1), (3, 2) shows that

$$|f(x)-f_s(x)|^2 \leqq 2\lambda |x|^2(1+2n^2/(n!\theta(s))^4) \\ +4\lambda |x| (\textstyle\sum(|x|^2+|v_i|^2)/n!\theta(s)|v_i|) .$$

Since $\alpha\delta(s) \leqq |x| \leqq \delta(s)$,

$$\begin{aligned}\sum(|x|^2+|v_i|^2/|v_i|) &= |x|\sum(|x|/|v_i|+|v_i|/|x|)\\ &\leqq |x|(n/n!\theta(s)+n/\alpha).\end{aligned}$$

Then one gets the evaluation.

Consider now the case $f(0)$ is not necessarily 0, and let $f(p)=q$, then a parallel translation $g(x)=f(x+p)-q$ maps 0 into 0 and the local Lipschitz condition for f at p yields that for g at 0. Therefore defining s-approximation f_s of f by the parallel translation of g_s, one gets properties of f_s similar to those in Lemmas 1–4 and in Proporition 3.

In particular, since, for any $x\in s$, there found a vertex v of s such that

$$|x-v| \geqq n!\theta(s)\delta(s)/2$$

(see (3, 1)). Lemma 4 applied to a parallel translation of f implies

Corollary 1. *Under the same condition as in Lemma* 4,

$$|f(x)-f_s(x)|^2 \leqq 2\lambda\delta^2(s)\{1+2n^2/(n!\theta(s))^4+6n/(n!\theta(s))^2\}$$

for all $x\in s$.

4. Simplexwise positive maps.

Let K be a pseudo n-manifold which may have boundary ∂K.

A map f of K into R^n is said to be *simplexwise positive* if for each n-simplex $s\in K$, f is smooth and one to one in s and Jacobian Jf of f in s is positive there.

Lemma 5. *If f is simplexwise positive in K, then for any interior point p of K there exists a neighbourhood $U(p)$ of p in K such that f restricted on $U(p)$ is one to one.*

Proof. Let p be an interior point of a simplex σ in K (dimension unspecified), then, since f is one to one in all simplexes in K, $f(p)$ is covered only once by $f(\overline{\mathrm{St}}(\sigma))$. Take a sufficiently fine subdivision K' of K for which the closed star $\overline{\mathrm{St}}'(\sigma')$ in K' of a simplex σ' having p in its intorior, is contained in $\mathrm{St}(\sigma)$. An application of LEMMA 15a of [Wy p. 369] to a simplexwise positive map f in K (therefore, so is in K' and in $\overline{\mathrm{St}}'(\sigma')$) and a combinatorial n-manifold $\overline{\mathrm{St}}'(\sigma')$, shows that f is one to one, when considered only in an inverse image $f^{-1}(R)$ of an open set R in $R^n-f\ (\partial(\overline{\mathrm{St}}(\sigma')))$ containing $f(p)$.

Let f, g be simplexwise differentiable maps of K into R^n. A homotopy h_t between f and g is called a *non degenerate homotopy*, if, for each

t, h_t is simplexwise differentiable and the non degenerate Jacobian Jh_t of h_t gives a homotopy between Jf and Jg. Obviously one gets

Lemma 6. *If a simplexwise one to one map f of K into R^n is non degenerately homotopic to a simplexwise positive map g of K into R^n then f itself is simplexwise positive.*

When one specialize the manifold K to that imbedded in R^n and satisfying

$$\theta \leq \theta(s) \quad \text{and} \quad \beta\delta \leq \delta(s) \leq \delta .$$

for each n-simplex s of K, one can apply Proposition 3 to get

Proposition 4. *Let K be a pseudo manifold as above and let $\mu(n, \beta, \theta)$ be the constant in Proposition 3. Then if a map g of K into R^n satisfies the Lipschitz condition with $\mu(n, \beta, \theta)$, the simplicial approximation g_K of g on K is simplexwise positive and, therefore, is locally homeomorphic at any interior point p of K.*

Corollary 2. *Using the same notations as in Proposition 4, if a simplexwise one to one map f of K into R^n is non degenerately homotopic to g_K, then f also is simplexwise positive, and therefore, is locally homeomorphic at any interior point p of K.*

5. A proof of the combinatorial equivalence.

First we fix notations; M is an orientable compact connected Rimannian n-manifold, isometrically imbedded in an Euclidean N-space R^N, and $V(M)$ is the tubular neighbourhood of M with the projection π along the normal plane field η.

The tangent n-plane at $p \in M$ is denoted by $T_p(M)$ or simply by T_p, then the local projection π_p along η of a part of T_p into M is defined in some neighbourhood of 0 in T_p, and so is the local orthogonal projection Π_p along the fixed plane $\eta(p)$ of a part of T_p into M.

If π_p^*, Π_p^* are defined to be local projections of a part of $V(M)$ into T_p along η and along $\eta(p)$, respectively, then obviously $\pi = \pi_p \cdot \pi_p^*$.

The following lemmas are elementary and their proofs are found, for instance, essentially in [Wy p. 117–174].

Lemma 7. *Given $\varepsilon > 0$, for each $p \in M$, there exists a neighbourhood $U_1(p)$ of p in M such that if a simplex σ of fullness $> \varepsilon$ in R^N has its all vertices in $U_1(p)$, then both π_p^*, Π_p^* are non degenerate and one to one on σ and, moreover, π_p^*, Π_p^* are non degenerately homotopic on σ.*

Lemma 8. *For any positive κ and for each $p \in M$, there exists a* *neighbourhood* $U_2(p)$ *of* p *in* M *such that both* π_p^* *and* Π_p^* *restricted on* $U_2(p)$, *satisfy the Lipschitz condition with* κ *relative to the Riemannian metric on* M *and the usual norm on* T_p.

The existence of a certain triangulation of M, which is very useful for our purpose, is proved in [Wy. p. 124–135].

Triangulation Theorem. *There are defined positive functions* $\beta(n, N) < 1$ *and* $\theta(n, N)$ *of* n, N *such that for any positive* ε, *there are a positive* $\delta < \varepsilon$ *and an oriented finite combinatorial n-manifold* $K(\varepsilon) \subset V(M)$ *which satisfies the following properties* 1), 2).

1) *The restriction of* π *to* $K(\varepsilon)$ *is a homeomorphism onto* M *and for each n–simplex* s *in* $K(\varepsilon)$, π *is differentiable and non degenerate on* s.

2) *For each point* $p \in M$, *there is a combinatorial n-submanifold* $K(p)$ *of* $K(\varepsilon)$ *satisfying the following*:

(2.1) *The simplicial approximation* $(\pi_p^*)_K$ *of* π_p^* *on* $K(p)$ *is isomorphic and the image contains a neighbourhood of* 0 *of* T_p.

(2.2) *The isomorphic image* $L(p)$ *of* $K(p)$ *by* $(\pi_p^*)_K$ *satisfies that*

$$\operatorname{diam}(L(p)) < 10\delta$$

and for all n–simplex s *in* $L(p)$,

$$\theta(n, N) \leq \theta(s),\ \beta(n, N)\delta \leq \delta(s) \leq \delta\,.$$

Now let M' be a second compact connected Riemannian n-manifold, isometrically imbedded in R^N, and let $V'(M')$ $\pi'\cdots$etc. denote the corresponding notions defined for $M' \subset R^N$ such as tubular neighbourhood, projection along the normal plane field etc.

Assume that a map f of M onto M' satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))/2$, (See Prop 2). Then by Lemmas 2, 7, 8 and by Triangulation Theorem, using the compactness of M, M', one easily verifies:

Assertion. For some $\varepsilon > 0$, $K(\varepsilon)$ satisfies the following properties 1)–3):

1) For all $p \in M$, π_p is defined on $L(p)$ and both $\pi_p^{*\prime}$, $\Pi_p^{*\prime}$ are defined on a neighbourhood of $q = f(p)$ in R^N containing both $f\pi_p(L(p))$ and $(f\pi_p)_L(L(p))$, moreover, π_p' is one to one on $\pi_q^{*\prime}(f\pi_p)_L(L(p))$.

2) For each simplex σ in $L(p)$, both $\pi_q^{*\prime}$, $\Pi_q^{*\prime}$ are one to one on the simplex $(f\pi_p)_L(\sigma)$ in R^N and are non degenerately homotopic to each other on it.

3) The map $\Pi_q^{*\prime} f\pi_p$ satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))$.

Hence, from Proposition 4 and the property 3) above, $(\Pi_q^{*\prime} f\pi_p)_L$ is simplexwise positive and, since $\Pi_q^{*\prime}$ is linear, simplexwise one to one map $\pi_q^{*\prime}$ $(f\pi_p)_L$ is non degenerately homotopic to $\Pi_q^{*\prime}(f\pi_p)_L=(\Pi_q^{*\prime}f\pi_p)_L$, (see 2) above). Therefore $\pi_q^{*\prime}(f\pi_p)_L$ is locally homeomorphic at $0\in L(p)$ and so is $\pi_q^{*\prime}(f\pi_p)_L\cdot(\pi_p^*)_K=\pi_q^{*\prime}(f\pi)_K$ at p^* in $K(\varepsilon)$ which is mapped to p by π (see (2.1) of Triangulation Theorem).

Thus the composition $\pi_q'\circ\pi_q^{*\prime}(f\pi)_K=\pi'(f\pi)_K$ is locally homeomorphic at $p^*\in K(\varepsilon)$.

These facts can be unified as follows:

Proposition 4. *Let M, M' be oriented compact connected Riemannian n-manifolds, isometrically imbedded in R^N and let f be a map of M onto M' which satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))/2$ relative to the Riemannian metrics, then there exists a triangulation $K(\varepsilon)$ of M such that the simplexwise differentiable and simplexwise non degenerate map $F=\pi'(f\pi)_K$ is a homeomorphism of $K(\varepsilon)$ onto M', that is, there is a combinatorial equivalence between $K(\varepsilon)$ aud M'.*

Proof. Since F is locally homeomorphic, $F: K(\varepsilon)\to M'$ is a covering of M', (see, for instance, [Hu, p. 105]). And since F is homotopic to a homeomorphic map $f\pi$, every point in M' is covered only once by $F(K(\varepsilon))$, indicating that F itself is a homeomorphism.

According to J. Nash and N. Kuiper [N], every Riemannian manifold M is isometrically imbedded in R^N, and an upper bound of N can be given as a function $N(n)$ of the dimension n of M. Thus letting $\lambda(n)=\mu(n, \beta(n, N(n))\theta(n, N(n))/4$, for instance, one gets

Theorem 1. *Let M, M' be oriented compact connected Riemannian n-manifolds. Then if there exists a map f on M onto M' of which Lipschitz constant $\mathfrak{l}(f)$ is less than a certain positive $\lambda(n)$, which is a function only in n, the differentiable manifolds M, M' admit a common triangulation.*

In particular Theorem 1 implies (see Prop 2)

Theorem 2. *Let σ_1, σ_2 be differentiable structures on an orientable compact connected topological n-manifold. Then there exists a positive $\rho(n)$ depending only on n such that $\rho(\sigma_1, \sigma_2)<\rho(n)$ implies the combinatorial equivalence of σ_1 and σ_2.*

PART II

1. ϕ approximation of a Lipschitz map.

Let ϕ be a non negative smooth function on R satisfying (1.1)-(1.3):

(1.1) $\text{Car}\,\phi\subset[-1, 1], \quad \max\phi = 1.$

(1.2) $\text{Car}\,\phi'\subset[-1, -1/2]\cup[1/2, 1], \quad \max|\phi'|\leq 4.$

(1.3) $\max|\phi''|\leq 32.$

For a positive δ, define $\kappa(n)$ and $\phi_\delta(x)$ by

$$\kappa(n) = \int\phi(|x|)\,dv,$$

$$\phi_\delta(x) = \phi(|x|/\delta)/\kappa(n)\delta^n, \tag{1.4}$$

where $\int dv$ denotes the integration over R^n by the standard volume element dv. Then obviously

$$\int\phi_\delta(x)dv = 1, \tag{1.5}$$

$$\text{Car}\,\phi_\delta\subset U_\delta(0), \quad \max\phi_\delta = 1/\kappa(n)\delta^n. \tag{1.1$'$}$$

Let $\partial_\xi f$ denote the differential in $\xi\in T_x(R^n)$:

$$\partial_\xi f = \lim_{t\to 0} f(x+\xi t)-f(x)/t.$$

Then an easy calculation shows that, for $\xi, \eta\in T_0(R^n)$,

$$\text{Car}\,\partial_\xi\phi_\delta\subset U_{\delta/2}(0)'\cap U_\delta(0), \tag{1.2$'$}$$

$$\max|\partial_\xi\phi_\delta(x)|\leqq 4|\xi|/\kappa(n)\delta^{n+1},$$

$$\max|\partial_\eta\partial_\xi\phi_\delta(x)|\leqq 64|\xi|\,|\eta|/\kappa(n)\delta^{n+2}. \tag{1.3$'$}$$

Define $\phi_\delta(x, p)$ (or simply $\phi(x, p)$) for $x, p\in R^n$ by

$$\phi_\delta(x, p) = \phi_\delta(x-p).$$

Denote simply by $\int f\,dv$ the componentwise integration over R^n of an R^N valued function f on R^n:

$$\int f\,dv = \left(\int f_1(x)dv, \cdots, \int f_N(x)dv\right).$$

And define the $\phi_\delta(x, p)$ approximation (or simply ϕ approximation) $\phi(f)$ of a map f of R^n into R^N by

$$(\phi(f))(p) = \int\phi_\delta(x, p)f(x)dv.$$

Then, if f satisfies the local Lipschitz condition with λ^2 on $U_\delta(0)$, that is, if

$$||f(x)-f(y)|^2-|x-y|^2|\leq\lambda^2|x-y|^2$$

for all $x, y\in U_\delta(0)$, one gets the following evaluations (1.6) (1.7):
(1.6) *For some positive* $\mu_0=\mu_0\ (n, N)$,

$$|\phi(f)(p)-f(p)|\leqq\mu_0(1+\lambda)\delta\,.$$

(1.7) *If* σ *is an n-simplex having* p *as one of its vertices and such that*

$$\rho(\sigma)\geqq\eta,\ \delta(\sigma)=\delta\,,$$

then there is a positive $\mu_1=\mu_1\ (n, N, \eta)$, *such that*

$$|\partial_\xi\phi(f)-f_\sigma(\xi)|\leqq\mu_1\lambda|\xi|$$

for $\xi\in T_0(R^n)$ *and for the* σ *approximation* f_σ *of* f.

Proof. (1.6) is easily deduced from (1.1)′, and μ_0 is given by

$$\mu_0=\sqrt{N}\gamma(n)/\kappa(n)\,,$$

where $\gamma(n)$ is the ratio of the volume of the n-ball $U_\delta(0)$ to δ^n. (1.7) is obtained with the aid of Lemma 4 of part I which shows that, if $\delta(\sigma)/2\leq|x-p|\leq\delta(\sigma)$, then

$$|f(x)-f_\sigma(x)|^2\leqq2\lambda^2\beta(n, \eta)|x|^2$$

for some $\beta=\beta(n, \eta)$. And one deduces (1.7) from (1.2)′ as follows:

$$\begin{aligned}|\partial_\xi\phi(f)-f_\sigma(\xi)|&=|\int\partial_\xi\phi(x, p)\{f(x)-f_\sigma(x)\}dv|\\&\leqq\lambda\sqrt{2N\beta}\,4\gamma(n)|\xi|/\kappa(n)\\&\leqq\lambda\mu_1(n, N, \eta)|\xi|\,,\end{aligned}$$

where $\mu_1(n, N, \eta)=4\sqrt{2N\beta}\,\gamma(n)/\kappa(n)$.

2. α-neighbourhood of ϕ.

If no confusion occurs, the notations Car g, $\int g\,dv$, and $\partial_\xi g$ of a function g of two variables x, p denote the carrier, integration in x of $g(x)=g(x, p_0)$, and the differential in p of $g(p)=g(x_0, p)$, respectively.

A smooth function g on $R^n\times R^n$ is said to be in α-neighbourhood of $\phi_\delta(x, p)$, if

(2.1) $$|g(x, p)-\phi_\delta(x, p)|<1/\delta^n,\ \text{Car}\ g(x, p)\subset U_1(p)\,,$$

(2.2) $$|\partial_\xi g(x, p)-\partial_\xi\phi_\delta(x, p)|<\alpha|\xi|/\delta^{n+1}\,,$$

(2.3) $$\int g(x, p)dv=1\,,$$

where $U_1(p)$ denotes the open ball

$$U_1(p) = \{y \in R^n / |y-p| < 2\delta\}.$$

Let $g(f)$ be the g-average of a map f of R^n into R^n:

$$g(f)(p) = \int g(x, p) f(x) dv.$$

Then if f satisfies the λ^2-Lipschitz condition on $U_1(p)$ and if g is in α-neighbourhood of ϕ_δ, one easily gets the following:

$$(2.4) \qquad |g(f)(p) - \phi_\delta(f)(p)| < \mu_2(1+\lambda)\delta,$$

$$(2.5) \qquad |\partial_\xi g(f) - \partial_\xi \phi_\delta(f)| < \mu_2(1+\lambda)\alpha |\xi|,$$

where μ_2 is given by

$$\mu_2 = 2^{n+1}\sqrt{N}\gamma(n).$$

Combining (2,4) (2,5) with (1,6) (1,7), one gets the following

Proposition 1. *Let $\phi_\delta(x, p)$ be the function defined in* 1) *and assume that a map f satisfies the λ^2-Lipschitz condition on $U_{2\delta}(p)$, then there exist positive numbers μ_0, μ_1, μ_2 such that, for any function g in α-neighbourhood of ϕ_δ,*

$$(2.6) \qquad |g(f)(p) - f(p)| < (\mu_0 + \mu_2)(1+\lambda)\delta,$$

$$(2.7) \qquad |\partial_\xi g(f) - f_\sigma(\xi)| < (\mu_1\lambda + \mu_2(1+\lambda)\alpha\} |\xi|.$$

Corollary 1. *There exists $\lambda_0 = \lambda_0 (n, N, \eta) > 0$, such that, if f satisfies the Lipschitz condition with λ_0^2 and if g is in α-neighbourhood of $\phi_\delta(x, p)$ for some small α, then the g-average $g(f)$ is non degenerate at p.*

3. Proof of main theorem.

Let M be a compact manifold with Riemannian metric ρ. Denote by dV the volume element on M and define a function $G_\delta(X, P)$ on $M \times M$ by

$$G_\delta(X, P) = \phi(\rho(X, P)/\delta) / \int_M \phi(\rho(X, P)/\delta) dV,$$

where ϕ is the function defined in 1).

The $G_\delta(X, P)$-average $G_\delta(F)$ of a map F of M into R^N is defined to be

$$G_\delta(F)(P) = \int_M G_\delta(X, P) F(X) dV.$$

Obviously $G_\delta(F)$ is a smooth map of M into R^N. If no confusion occurs, denote simply by corresponding small letters $x, p, \cdots$ the points in $T_Q(M)$ which are mapped by the exponential map E_Q to $X, P, \cdots$ in M, and let $f_Q(x)$, $\rho_Q(x, p) \cdots$ denote the composition maps $F(E_Q(x))$, $\rho(E_Q(p))$. Then if δ is sufficiently small and if Q is near to P, one gets

$$G_\delta(X, P) = \phi(\rho_Q(x, p)/\delta) \Big/ \int \phi(\rho_Q(x, p)/\delta) e_Q(x) dv , \tag{3.1}$$

$$G_\delta(F)(P) = \int g_\delta(x, p) f_Q(x) e_Q(x) dv , \tag{3.2}$$

where dv is the volume element in $T_p(M)$ and $e_p(x)$ is given by

$$e_Q(x) = \det dE_Q(x) .$$

Assertion. For any $\alpha > 0$, there exists $\delta > 0$, such that the function $g(x, p)$ of two variables defined by

$$g(x, p) = g_\delta(x, p) e_Q(x)$$

is in α-neighbourhood of $\phi_\delta(x, p)$, provided p is sufficiently near to 0.

Proof. (2, 3) for g is obvious. Also the following (3, 3) (3, 4) are well known

$$dE_Q(0) = id , \tag{3.3}$$

$$\textit{if } x=0 \textit{ or } p=0 \textit{ or } x=p, \textit{ then } \rho_Q(x, p) = |x-p| . \tag{3.4}$$

Since $\rho_Q(x, p)$, $|x-p|$ both are smooth, (3, 4) implies the following:

Given $\varepsilon > 0$, there is $\gamma > 0$ such that if $|x| < \gamma$ then

$$|\rho_Q(x, p) - |x-p|| < \varepsilon |x-p| ,$$
$$|\partial_\xi \rho_Q(x, p) - \partial_\xi(|x-p|)| < \varepsilon |\xi| .$$

Therefore one gets the following evaluations:

$$|\phi(\rho_Q(x, p)/\delta) - \phi(|x-p|/\delta)| < 8\varepsilon ,$$
$$|\partial_\xi \phi(\rho_Q(x, p)/\delta) - \partial_\xi \phi(|x-p|/\delta) < 32\varepsilon |\xi| / \delta .$$

Taking (3, 3) into consideration, if δ and $|p|$ are sufficiently small, one also gets

$$\left| \int \phi(\rho_Q(x, p)/\delta) e_Q(x) dv - \int \phi(|x-p|/\delta) dv \right| < 8\mu_2 \varepsilon' \delta^n ,$$

$$\left| \partial_\xi \int \phi(\rho_Q(x, p)/\delta) d_Q(x) dv - \partial_\xi \int \phi(|x-p|/\delta) dv \right| < 32\mu_2 \varepsilon' \delta^{n-1} |\xi| .$$

From these inequalities, (2, 1) (2, 2) can be deduced easily, and this finishes the proof.

(3. 3) also yields that if $F(X)$ satisfies the Lipschitz condition with $\lambda^2/4$ at $Q\in M$ then $f_Q(x)=F(E_Q(x))$ satisfies that with λ^2 at $0\in T_Q(M)$. Thus from Proposition 1, Corollary 1 and Assertion, one easily gets

Proposition 2. *If $F(X)$ satisfies the Lipschitz condition with $\lambda_0^2/4$ at $Q\in M$, there are δ_0 and a neighbourhood $V(Q)$, in M of Q such that, for any $0<\delta\leqq\delta_0$ and for any $P\in V(Q)$, $G_\delta(X, P)$ is non degenerate and*

$$|G_\delta(F)(P)-F(P)|<(\mu_0+\mu_2)(1+\lambda_0)\delta\,. \tag{3. 4}$$

Now let F be a map of M onto a Riemannian manifold M' isometrically imbedded in R^N and assume that F satisfies the Lipschitz condition with $\lambda_0^2/4$ for each point $Q\in M$. Making δ small, $G_\delta(F)(M)$ is in the tubular neighbourhood of M' and the composition $\pi' G_\delta(F)$ is defined, where π' is the projection of the tubular neighbourhood onto M'.

Proposition 3. *If $F: M\to M'$ satisfies the Lipschitz condition with $\lambda_1=\lambda_1(n)$ for each $Q\in M$, then $\pi' G_\delta(X, P)$ is non degenerate at any point in M.*

Proof. Make $\lambda_1'\leqq\lambda_0^2/4$ so small that f of the λ_1-Lipschitz condition satisfies

$$\theta(f_\sigma(\sigma))\geqq\varepsilon/2\,,$$

for $\sigma\subset T_Q(M)$ of fullness $\geqq\varepsilon$ and of diameter δ and for the σ-approximation f_σ of f_Q. Then by LEMMA IIa of [Wy p. 123], one concludes that, for some δ, the plane $\Pi(f_\sigma(\sigma))$ is near to $T_{Q'}(M')$ and any vector $\xi\in R^N$ satisfying

$$|\xi-\eta|<\kappa|\eta| \qquad \text{for some} \quad \eta\in\Pi(f_\sigma)\,, \quad \kappa>0\,,$$

is not in $d\pi'$-kernel near Q', therefore, by (2, 7), for a small $\lambda_1\leqq\lambda_1'$ and δ,

$$d\pi' dG_\delta(F)(\xi)\neq 0, \qquad \text{for any} \quad \xi\in T_Q(M)\,.$$

Thus the compactness arguement finishes the proof.

Using the same notations as in Proposition 3, define $H_t(P)$ by

$$H_t(P) = \begin{cases} F(P) & t=0\,, \\ \pi' G_{t\delta}(F)(P) & 0<t\leqq 1\,, \end{cases}$$

then $H_t(P)$ gives a homotopy between $F(P)$ and $\pi' G_\delta(F)(P)$, because the continuity of $H_t(P)$ at $t=0$ is given by (3, 4). Since F is homeomorphic, the fiber of the covering $(M, M', \pi' G_\delta(F))$ (see [Hu p. 105]) consists of a single point, that is, $\pi' G_\delta(F)$ itself is homeomorphic.

Theorem 1. *Let M, M' be compact connected Riemannian n-manifolds. Then, if there exists a map f of M onto M' whose Lipschitz constant $l(f)$ is less than a certain positive, which is a function only in n, the differentiable manifolds are diffeomorphic.*

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