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# MULTISAMPLE AND MULTIVARIATE NONPARAMETRIC TESTS BASED ON *U* STATISTICS AND THEIR ASYMPTOTIC EFFICIENCIES

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Summary: In this paper some new nonparametric several-sample tests for location or scale are proposed, and their asymptotic efficiencies for some parametric alternatives are calculated. A multivariate nonparametric two-sample test for location which may be regarded as a multivariate generalization of the Wilcoxon test, is also proposed with its asymptotic efficiency for normal alternative. Finally we shall generalize the several-sample tests for location to the multivariate case and multivariate Kruskal and Wallis' test is also proposed.

## 1. Introduction

We shall first propose a family of nonparametric several-sample tests for location constructed from generalized $U$ statistics which includes as special cases, Bhapkar's $V$-test [4], Deshpande's $L$-test [10] and a test asymptotically as efficient as Kruskal and Wallis' $H$-test [18]. As a consequence, if the number of populations is greater than three, we can select a test whose asymptotic efficiency is higher than Kruskal and Wallis' $H$-test [18] against the normal, exponential and uniform distribution.

Secondly, for the nonparametric several-sample problem of scale we shall propose a family of tests constructed from generalized $U$ statistics which includes Deshpande's $D$-test [11] as a special case, and further we shall generalize Tamura's $Q$-test [29] for the two-sample problem of scale. We can find a test whose asymptotic relative efficiency with respect to the asymptotically UMP invariant test (Lehmann [20, p. 275]) for normal distribution is $15/2\pi^2$ independently of the number of populations. This value is equal to the asymptotic efficiency of Mood's two-sample test for dispersion [24] and Tamura's $Q$-test [29] against the $F$-test for normal alternative. Again we can also find a test whose asymptotic efficiency for normal alternative is higher than $15/2\pi^2$, if the number of populations is greater than four.

Thirdly we shall propose a multivariate Wilcoxon test which is a natural extension of the ordinary Wilcoxon test [32], [23]. Its asymptotic efficiency against Hotelling's $T^2$-test depends both on the direction of the sequence of alternatives and on the population covariances, and is shown to be equal to that of Bickel's $\hat{W}_n$-test [7] against Hotelling's $T^2$-test for the one-sample problem. Hence the asymptotic efficiency of our bivariate Wilcoxon test against Hotelling's $T^2$ for normal distribution is always higher than $\sqrt{3}/2$ $(=.866)$. The multivariate Wilcoxon test may be regarded as a two-sample analogue of Bickel's $\hat{M}_n$-test [7].

Finally we shall generalize the several-sample test for location to the multivariate case. Multivariate Kruskal and Wallis' test is also considered and their limiting distributions under the hypothesis and sequence of alternatives near hypothesis are obtained.

## 2. Preliminaries

The following Theorem 2.1 concerning a multivariate generalized $U$ statistics introduced by Hoeffding [13] and later improved by Lehmann [19], may be well known as Bhapkar [4] and Lehmann [21] stated. But as far as the author is aware, its proof in general case seems not to be given in the literature. So we shall state the theorem with a sketch of its proof, which is a modification of that of Fraser [12, p. 225].

Let $p\times 1$ vectors $x_{j\alpha}$, $\alpha=1,2,\cdots,n_j$ be a random sample from the $p$-variate distribution $F_j(x)$ $(j=1,2,\cdots,c)$ and $\phi^{(i)}(x_{11},\cdots,x_{1m_1^{(i)}};\cdots;x_{c1},\cdots,x_{cm_c^{(i)}})$ $(i=1,2,\cdots,r)$ be a real-valued function which is symmetric in each set of variables $x_{k1},\cdots,x_{km_k^{(i)}}$ $(k=1,2,\cdots,c)$. Put

$$(2.1)\quad U^{(i)}=\left[\prod_{\alpha=1}^{c}\binom{n_\alpha}{m_\alpha^{(i)}}\right]^{-1}\sum_{\alpha}\cdots\sum_{\delta}\phi^{(i)}(x_{1\alpha_1},\cdots,x_{1\alpha_{m_1^{(i)}}};\cdots;x_{c\delta_1},\cdots,x_{c\delta_{m_c^{(i)}}})$$

where the summation $\sum\limits_{\alpha}\cdots\sum\limits_{\delta}$ extends over all possible sets of subscripts, $(\alpha_1,\cdots,\alpha_{m_1^{(i)}}),\cdots,(\delta_1,\cdots,\delta_{m_c^{(i)}})$ such that $1\leqq\alpha_1<\cdots<\alpha_{m_1^{(i)}}\leqq n_1,\cdots,$ $1\leqq\delta_1<\cdots<\delta_{m_c^{(i)}}\leqq n_c$. Then $U^{(i)}$ is called a multivariate generalized $U$ statistic.

**Theorem 2.1.** *Suppose that there are r multivariate generalized U satistics $U^{(i)}$ defined by* (2.1).

(i) *If $E[\phi^{(i)}]=\eta^{(i)}$, then $E[U^{(i)}]=\eta^{(i)}$.*

(ii) *If $E[\{\phi^{(i)}\}^2]<\infty$, then for every i and j,*

$$(2.2)\quad \mathrm{Cov}\,[U^{(i)},U^{(j)}]=\left[\prod_{\alpha=1}^{c}\binom{n_\alpha}{m_\alpha^{(j)}}\right]^{-1}\sum_{d_1=0}^{m_1^{(i,j)}}\cdots\sum_{d_c=0}^{m_c^{(i,j)}}\left[\prod_{\alpha=1}^{c}\binom{m_\alpha^{(i)}}{d_\alpha}\binom{n_\alpha-m_\alpha^{(i)}}{m_\alpha^{(j)}-d_\alpha}\right]\zeta_{d_1,\cdots,d_c}^{(i,j)}$$

*where* $m_\alpha^{(i,j)}=\min(m_\alpha^{(i)}, m_\alpha^{(j)})$ *and* $\zeta_{d_1\cdots d_c}^{(i,j)}$ *means the covariance of* $\phi^{(i)}(x_{11}, \cdots, x_{1m_1^{(i)}}; \cdots; x_{c1}, \cdots, x_{cm_c^{(i)}})$ *and* $\phi^{(j)}(x_{11}, \cdots, x_{1d_1}, x'_{1d_1+1}, \cdots, x'_{1m_1^{(j)}}; \cdots; x_{c1}, \cdots, x_{cd_c}, x'_{cd_c+1}, \cdots, x'_{cm_c^{(j)}})$ *with* $x_{ij}$ *and* $x'_{ij}$ *being assumed to be independently distributed as* $F_i(x)$ *for varying* $j$.

(iii)* *Suppose that* $E[\phi^{(i)}]=\eta^{(i)}$ *and* $E[\{\phi^{(i)}\}^2]<\infty$. *Further let* $n_i=\rho_i N$ *with* $\rho_i$ *being positive constant independent of* $N$, *then as* $N\to\infty$ *the* $r$-*dimensional statistic*

$$\sqrt{N}(U^{(1)}-\eta^{(1)}, \cdots, U^{(r)}-\eta^{(r)}) \tag{2.3}$$

*is distributed asymptotically according to* $r$-*variate normal distribution, whose mean vector is* $\mathbf{0}$ *and the covariance matrix* $\boldsymbol{\Sigma}=(\sigma_{ij})$ *is given by*

$$\sigma_{ij} = \frac{m_1^{(i)}m_1^{(j)}}{\rho_1}\zeta_{10,\cdots,0}^{(i,j)} + \cdots + \frac{m_c^{(i)}m_c^{(j)}}{\rho_c}\zeta_{00,\cdots,1}^{(i,j)} . \tag{2.4}$$

Proof. (i) is obvious. From (2.1) the covariance of $U^{(i)}$ and $U^{(j)}$ is given by

$$\left[\prod_{\alpha=1}^{c}\binom{n_\alpha}{m_\alpha^{(i)}}\binom{n_\alpha}{m_\alpha^{(j)}}\right]^{-1}\sum_{\alpha}\cdots\sum_{\delta}\sum_{\alpha'}\cdots\sum_{\delta'}\operatorname{Cov}\Big[\phi^{(i)}(x_{1\alpha_1}, \cdots, x_{1\alpha_{m_1^{(i)}}}; \cdots; x_{c\delta_1}, \cdots, x_{c\delta_{m_c^{(i)}}}), \phi^{(j)}(x_{1\alpha'_1}, \cdots, x_{1\alpha'_{m_1^{(j)}}}; \cdots; x_{c\delta'_1}, \cdots, x_{c\delta'_{m_c^{(j)}}})\Big] . \tag{2.5}$$

In the multiple summation of the above expression the number of items such as just $d_1$ $x_{1k}$'s are present in common among $x_{1\alpha_1}, \cdots, x_{1\alpha_{m_1^{(i)}}}$ and $x_{1\alpha'_1}, \cdots, x_{1\alpha'_{m_1^{(j)}}}, \cdots$, and just $d_c$ $x_{ck}$'s are present in common among $x_{c\delta_1}, \cdots, x_{c\delta_{m_c^{(i)}}}$ and $x_{c\delta'_1}, \cdots, x_{c\delta'_{m_c^{(j)}}}$ is

$$\prod_{\alpha=1}^{c}\binom{n_\alpha}{m_\alpha^{(i)}}\binom{m_\alpha^{(i)}}{d_\alpha}\binom{n_\alpha-m_\alpha^{(i)}}{m_\alpha^{(j)}-d_\alpha} .$$

For this combination of $x_{ik}$'s and $x_{ik'}$'s the covariance of $\phi^{(i)}$ and $\phi^{(j)}$ is equal to $\zeta_{d_1\cdots d_c}^{(i,j)}$. Hence we get (ii). To prove (iii) we shall first note the following theorem which is a straightforward generalization of a theorem due to Cramér [9, p. 254].

**Theorem 2.2.** *If* $(x_1, y_1), (x_2, y_2), \cdots$ *is a sequence of independent random vectors such that* $x_1, x_2, \cdots$ *has a limiting distribution* $F(x)$ *and* $\lim_{N\to\infty} E[||y_N-c||^2]=0$ *holds for some constant vector* $c$, *where* $||c||$ *means the euclidean length of the real vector* $c$. *Then* $x_N+y_N$ *has the limiting distribution* $F(x-c)$.

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* The same result as (iii) holds when the distribution function $F_i(x)$ depends on $N$ and the righthand side of (2.4) converges to $\sigma_{ij}$ as $N\to\infty$. The proof of this fact is also the same as that of (iii).

Now we shall prove (iii) of Theorem 2.1. Put

$$(2.6)\qquad Y_N^{(j)} = \frac{1}{\sqrt{N}}\left[\frac{m_1^{(j)}}{\rho_1}\sum_{\alpha=1}^{n_1}(\psi_1^{(j)}(x_{1\alpha})-\eta^{(1)})+\cdots+\frac{m_c^{(j)}}{\rho_c}\sum_{\delta=1}^{n_c}(\psi_c^{(j)}(x_{c\delta})-\eta^{(c)})\right]$$

where

$$\psi_i^{(j)}(x_{i\alpha}) = E[\phi^{(j)}(x_{11}, \cdots, x_{1m_1^{(j)}}; \cdots; x_{i\alpha}, x_{i2}, \cdots, x_{im_i^{(j)}}; \cdots; x_{c1}, \cdots, x_{cm_c^{(j)}})]$$

and the expectation in the righthand side means the conditional expectation for given $x_{i\alpha}=x_{i\alpha}$. Then by the central limit theorem the random vector $y_N=(Y_N^{(1)}, \cdots, Y_N^{(r)})$ is distributed asymptotically normal as $N\to\infty$, with the mean $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$;

$$(2.7)\qquad \sigma_{ij} = \frac{m_1^{(i)}m_1^{(j)}}{\rho_1}\zeta_{10,\cdots,0}^{(i,j)}+\cdots+\frac{m_c^{(i)}m_c^{(j)}}{\rho_c}\zeta_{00,\cdots,1}^{(i,j)}\,.$$

Now we shall show that the random vector $z_N=(Z_N^{(1)}, \cdots, Z_N^{(r)})$ where $Z_N^{(i)}=\sqrt{N}(U^{(i)}-\eta^{(i)})$, is asymptotically equivalent to the random vector $y_N$, that is, $\lim_{N\to\infty} E[\|y_N-z_N\|^2]=0$. From (2.7) we have

$$(2.8)\qquad E[\{Y_N^{(j)}\}^2] = \frac{\{m_1^{(j)}\}^2}{\rho_1}\zeta_{10,\cdots,0}^{(j,j)}+\cdots+\frac{\{m_c^{(j)}\}^2}{\rho_c}\zeta_{00,\cdots,1}^{(j,j)}\,.$$

Also from (2.2) we have

$$(2.9)\qquad \mathrm{Var}[U^{(j)}] = \frac{\{m_1^{(j)}\}^2}{n_1}\zeta_{10,\cdots,0}^{(j,j)}+\cdots+\frac{\{m_c^{(j)}\}^2}{n_c}\zeta_{00,\cdots,1}^{(j,j)}+O\left(\frac{1}{N^2}\right).$$

Hence we can conclude

$$(2.10)\qquad E[\{Z_N^{(j)}\}^2] = \frac{\{m_1^{(j)}\}^2}{\rho_1}\zeta_{10,\cdots,0}^{(j,j)}+\cdots+\frac{\{m_c^{(j)}\}^2}{\rho_c}\zeta_{00,\cdots,1}^{(j,j)}+O\left(\frac{1}{N}\right).$$

From (2.1) and (2.6) we can see that $E(Y_N^{(j)}Z_N^{(j)})$ is equal to the righthand side of (2.8). Using the relation $E[\{Y_N^{(j)}-Z_N^{(j)}\}^2]=E[\{Y_N^{(j)}\}^2]+E[\{Z_N^{(j)}\}^2]-2E[Y_N^{(j)}Z_N^{(j)}]$, we can conclude that $\lim_{N\to\infty} E[\{Y_N^{(j)}-Z_N^{(j)}\}^2]=0$. Hence we have $\lim_{N\to\infty} E[\|y_N-z_N\|^2]=0$. From Theorem 2.2 the limiting distribution of $z_N$ is the same as that of $y_N=z_N+(y_N-z_N)$, so we get the desired result (iii).

## 3. Nonparametric several-sample tests for location

Let $X_{ij}$ $(j=1, 2, \cdots, n_i)$ denote a random sample from the continuous univariate distribution $F_i(x)$ $(i=1, 2, \cdots, c)$. From these samples we want to test the hypothesis $F_1(x)=\cdots=F_c(x)$ against the location alternative $F_i(x)=F(x-\theta_i)$ for some real number $\theta_i$ (not all $\theta_i$'s are equal), where the functional form of $F(x)$ is unknown. Nonparametric tests for this problem have been proposed by many authors such as Kruskal and Wallis [18], Kruskal [17], Bhapkar [4], Deshpande [10], Puri [25], Yen [33], etc. Now we shall propose a family of tests which includes as special cases both Bhapkar's $V$-test and Deshpande's $L$-test. Put for $i=1, 2, \cdots, c$

$$(3.1)\qquad \begin{aligned} U^{(i)} &= \frac{1}{n_1\cdots n_c}\sum_{\alpha_1=1}^{n_1}\cdots\sum_{\alpha_c=1}^{n_c}\phi^{(i)}(X_{1\alpha_1}, \cdots, X_{c\alpha_c}) \\ \phi^{(i)}(X_1, \cdots, X_c) &= \frac{(j-1)_r}{(c-1)_r}-\frac{(c-j)_s}{(c-1)_s} \quad \begin{array}{l}\text{if } X_i \text{ is the } j\text{-th} \\ \text{smallest among } X_1, \cdots, X_c\,,\end{array} \end{aligned}$$

where $(k)_r=k(k-1)\cdots(k-r+1)$, $(k)_0=1$ and $r, s=0, 1, 2, \cdots, c-1$, except for $(r, s)=(0, 0)$. Then $U^{(i)}$ is a generalized $U$ statistic stated in section 2. We shall construct nonparametric tests from these $U^{(i)}$.

In case $r=0$ and $s=c-1$,

$$(3.2)\qquad \phi^{(i)}(X_1, \cdots, X_c) = \begin{cases} 0 & \text{if } X_i<X_j \text{ for any } j\neq i \\ 1 & \text{otherwise} \end{cases}$$

which leads to Bhapkar's $V$-statistic. In case $r=s=c-1$,

$$(3.3)\qquad \phi^{(i)}(X_1, \cdots, X_c) = \begin{cases} 1 & \text{if } X_i>X_j \text{ for any } j\neq i \\ -1 & \text{if } X_i<X_j \text{ for any } j\neq i \\ 0 & \text{otherwise} \end{cases}$$

which leads to Deshpande's $L$-statistic. In these two cases special attention is paid to the largest or the smallest item in the $c$-plets: $(X_{1\alpha_1}, X_{2\alpha_2}, \cdots, X_{c\alpha_c})$, whereas our test gives some monotone weight to the $(r+1)$-th smallest $\cdots$ $c$-th smallest and to the $(s+1)$-th largest $\cdots$ $c$-th largest item for given $r$ and $s$.

Since under the null hypothesis $H$: $F_1(x)=\cdots=F_c(x)$, the events that the random variable $X_i$ is the $j$-th smallest among $X_1, \cdots, X_c$ for $j=1, 2, \cdots, c$ are equally probable, we get

$$(3.4)\qquad E[\phi^{(i)}(X_1, \cdots, X_c)] = \frac{1}{c}\sum_{j=1}^{c}\left[\frac{(j-1)_r}{(c-1)_r}-\frac{(j-1)_s}{(c-1)_s}\right].$$

Substituting $\sum_{j=1}^{c}(j-1)_r=(c)_{r+1}/(r+1)$ into (3.4), we have

$$(3.5)\qquad E[U^{(i)}] = \frac{1}{r+1}-\frac{1}{s+1}.$$

From Theorem 2.1 we can conclude that under the hypothesis $H$, the statistic

$$(3.6)\qquad \sqrt{N}(U^{(1)}-E[U^{(1)}], \cdots, U^{(c)}-E[U^{(c)}])$$

is distributed asymptotically normal as $N\to\infty$ with the mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$;

$$\sigma_{ij}=\frac{1}{\rho_1}\zeta^{(i,j)}_{10,\cdots,0}+\cdots+\frac{1}{\rho_c}\zeta^{(i,j)}_{00,\cdots,1}, \tag{3.7}$$

where $\zeta^{(i,j)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $k$-th place) is the covariance of $\phi^{(i)}(X_1,\cdots,X_c)$ and $\phi^{(j)}(X'_1,\cdots,X'_{k-1},X_k,X'_{k+1},\cdots,X'_c)$. Under the null hypothesis, we shall calculate $\zeta^{(i,j)}_{0,\cdots,1,\cdots,0}$ by considering the following three cases (i), (ii) and (iii).

(i) $\zeta^{(i,i)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $i$-th place)

is equal to

$$E[\phi^{(i)}(X_1,\cdots,X_c)\phi^{(i)}(X'_1,\cdots,X'_{i-1},X_i,X'_{i+1},\cdots,X'_c)]-\left(\frac{1}{r+1}-\frac{1}{s+1}\right)^2. \tag{3.8}$$

The first term can be written as

$$\int_{-\infty}^{\infty}E[\phi^{(i)}(X_1,\cdots,X_c)\,|\,X_i=x]\cdot E[\phi^{(i)}(X'_1,\cdots,X'_{i-1},X_i,X'_{i+1},\cdots,X'_c)\,|\,X_i=x]\,dF(x)$$

$$=\sum_{k=1}^{c}\sum_{l=1}^{c}\left[\frac{(k-1)_r}{(c-1)_r}-\frac{(c-k)_s}{(c-1)_s}\right]\cdot\left[\frac{(l-1)_r}{(c-1)_r}-\frac{(c-l)_s}{(c-1)_s}\right]$$
$$\times\int_{-\infty}^{\infty}\binom{c-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{c-k}\cdot\binom{c-1}{l-1}[F(x)]^{l-1}[1-F(x)]^{c-l}dF(x). \tag{3.9}$$

Differentiating $r$-times the identity $(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^jy^{n-j}$ with respect to $x$, we have

$$(n)_r(x+y)^{n-r}=\sum_{j=0}^{n}\binom{n}{j}(j)_rx^{j-r}y^{n-j}. \tag{3.10}$$

Hence we can simplify (3.9) to get

$$\int_{-\infty}^{\infty}\{[F(x)]^r-[1-F(x)]^s\}^2dF(x). \tag{3.11}$$

Combining this with (3.8), we can conclude

(3.12) $\zeta^{(i,i)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $i$-th place)

$$=\frac{r^2}{(2r+1)(r+1)^2}+\frac{s^2}{(2s+1)(s+1)^2}+\frac{2}{(r+1)(s+1)}-2B(r+1,s+1),$$

where $B(p,q)$ means the Beta function and $B(r+1,s+1)=(r!\,s!)/(r+s+1)!$.

(ii) $\zeta^{(i,j)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $\alpha$-th place, where $\alpha \neq i, j$)

is equal to

$$E[\phi^{(i)}(X_1, \cdots, X_c)\phi^{(j)}(X'_1, \cdots, X'_{\alpha-1}, X_\alpha, X'_{\alpha+1}, \cdots, X'_c)] - \left(\frac{1}{r+1} - \frac{1}{s+1}\right)^2. \tag{3.13}$$

The first term of (3.13) can be written as

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\phi^{(i)}(X_1, \cdots, X_c) \mid X_i = x \text{ and } X_\alpha = z] \\ &\quad \times E[\phi^{(j)}(X'_1, \cdots, X'_{\alpha-1}, X_\alpha, X'_{\alpha+1}, \cdots, X'_c) \mid X'_j = y \text{ and } \\ &\quad X_\alpha = z] dF(x)dF(y)dF(z) \\ &= \sum_{k=1}^{c}\sum_{l=1}^{c}\left[\frac{(k-1)_r}{(c-1)_r} - \frac{(c-k)_s}{(c-1)_s}\right]\left[\frac{(l-1)_r}{(c-1)_r} - \frac{(c-l)_s}{(c-1)_s}\right] \\ &\quad \times \left[\iiint_{\substack{z<x \\ z<y}} A_1(x, y) + 2\iiint_{x<z<y} A_2(x, y) + \iiint_{\substack{x<z \\ y<z}} A_3(x, y)\right] \\ &\quad \times dF(x)dF(y)dF(z), \end{aligned} \tag{3.14}$$

where

(3.15)

$$\begin{aligned} A_1(x, y) &= \binom{c-2}{k-2}[F(x)]^{k-2}[1-F(x)]^{c-k}\binom{c-2}{l-2}[F(y)]^{l-2}[1-F(y)]^{c-l} \\ A_2(x, y) &= \binom{c-2}{k-1}[F(x)]^{k-1}[1-F(x)]^{c-k-1}\binom{c-2}{l-2}[F(y)]^{l-2}[1-F(y)]^{c-l} \\ A_3(x, y) &= \binom{c-2}{k-1}[F(x)]^{k-1}[1-F(x)]^{c-k-1}\binom{c-2}{l-1}[F(y)]^{l-1}[1-F(y)]^{c-l-1}. \end{aligned}$$

Using the identity $(k-1)_r = (k-2)_r + r(k-2)_{r-1}$ together with (3.10), we can simplify the above summation of (3.14), and integrate with respect to $x$ and $y$ to get

$$\begin{aligned} &\int_0^1\left[\left(1-\frac{r}{c-1}\right)\frac{1-F^{r+1}}{r+1} + \frac{1-F^r}{c-1} - \left(1-\frac{s}{c-1}\right)\frac{(1-F)^{s+1}}{s+1}\right]^2 dF \\ &\quad + 2\int_0^1\left[\left(1-\frac{r}{c-1}\right)\frac{1-F^{r+1}}{r+1} + \frac{1-F^r}{c-1} - \left(1-\frac{s}{c-1}\right)\frac{(1-F)^{s+1}}{s+1}\right] \\ &\quad \times \left[\left(1-\frac{r}{c-1}\right)\frac{F^{r+1}}{r+1} - \left(1-\frac{s}{c-1}\right)\frac{1-(1-F)^{s+1}}{s+1} - \frac{1-(1-F)^s}{c-1}\right] dF \\ &\quad + \int_0^1\left[\left(1-\frac{r}{c-1}\right)\frac{F^{r+1}}{r+1} - \left(1-\frac{s}{c-1}\right)\frac{1-(1-F)^{s+1}}{s+1} - \frac{1-(1-F)^s}{c-1}\right]^2 dF, \end{aligned} \tag{3.16}$$

which is equal to

$$(3.17)\qquad \int_0^1\left[\frac{1}{r+1}\left(1-\frac{r}{c-1}\right)-\frac{1}{s+1}\left(1-\frac{s}{c-1}\right)-\frac{F^r-(1-F)^s}{c-1}\right]^2 dF\,.$$

Further calculating (3.17) and combining with (3.13), we can conclude that

$$(3.18)\qquad \begin{aligned}&\zeta^{(i,j)}_{0,\cdots,1,\cdots,0} \quad (1 \text{ lies at the } \alpha\text{-th place, where } \alpha\neq i, j)\\ &=\frac{1}{(c-1)^2}\left[\frac{r^2}{(2r+1)(r+1)^2}+\frac{s^2}{(2s+1)(s+1)^2}+\frac{2}{(r+1)(s+1)}\right.\\ &\qquad\left.-2B(r+1,\, s+1)\right].\end{aligned}$$

An analogous calculation leads us to

(iii)

$$(3.19)\qquad \begin{aligned}&\zeta^{(i,j)}_{0,\cdots,1,\cdots,0} \quad (1 \text{ lies at the } i\text{-th place, where } i\neq j)\\ &=-\frac{1}{c-1}\left[\frac{r^2}{(2r+1)(r+1)^2}+\frac{s^2}{(2s+1)(s+1)^2}+\frac{2}{(r+1)(s+1)}\right.\\ &\qquad\left.-2B(r+1,\, s+1)\right].\end{aligned}$$

Substituting (3.12), (3.18) and (3.19) into (3.7), we can get

$$(3.20)\qquad \sigma_{ij}=\frac{K(r,s)}{(c-1)^2}\left[\sum_{\alpha=1}^{c}\frac{1}{\rho_\alpha}-\frac{c}{\rho_i}-\frac{c}{\rho_j}+\frac{c^2\delta_{ij}}{\rho_i}\right],$$

where

$$(3.21)\qquad \begin{aligned}K(r,s)&=\frac{r^2}{(2r+1)(r+1)^2}+\frac{s^2}{(2s+1)(s+1)^2}+\frac{2}{(r+1)(s+1)}\\ &\quad -2B(r+1,\, s+1)\,.\end{aligned}$$

It is easily seen that the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$ is singular, since $\sum_{j=1}^{c}\sigma_{ij}=0$ for every $i$, and that the rank of $\boldsymbol{\Sigma}$ is $c-1$ as is easily checked by calculating the minor determinant. To construct a test statistic we shall state the following lemma which will also be used in later sections.

**Lemma 3.1.** *Let the distribution of c-dimensional column vector* $\boldsymbol{x}$ *is normal with the mean* $\mathbf{0}$ *and the covariance matrix* $\boldsymbol{\Sigma}=(\sigma_{ij})$ *where*

$$(3.22)\qquad \sigma_{ij}=K\left[\sum_{\alpha=1}^{c}\frac{1}{\rho_\alpha}-\frac{c}{\rho_i}-\frac{c}{\rho_j}+\frac{c^2\delta_{ij}}{\rho_i}\right].$$

*Then the statistic* $(1/c^2K)\sum_{i=1}^{c}\rho_i(X_i-\tilde{X})^2$ *is distributed as* $\chi^2$ *with* $c-1$ *degrees of freedom where* $\boldsymbol{x}=(X_1,\cdots,X_c)'$ *and* $\tilde{X}=\sum_{\alpha=1}^{c}\rho_\alpha X_\alpha/\sum_{\alpha=1}^{c}\rho_\alpha$.

Proof. By Lemma 2 in Sugiura [26] which is restated in a more general form in Lemma 4.2 in the next section, it is sufficient to solve the following set of equations with respect to $\Lambda=(x_{ij})$ and construct $\boldsymbol{x}'\Lambda\boldsymbol{x}$.

$$(3.23)\qquad \begin{cases} \sum_{j=1}^{c} \sigma_{ij} x_{jk} = \delta_{ik} - \dfrac{1}{c} \\ \sum_{i=1}^{c} x_{ij} = 0\,. \end{cases}$$

The solution of the above equations is given by

$$(3.24)\qquad x_{ik} = \frac{1}{c^2K}\left[\delta_{ik}\rho_i - \frac{\rho_i\rho_k}{\sum_{\alpha=1}^{c}\rho_\alpha}\right].$$

Hence we have $\boldsymbol{x}'\Lambda\boldsymbol{x}=(1/c^2K)\sum_{i=1}^{c}\rho_i(X_i-\tilde{X})^2$.

**Lemma 3.2.** (Mann and Wald) *If the random vector $\boldsymbol{x}_N$ converges in law to the random vector $\boldsymbol{x}$, then the random variable $g(\boldsymbol{x}_N)$ converges in law to the random variable $g(\boldsymbol{x})$ for any continuous function $g$.*

This Lemma is a special case of Theorem 5 in Mann and Wald [22]. Putting $\boldsymbol{x}_N=\sqrt{N}(U^{(1)}-E[U^{(1)}], \cdots, U^{(c)}-E[U^{(c)}])$ and $g(X_1, \cdots, X_n)=(1/c^2K)\sum_{i=1}^{c}\rho_i(X_i-\tilde{X})^2$ in Lemma 3.2 and noticing Lemma 3.1 we have the following theorem.

**Theorem 3.1.** *Put $n_i=\rho_i N$ $(i=1, 2, \cdots, c)$ and let $N$ tend to infinity with $\rho_i>0$ fixed. Then under the hypothesis $H: F_1=\cdots=F_c$ the statistic $V_{rs}$ defined by*

$$(3.25)\qquad \begin{aligned} V_{rs} &= \frac{(c-1)^2}{c^2K(r, s)}\sum_{i=1}^{c} n_i(U^{(i)}-\tilde{U})^2 \qquad (0\leqq r, s\leqq c-1 \text{ and } (r, s)\neq(0, 0)) \\ \tilde{U} &= \sum_{i=1}^{c} n_i U^{(i)}\Big/\sum_{i=1}^{c} n_i\,, \end{aligned}$$

*where $U^{(i)}$ is defined by* (3.1) *and $K(r, s)$ is given by* (3.21), *is distributed asymptotically as $\chi^2$ with $c-1$ degrees of freedom.*

By the definition of $U^{(i)}$ given by (3.1), we can easily see that the distribution of the statistic $V_{rs}$ is the same as that of $V_{sr}$ under the alternative that $F_i(x)$ is symmetric at the origin for every $i$. We can also see that the $V_{10}$, $V_{01}$, $V_{11}$ and $V_{22}$-statistic are equivalent. As special cases of Theorem 3.1 we get the following corollaries.

**Corollary 1.** *The statistic*

$$(3.26)\qquad V_{0,c-1} = (2c-1)\sum_{i=1}^{c} n_i(U^{(i)}-\tilde{U})^2$$

*is distributed asymptotically as $\chi^2$ with $c-1$ degrees of freedom as $N\to\infty$.*

This is Bhapkar's $V$-statistic proposed in [4].

**Corollary 2.** *The statistic*

$$V_{c-1,c-1}=\frac{(c-1)^2(2c-1)}{2c^2\left[1-\binom{2c-2}{c-1}^{-1}\right]}\sum_{i=1}^{c}n_i(U^{(i)}-\tilde{U})^2 \tag{3.27}$$

*is distributed asymptotically as* $\chi^2$ *with* $c-1$ *degrees of freedom as* $N\to\infty$.
This is Deshpande's $L$-statistic proposed in [10].

**Corollary 3.** *The statistic*

$$V_{11}=\frac{3(c-1)^2}{c^2}\sum_{i=1}^{c}n_i(U^{(i)}-\tilde{U})^2 \tag{3.28}$$

*is distributed asymptotically as* $\chi^2$ *with* $c-1$ *degrees of freedom as* $N\to\infty$.

## 4. Asymptotic efficiency of the $V_{rs}$-test for the location alternative

Now we shall consider the limiting distribution of the statistic $V_{rs}$ given by (3.25) under the following sequence of alternatives,

$$K_N: F_i(x)=F(x-N^{-1/2}\theta_i) \qquad (i=1,2,\cdots,c), \tag{4.1}$$

where $\theta_i$ is some constant (not all $\theta_i$'s are equal).

**Theorem 4.1.** *Put* $n_i=\rho_i N$ *where* $\rho_i>0$ *is independent of* $N$, *and suppose that the distribution function* $F(x)$ *has the derivative* $f(x)$ *except for a set of* $F$*-measure zero and further there exists a function* $g(x)$ *such that* $\int_{-\infty}^{\infty}g(x)dF(x)<\infty$ *and that*

$$\left|\frac{1}{h}[F(x+h)-F(x)]\right|\leqq g(x) \tag{4.2}$$

*holds for any* $x$ *and any sufficiently small* $h$. *Then under the sequence of alternatives* $K_N$ *the limiting distribution of* $V_{rs}$ *is noncentral* $\chi^2$ *with* $c-1$ *degrees of freedom and the noncentrality parameter*

$$\lambda_{rs}^2=\frac{1}{K(r,s)}\left(\int_{-\infty}^{\infty}\{r[F(x)]^{r-1}+s[1-F(x)]^{s-1}\}f(x)dF(x)\right)^2 \times\sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2, \tag{4.3}$$

*where* $K(r,s)$ *is defined by* (3.21) *and* $\tilde{\theta}=\sum_{\alpha=1}^{c}\rho_\alpha\theta_\alpha/\sum_{\alpha=1}^{c}\rho_\alpha$.
As a special case we get the following corollary.

**Corollary 1.** *Under the sequence of alternatives* $K_N$, *the limiting distributions of either* $V_{0,c-1}$ (*Bhapkar's V-statistic*) *or* $V_{c-1,c-1}$ (*Deshpande's*

*L-statistic) is noncentral $\chi^2$ with $c-1$ degrees of freedom with noncentrality parameters given respectively by*

$$\lambda^2_{0,c-1} = c^2(2c-1)\left(\int_{-\infty}^{\infty}[1-F(x)]^{c-2}f(x)dF(x)\right)^2 \sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2, \tag{4.4}$$

$$\lambda^2_{c-1,c-1}=\frac{(2c-1)(c-1)^2}{2\left[1-\binom{2c-2}{c-1}^{-1}\right]}\left(\int_{-\infty}^{\infty}\{[F(x)]^{c-2}+[1-F(x)]^{c-2}\}f(x)dF(x)\right)^2 \times\sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2. \tag{4.5}$$

This result was obtained by Bhapkar [4] and Deshpande [10], respectively. To prove Theorem 4.1 we shall first show the following lemma.

**Lemma 4.1.** *Let $F(x)$ be a distribution function. Suppose that the distribution function $G_i(x)$ has the derivative $g_i(x)$ except for a set of $F$-measure 0 and further there exists a function $g(x)$ such that $\int_{-\infty}^{\infty} g(x)dF(x)<\infty$ and that*

$$\left|\frac{1}{h_i}[G_i(x+h_i)-G_i(x)]\right| \leqq g(x) \qquad (i=1, 2, \cdots, n) \tag{4.6}$$

*hold for any $x$ and any sufficiently small $h_i$ $(=\alpha_i h$ and $h$ is small). Then*

$$\int_{-\infty}^{\infty}\prod_{i=1}^{n} G_i(x+h_i)dF(x) = \int_{-\infty}^{\infty}\prod_{i=1}^{n} G_i(x)dF(x)+\sum_{i=1}^{n} h_i\int_{-\infty}^{\infty} g_i(x)\prod_{j\neq i} G_j(x)dF(x)+o(h). \tag{4.7}$$

Proof. It is sufficient to show (4.7) for $n=2$. We can write

$$\frac{1}{h}\int_{-\infty}^{\infty}[G_1(x+h_1)G_2(x+h_2)-G_1(x)G_2(x)]dF(x)$$
$$=\int_{-\infty}^{\infty}G_1(x+h_1)\left[\frac{G_2(x+h_2)-G_2(x)}{h}\right]dF(x)+\int_{-\infty}^{\infty}G_2(x)\left[\frac{G_1(x+h_1)-G_1(x)}{h}\right]dF(x).$$

By the Lebesgue's bounded convergence theorem, the first integral is equal to $\alpha_2\int_{-\infty}^{\infty}G_1(x)g_2(x)dF(x)+o(1)$ and the second integral is equal to $\alpha_1\int_{-\infty}^{\infty}G_2(x)g_1(x)dF(x)+o(1)$. Hence we have (4.7).

Now we shall prove Theorem 4.1. Under the alternative $K_N : F_i(x) = F(x-N^{-1/2}\theta_i)$, the expectation of $U^{(i)}$ defined by (3.1) is given by

$$E[U^{(i)}|K_N] = \sum_{k=1}^{c}\left[\frac{(k-1)_r}{(c-1)_r}-\frac{(c-k)_s}{(c-1)_s}\right] \times\sum_{(l)}\int_{-\infty}^{\infty}F_{l_1}(x)\cdots F_{l_{k-1}}(x)[1-F_{l_{k+1}}(x)]\cdots[1-F_{l_c}(x)]dF_i(x), \tag{4.8}$$

where the summation $\sum_{(l)}$ extends over all combinations of possible integers $(l_1, \cdots, l_{k-1}, l_{k+1}, \cdots, l_c)$ such that $l_1 < \cdots < l_{k-1}$ and $l_{k+1} < \cdots < l_c$, and the set $\{l_1, \cdots, l_{k-1}, l_{k+1}, \cdots, l_c\}$ is equal to the set $\{1, 2, \cdots, i-1, i+1, \cdots, c\}$. Since the number of such combinations is $\binom{c-1}{k-1}$, by Lemma 4.1 we can express (4.8) as

$$(4.9)\quad \sum_{k=1}^{c}\left[\frac{(k-1)_r}{(c-1)_r}-\frac{(c-k)_s}{(c-1)_s}\right]\sum_{(l)}\int_{-\infty}^{\infty}\Big\{[F(x)]^{k-1}[1-F(x)]^{c-k}$$
$$+N^{-1/2}[(\theta_i-\theta_{l_1})+\cdots+(\theta_i-\theta_{l_{k-1}})][F(x)]^{k-2}[1-F(x)]^{c-k}f(x)$$
$$-N^{-1/2}[(\theta_i-\theta_{l_{k+1}})+\cdots+(\theta_i-\theta_{l_c})][F(x)]^{k-1}[1-F(x)]^{c-k-1}f(x)\Big\}$$
$$\times dF(x)+o(N^{-1/2}).$$

It is easy to see that $\sum_{(l)}[(\theta_i-\theta_{l_1})+\cdots+(\theta_i-\theta_{l_{k-1}})]=\binom{c-2}{k-2}\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)$ and $\sum_{(l)}[(\theta_i-\theta_{l_{k+1}})+\cdots+(\theta_i-\theta_{l_c})]=\binom{c-2}{c-k-1}\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)$. So we can simplify (4.9) to get

$$(4.10)\quad E[U^{(i)}\,|\,K_N]=\frac{1}{r+1}-\frac{1}{s+1}+\frac{\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)}{(c-1)\sqrt{N}}$$
$$\times\int_{-\infty}^{\infty}\{r[F(x)]^{r-1}+s[1-F(x)]^{s-1}\}f(x)dF(x)+o\Big(\frac{1}{\sqrt{N}}\Big).$$

It follows from Theorem 2.1 that under the alternative $K_N$, the statistic $\sqrt{N}(U^{(1)}-E[U^{(1)}], \cdots, U^{(c)}-E[U^{(c)}])$ defined by (3.6) is distributed asymptotically normal as $N\to\infty$, with the mean vector $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_c)$ and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$ where

$$(4.11)\quad \begin{aligned}\mu_i&=\frac{\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)}{(c-1)}\int_{-\infty}^{\infty}\{r[F(x)]^{r-1}+s[1-F(x)]^{s-1}\}f(x)dF(x)\\ \sigma_{ij}&=\frac{K(r,s)}{(c-1)^2}\left[\sum_{\alpha=1}^{c}\frac{1}{\rho_\alpha}-\frac{c}{\rho_i}-\frac{c}{\rho_j}+\frac{c^2\delta_{ij}}{\rho_i}\right].\end{aligned}$$

To obtain the limiting distribution of $V_{rs}$ under the alternative $K_N$, we must generalize Lemma 2 stated in Sugiura [26] to the noncentral case.

**Lemma 4.2.** *Suppose that the distribution of the c-variate column vector $\boldsymbol{x}$ is normal with the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ of rank r ($r\leqq c$). Then there exists the unique $c\times c$ matrix $\boldsymbol{\Lambda}$ such that*

$$(4.12)\quad \begin{cases}\boldsymbol{B\Lambda}=\boldsymbol{0}\\ \boldsymbol{\Sigma\Lambda}=\boldsymbol{I}-\boldsymbol{B},\end{cases}$$

*where $\boldsymbol{B}$ is the projection of the c-dimensional euclidean space to the* *eigenspace belonging to the eigenvalue zero of* $\boldsymbol{\Sigma}$. *This* $\boldsymbol{A}$ *is symmetric and* $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ *is distributed as noncentral* $\chi^2$ *with* $r$ *degrees of freedom and noncentrality parameter* $\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$.

Proof. It is shown by Sugiura [26] that the unique solution of $\boldsymbol{A}$ is given by $\boldsymbol{A}=\frac{1}{\alpha_1}\boldsymbol{A}_1+\cdots+\frac{1}{\alpha_s}\boldsymbol{A}_s$, where $\alpha_i$ is the nonzero eigenvalue of $\boldsymbol{\Sigma}$ and $\boldsymbol{A}_i$ is the projection to the eigenspace of $\boldsymbol{\Sigma}$ belonging to the eigenvalue $\alpha_i$. We can easily see that $\frac{1}{\alpha_i}\boldsymbol{x}'\boldsymbol{A}_i\boldsymbol{x}$ is distributed as noncentral $\chi^2$ with the number of degrees of freedom being equal to the rank of $\boldsymbol{A}_i$ and noncentrality parameter $\lambda_i^2=\frac{1}{\alpha_i}\boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{\mu}$. Since $\boldsymbol{x}'\boldsymbol{A}_i\boldsymbol{x}$ $(i=1,2,\cdots,c)$ are stochastically independent, $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is distributed as noncentral $\chi^2$ with $r$ degrees of freedom and noncentrality parameter $\lambda^2=\sum_{i=1}^c\lambda_i^2=\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$.

Now we shall return to search for the distribution of the statistic $V_{rs}$ under the alternative $K_N$. Calculating the noncentrality parameter by Lemma 4.2, where the transpose of $\boldsymbol{x}$ is given by (3.6), $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_c)'$ and $\boldsymbol{\Sigma}=(\sigma_{ij})$ are given by (4.11), we can see that the projection $\boldsymbol{B}$ is given by

$$\text{(4.13)} \qquad \boldsymbol{B}=\begin{pmatrix} \frac{1}{c}, & \frac{1}{c} & \cdots & \frac{1}{c} \\ \frac{1}{c}, & \frac{1}{c} & \cdots & \frac{1}{c} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{c}, & \frac{1}{c} & \cdots & \frac{1}{c} \end{pmatrix}$$

and the solution of (4.12) is given by (3.24) where $K=K(r,s)/(c-1)^2$. Hence by Lemma 3.2 we can immediately obtain Theorem 4.1.

From Theorem 3.1 and Theorem 4.1 the uniformly most powerful test due to the $V_{rs}$-statistic in the limiting distribution is given by rejecting the hypothesis if $V_{rs}$ is larger than a preassigned constant, since the hypothesis is $\lambda_{rs}^2=0$ and the alternatives are $\lambda_{rs}^2\neq 0$ in the noncentral $\chi^2$ distribution. This test will be called the $V_{rs}$-test which depends on the pair $(r,s)$ of integers.

From Theorem 4.1 we can see that the statistic $V_{11}$ is distributed asymptotically noncentral $\chi^2$ as $N\to\infty$, with $c-1$ degrees of freedom and noncentrality parameter $\lambda_{11}^2=12\sum_{i=1}^c\rho_i(\theta_i-\tilde{\theta})^2\left[\int_{-\infty}^{\infty}f(x)\,dF(x)\right]^2$ which is the case with Kruskal and Wallis' $H$-test as is shown by Andrews [2]. Hence we have the following theorem.

**Theorem 4.2.** *Under the assumption of Theorem* 4.1 *the limiting distribution of the statistic* $V_{11}$ *defined by* (3.28) *is the same as Kruskal and Wallis' H-statistic.*

Now we shall consider the asymptotic relative efficiency of the $V_{rs}$-test for general $r, s$ ($r, s=0, 1, \cdots, c-1$) against Kruskal and Wallis' $H$-test. By Andrews [2], this efficiency is given by the ratio of the noncentrality parameters of the statistic $V_{rs}$ to that of $V_{11}$ in the limiting distribution. Hence we have from Theorem 4.1 and Theorem 4.2,

**Theorem 4.3.** *Under the same assumptions as in Theorem* 4.1 *the asymptotic relative efficiency of the $V_{rs}$-test against Kruskal and Wallis' H-test is*

$$\varepsilon_{V_{rs}/H}=\frac{\left(\int_{-\infty}^{\infty}\{r[F(x)]^{r-1}+s[1-F(x)]^{s-1}\}f(x)dF(x)\right)^2}{12K(r, s)\left[\int_{-\infty}^{\infty}f(x)dF(x)\right]^2} \tag{4.14}$$

*where $K(r, s)$ is given by* (3.21).

In particular $\varepsilon_{V_{11}/H}=\varepsilon_{V_{10}/H}=\varepsilon_{V_{01}/H}=\varepsilon_{V_{22}/H}=1$, and this fact is alluded recently by Bhapkar [5]. The $V_{10}$, $V_{01}$, $V_{11}$, $V_{22}$-test are equivalent, since their statistics are equivalent.

Let us now specify the distribution function $F(x)$ or the density function $f(x)$ in the alternative as follows.

EXAMPLE 1. Uniform distribution: $f(x)=1$ for $0<x<1$ and zero otherwise.

$$\varepsilon_{V_{rs}/H}=\begin{cases}1/3K(r, s) & \text{if } r, s\geqq 1\\ (2r+1)(r+1)^2/12r^2 & \text{if } s=0\\ (2s+1)(s+1)^2/12s^2 & \text{if } r=0.\end{cases} \tag{4.15}$$

EXAMPLE 2. Expontial distribution: $f(x)=e^{-x}$ for $x>0$ and zero otherwise.

$$\varepsilon_{V_{rs}/H}=\begin{cases}\left(\frac{1}{r+1}+\frac{s}{s+1}\right)^2/3K(r, s) & \text{if } r\geqq 1\\ (2s+1)/3 & \text{if } r=0.\end{cases} \tag{4.16}$$

EXAMPLE 3. Normal distribution: $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$ for $-\infty<x<\infty$.

$$\varepsilon_{V_{rs}/H}=\frac{\pi}{3K(r, s)}\left[\frac{1}{r+1}E(X_{r+1/r+1})+\frac{1}{s+1}E(X_{s+1/s+1})\right]^2, \tag{4.17}$$

where $X_{j/n}$ means the $j$-th smallest observation among the random sample of size $n$ from the standard normal distribution.

EXAMPLE 4. Double exponential distribution: $f(x)=\frac{1}{2}e^{-|x|}$ for $-\infty<x<\infty$.

$$\varepsilon_{V_{rs}/H}=\frac{4}{3K(r,s)}\left[\frac{1}{r+1}\left(1-\frac{1}{2^r}\right)+\frac{1}{s+1}\left(1-\frac{1}{2^s}\right)\right]^2. \tag{4.18}$$

EXAMPLE 5. Logistic distribution: $f(x)=e^{-x}/(1+e^{-x})^2$ for $-\infty<x<\infty$.

$$\varepsilon_{V_{rs}/H}=\frac{3}{K(r,s)}\left[\frac{r}{(r+1)(r+2)}+\frac{s}{(s+1)(s+2)}\right]^2. \tag{4.19}$$

Some numerical values of $\varepsilon_{V_{rs}/H}$ given by the above examples are shown in the following Tables. The value of $\varepsilon_{V_{rs}/H}$ is symmetric with respect to $r$ and $s$, if the distribution $F(x)$ is symmetric at the origin, as is the case with Example 3, 4 and 5.
From these Tables we can see that the asymptotic efficiency of the $V_{rs}$-test against Kruskal and Wallis' $H$-test for the range $(r, s)=(3.3)\sim(6.6)$ is larger than one for each of the uniform, exponential and normal distributions, but less than one for either the double exponential or the logistic distribution. The above statement for the normal distribution agrees well with the result by Deshpande [10]. For the double exponential and the logistic distribution, the $V_{11}$-test, which is asymptotically as efficient as Kruskal and Wallis' $H$-test, seems to be most efficient within the class of the $V_{rs}$-test.

Table 1. Asymptotic relative efficiency $\varepsilon_{V_{rs}/H}$ for the uniform distribution

| $r$ \ $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | — | 1.000 | .938 | 1.037 | 1.172 | 1.320 | 1.475 | 1.633 | 1.793 | 1.955 | 2.118 |
| 1 | | 1.000 | .984 | 1.063 | 1.158 | 1.255 | 1.350 | 1.439 | 1.524 | 1.604 | 1.680 |
| 2 | | | 1.000 | 1.102 | 1.215 | 1.327 | 1.432 | 1.531 | 1.623 | 1.708 | 1.787 |
| 3 | | | | 1.228 | 1.364 | 1.496 | 1.619 | 1.733 | 1.838 | 1.935 | 2.024 |
| 4 | | | | | 1.522 | 1.672 | 1.813 | 1.942 | 2.061 | 2.171 | 2.272 |
| 5 | | | | | | 1.841 | 1.997 | 2.142 | 2.275 | 2.398 | 2.511 |
| 6 | | | | | | | 2.169 | 2.328 | 2.475 | 2.610 | 2.736 |
| 7 | | | | | | | | 2.501 | 2.661 | 2.809 | 2.947 |
| 8 | | | | | | | | | 2.834 | 2.994 | 3.144 |
| 9 | | | | | | | | | | 3.167 | 3.328 |
| 10 | | | | | | | | | | | 3.500 |

Table 2. Asymptotic relative efficiency $\varepsilon_{V_{rs}/H}$ for the exponential distribution

| r \ s | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | — | 1.000 | 1.667 | 2.333 | 3.000 | 3.667 | 4.333 | 5.000 | 5.667 | 6.333 | 7.000 |
| 1 | 1.000 | 1.000 | 1.339 | 1.660 | 1.958 | 2.232 | 2.486 | 2.721 | 2.940 | 3.144 | 3.335 |
| 2 | .417 | .683 | 1.000 | 1.293 | 1.561 | 1.806 | 2.030 | 2.236 | 2.424 | 2.598 | 2.759 |
| 3 | .259 | .598 | .926 | 1.228 | 1.504 | 1.755 | 1.984 | 2.193 | 2.384 | 2.559 | 2.720 |
| 4 | .188 | .568 | .913 | 1.231 | 1.522 | 1.786 | 2.026 | 2.244 | 2.444 | 2.627 | 2.795 |
| 5 | .147 | .558 | .921 | 1.257 | 1.563 | 1.841 | 2.093 | 2.324 | 2.535 | 2.728 | 2.906 |
| 6 | .120 | .558 | .939 | 1.290 | 1.611 | 1.903 | 2.169 | 2.412 | 2.634 | 2.839 | 3.028 |
| 7 | .102 | .562 | .960 | 1.327 | 1.662 | 1.967 | 2.246 | 2.501 | 2.735 | 2.951 | 3.151 |
| 8 | .089 | .569 | .982 | 1.363 | 1.711 | 2.029 | 2.320 | 2.587 | 2.834 | 3.061 | 3.272 |
| 9 | .078 | .578 | 1.004 | 1.398 | 1.758 | 2.089 | 2.391 | 2.670 | 2.928 | 3.167 | 3.389 |
| 10 | .070 | .587 | 1.026 | 1.431 | 1.804 | 2.145 | 2.459 | 2.749 | 3.018 | 3.268 | 3.500 |

Table 3. Asymptotic relative efficiency $\varepsilon_{V_{rs}/H}$ for the normal distribution

| r \ s | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | — | 1.000 | .938 | .863 | .797 | .740 | .691 | .650 | .613 | .582 | .553 |
| 1 | | 1.000 | .984 | .971 | .964 | .960 | .958 | .957 | .957 | .958 | .959 |
| 2 | | | 1.000 | 1.007 | 1.011 | 1.014 | 1.016 | 1.018 | 1.019 | 1.020 | 1.020 |
| 3 | | | | 1.022 | 1.029 | 1.032 | 1.032 | 1.031 | 1.030 | 1.028 | 1.025 |
| 4 | | | | | 1.035 | 1.035 | 1.032 | 1.028 | 1.024 | 1.019 | 1.014 |
| 5 | | | | | | 1.032 | 1.026 | 1.019 | 1.011 | 1.004 | .997 |
| 6 | | | | | | | 1.017 | 1.007 | .997 | .988 | .978 |
| 7 | | | | | | | | .995 | .983 | .972 | .961 |
| 8 | | | | | | | | | .969 | .956 | .944 |
| 9 | | | | | | | | | | .942 | .929 |
| 10 | | | | | | | | | | | .915 |

Table 4. Asymptotic relative efficiency $\varepsilon_{V_{rs}/H}$ for the double exponential distribution

| r \ s | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | — | 1.000 | .938 | .794 | .659 | .551 | .467 |
| 1 | | 1.000 | .984 | .934 | .887 | .850 | .824 |
| 2 | | | 1.000 | .968 | .930 | .899 | .874 |
| 3 | | | | .940 | .901 | .865 | .836 |
| 4 | | | | | .856 | .815 | .781 |
| 5 | | | | | | .768 | .729 |
| 6 | | | | | | | .686 |

Table 5. Asymptotic relative efficiency $e_{V_{rs}/H}$ for the logistic distribution

| $r$ \ $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | — | 1.000 | .938 | .840 | .750 | .673 | .609 |
| 1 | | 1.000 | .984 | .959 | .938 | .922 | .911 |
| 2 | | | 1.000 | .994 | .984 | .975 | .967 |
| 3 | | | | .995 | .986 | .974 | .963 |
| 4 | | | | | .974 | .959 | .943 |
| 5 | | | | | | .939 | .920 |
| 6 | | | | | | | .896 |

## 5. Nonparametric several-sample tests for scale

Although various nonparametric two-sample tests for the problem of scale have been proposed by many authors such as Mood [24], Kamat [14], Sukhatme [27, 28], Tamura [29, 30, 31] and Ansari and Bladrey [3] with their asymptotic efficiencies investigated by Capon [8], Klotz [16], etc., but a few several-sample tests for scale are available. In the following four sections we shall treat this problem.

Let $X_{ij}$ $(j=1, 2, \cdots, n_i)$ be a random sample from the continuous univariate distribution $F_i(x)$ $(i=1, 2, \cdots, c)$. From these samples we want to test the hypothesis $H: F_1=\cdots=F_c$ against the scale alternative $K: F_i(x)=F(x/\sigma_i)$, where the functional from of $F$ is unknown and $\sigma_i$ is some constant (not all $\sigma_i$'s are equal).

Corresponding to the statistic (3.1), we shall put at this time for $i=1, 2, \cdots, c$

$$\text{(5.1)}\quad \begin{aligned} U^{(i)} &= \frac{1}{n_1\cdots n_c}\sum_{\alpha_1=1}^{n_1}\cdots\sum_{\alpha_c=1}^{n_c}\phi^{(i)}(X_{1\alpha_1}, \cdots, X_{c\alpha_c}) \\ \phi^{(i)}(X_1, \cdots, X_c) &= \frac{(j-1)_r}{(c-1)_r}+\frac{(c-j)_s}{(c-1)_s} \quad \text{if } X_i \text{ is the } j\text{-th smallest among } X_1, \cdots, X_c \end{aligned}$$

where $0\leqq r,\ s\leqq c-1$ except for $(r, s)=(0, 0)$ and $(1, 1)$.
Then the statistic $U^{(i)}$ is a generalized $U$ statistic stated in section 2 and we shall construct a nonparametric test from these $U^{(i)}$ $(i=1, 2, \cdots, c)$. In case $r=s=c-1$

$$\text{(5.2)}\quad \phi^{(i)}(X_1, \cdots, X_c) = \begin{cases} 1 & \text{if } X_i<X_j \text{ or } X_i>X_j \text{ for any } j\neq i \\ 0 & \text{otherwise} \end{cases}$$

which reduces to Deshpande's $D$-statistic [11]. Since our method of construction is quite analogous to that in section 3, we shall only show the results, leaving the details to readers. The expectation of the $U^{(i)}$ under the hypothesis $H$ is given by

$$E[U^{(i)}] = \frac{1}{r+1}+\frac{1}{s+1}. \tag{5.3}$$

Putting $n_i=\rho_i N$ where $\rho_i>0$ is independent of $N$, we can see that the statistic

$$\sqrt{N}(U^{(1)}-E[U^{(1)}], \cdots, U^{(c)}-E[U^{(c)}]) \tag{5.4}$$

is distributed asymptotically normal as $N\to\infty$, with the mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$ where $\sigma_{ij}$ is given by (3.7) as before and

$$\begin{aligned} &\zeta^{(i,i)}_{0,\dots,1,\dots,0} \quad (1 \text{ lies at the } i\text{-th place}) = L(r, s) \\ &\zeta^{(i,j)}_{0,\dots,1,\dots,0} \quad (1 \text{ lies at the } k\text{-th place, where } k\neq i, j) = \frac{1}{(c-1)^2}L(r, s) \\ &\zeta^{(i,j)}_{0,\dots,1,\dots,0} \quad (1 \text{ lies at the } i\text{-th place, where } i\neq j) = -\frac{1}{c-1}L(r, s), \end{aligned} \tag{5.5}$$

where

$$L(r, s)=\frac{r^2}{(2r+1)(r+1)^2}+\frac{s^2}{(2s+1)(s+1)^2}-\frac{2}{(r+1)(s+1)}+2B(r+1, s+1). \tag{5.6}$$

Hence we have

$$\sigma_{ij} = \frac{L(r, s)}{(c-1)^2}\left[\sum_{\alpha=1}^{c}\frac{1}{\rho_\alpha}-\frac{c}{\rho_i}-\frac{c}{\rho_j}+\frac{c^2\delta_{ij}}{\rho_i}\right]. \tag{5.7}$$

From Lemma 3.1 and Lemma 3.2, Theorem 5.1 follows immediately.

**Theorem 5.1.** *Put $n_i=\rho_i N$ where $\rho_i>0$ is independent of $N$. Then under the hypothesis $H: F_1=\cdots=F_c$, the statistic defined by*

$$\begin{aligned} D_{rs} &= \frac{(c-1)^2}{c^2L(r, s)}\sum_{i=1}^{c} n_i(U^{(i)}-\tilde{U})^2 \qquad (0\leqq r,\ s\leqq c-1 \text{ and } (r, s)\neq(0, 0), (1, 1)) \\ \tilde{U} &= \textstyle\sum_{i=1}^{c} n_i U^{(i)}/\sum_{i=1}^{c} n_i, \end{aligned} \tag{5.8}$$

*where $U^{(i)}$ is given by* (5.1) *and $L(r, s)$ is defined by* (5.6), *is distributed asymptotically as $\chi^2$ with $c-1$ degrees of freedom as $N\to\infty$.*

It is easily seen by the definition of $U^{(i)}$ in (5.1) that the $D_{12}$, $D_{21}$, $D_{22}$ and the $D_{33}$-statistic are equivalent. We can also remark that the distribution of $D_{rs}$ is the same as that of $D_{sr}$ under the alternative that $F_i(x)$ is symmetric at the origin for every $i$.

As a special case of Theorem 5.1 we have

**Corollary 1.** *The statistic*

$$D_{c-1,\,c-1} = \frac{(2c-1)(c-1)^2}{2\left[\dfrac{c^2-4c+2}{c^2} + \left(\dfrac{2c-2}{c-1}\right)^{-1}\right]} \sum_{i=1}^{c} n_i (U^{(i)} - \tilde{U})^2 \tag{5.9}$$

*is distributed asymptotically as $\chi^2$ with $c-1$ degrees of freedom as $N \to \infty$.*

This is Deshpande's $D$-statistic proposed in [11], though some misprints seem to scatter in his paper.

**Corollary 2.** *The statistic*

$$D_{22} = \frac{45(c-1)^2}{c^2} \sum_{i=1}^{c} n_i (U^{(i)} - \tilde{U})^2 \tag{5.10}$$

*is distributed asymptotically as $\chi^2$ with $c-1$ degrees of freedom as $N \to \infty$.*

This statistic $D_{22}$ is equivalent to the one defined by $\phi^{(i)}(X_1, \cdots, X_c) = 2\left(j - \frac{c+1}{2}\right)^2 / (c-1)(c-2)$, if $X_i$ is the $j$-th smallest among $X_1, \cdots, X_c$ which assigns to each $j$ the quadratic weight centered at $(c+1)/2$. Correspondingly if we assign any linear weight to each $j$, it is equivalent to $V_{11}$-test.

## 6. Asymptotic efficiency of $D_{rs}$-test for the scale alternative

Now we shall consider the limiting distribution of the statistic $D_{rs}$ given by (5.8) under the following sequence of alternatives;

$$K_N : F_i(x) = F(x/(\sigma + N^{-1/2}\theta_i)) \qquad (i = 1, 2, \cdots, c)\,, \tag{6.1}$$

where $\sigma$ and $\theta_i$ is some constant. (not all $\theta_i$'s are equal).

**Theorem 6.1.** *Put $n_i = \rho_i N$ where $\rho_i > 0$ is independent of $N$ and suppose that the distribution function $F(x)$ has the derivative $f(x)$ except for a set of $F$-measure zero and further that there exists a function $g(x)$ such that $\int_{-\infty}^{\infty} x g(x) dF(x) < \infty$ and that*

$$\left| \frac{1}{h} [F(x+h) - F(x)] \right| \leqq g(x) \tag{6.2}$$

*holds for any $x$ and any sufficiently small $h$. Then under the sequence of alternatives $K_N$, the limiting distribution of $D_{rs}$ as $N \to \infty$, is non-central $\chi^2$ with $c-1$ degrees of freedom and the noncentrality parameter*

$$\lambda_{rs}^2 = \frac{\sum_{i=1}^{c} \rho_i (\theta_i - \tilde{\theta})^2}{\sigma^2 L(r, s)} \left( \int_{-\infty}^{\infty} x \left\{ r[F(x)]^{r-1} - s[1-F(x)]^{s-1} \right\} f(x) dF(x) \right)^2 \tag{6.3}$$

*where* $L(r, s)$ *is given by* (5.6) *and* $\tilde{\theta}=\sum_{\alpha=1}^{c}\rho_{\alpha}\theta_{\alpha}/\sum_{\alpha=1}^{c}\rho_{\alpha}$.

As a special case of Theorem 6.1 we get the following Corollary 1.

**Corollary 1.** *Under the sequence of alternatives* $K_N$, *the limiting distribution of* $D_{c-1,c-1}$ *and* $D_{22}$ *are noncentral* $\chi^2$ *with* $c-1$ *degrees of freedom with noncentrality parameter given respectively by*

$$\lambda_{c-1,c-1}^2 = \frac{(2c-1)(c-1)^2\sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2}{2\sigma^2\left[\dfrac{c^2-4c+2}{c^2}+\left(\dfrac{2c-2}{c-1}\right)^{-1}\right]} \times\left(\int_{-\infty}^{\infty}x\left\{[F(x)]^{c-2}-[1-F(x)]^{c-2}\right\}f(x)\,dF(x)\right)^2 \tag{6.4}$$

$$\lambda_{22}^2 = \frac{180\sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2}{\sigma^2}\left(\int_{-\infty}^{\infty}x[2F(x)-1]f(x)\,dF(x)\right)^2. \tag{6.5}$$

To prove Theorem 6.1 we shall first show the following Lemma 6.1 which is similar to Lemma 4.1.

**Lemma 6.1.** *Let* $F(x)$ *be a distribution function. Suppose that the distribution function* $G_i(x)$ *has the derivative* $g_i(x)$ *and further there exist a function* $g(x)$ *such that* $\int xg(x)dF(x)<\infty$ *and that*

$$\left|\frac{1}{h_i}[G_i(x+h_i)-G_i(x)]\right|\leqq g(x) \qquad (i=1,2,\cdots,n) \tag{6.6}$$

*hold for any* $x$ *and any sufficiently small* $h_i$ $(=\alpha_i h$ *for small* $h)$. *Then*

$$\int_{-\infty}^{\infty}\prod_{i=1}^{n}G_i(x+h_ix)\,dF(x)=\int_{-\infty}^{\infty}\prod_{i=1}^{n}G_i(x)\,dF(x) +\sum_{i=1}^{n}h_i\int_{-\infty}^{\infty}xg_i(x)\prod_{j\neq i}G_j(x)\,dF(x)+o(h). \tag{6.7}$$

Proof. By the assumption (6.6), it follows that $|[G_i(x+h_ix)-G_i(x)]/h_i|\leqq|x|g(x)$. Since $xg(x)$ is integrable with respect to $F(x)$, we can apply Lebesgue's bounded convergence theorem to get $\lim_{h_i\to 0}\int_{-\infty}^{\infty}\left\{[G_i(x+h_ix)-G_i(x)]/h_i\right\}dF(x)=\int_{-\infty}^{\infty}xg_i(x)\,dF(x)$. The same argument as in Lemma 4.1. completes the proof.

Now we shall prove Theorem 6.1. Under the alternative $K_N$ the expectation of the statistic $U^{(i)}$ defined by (5.1) is

$$E[U^{(i)}|K_N]=\sum_{k=1}^{c}\left[\frac{(k-1)_r}{(c-1)_r}+\frac{(c-k)_s}{(c-1)_s}\right] \times\sum_{(l)}\int_{-\infty}^{\infty}F_{l_1}(x)\cdots F_{l_{k-1}}(x)[1-F_{l_{k+1}}(x)]\cdots[1-F_{l_c}(x)]\,dF_i(x), \tag{6.8}$$

where the meaning of $\sum_{(l)}$ is the same as in (4.8). The integral in the righthand side can be written as

$$\int_{-\infty}^{\infty}\prod_{j=1}^{k-1}F\left(x+x\frac{\theta_i-\theta_{l_j}}{\sigma\sqrt{N}+\theta_{l_j}}\right)\prod_{j=k+1}^{c}\left[1-F\left(x+x\frac{\theta_i-\theta_{l_j}}{\sigma\sqrt{N}+\theta_{l_j}}\right)\right]dF(x). \tag{6.9}$$

By Lemma 6.1 it is equal to

$$\begin{aligned}\int_{-\infty}^{\infty}&[F(x)]^{k-1}[1-F(x)]^{c-k}dF(x)+\frac{(\theta_i-\theta_{l_1})+\cdots+(\theta_i-\theta_{l_{k-1}})}{\sigma\sqrt{N}}\\&\times\int_{-\infty}^{\infty}xf(x)[F(x)]^{k-2}[1-F(x)]^{c-k}dF(x)\\&-\frac{(\theta_i-\theta_{l_{k+1}})+\cdots+(\theta_i-\theta_{l_c})}{\sigma\sqrt{N}}\\&\times\int_{-\infty}^{\infty}xf(x)[F(x)]^{k-1}[1-F(x)]^{c-k-1}dF(x)+o\left(\frac{1}{\sqrt{N}}\right).\end{aligned} \tag{6.10}$$

Hence by the same argument as in (4.10) we have

$$\begin{aligned}E[U^{(i)}|K_N]&=\sum_{k=1}^{c}\left[\frac{(k-1)_r}{(c-1)_r}+\frac{(c-k)_s}{(c-1)_s}\right]\left(\binom{c-1}{k-1}\right.\\&\times\int_{-\infty}^{\infty}[F(x)]^{k-1}[1-F(x)]^{c-k}dF(x)+\frac{\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)}{\sigma\sqrt{N}}\\&\times\int_{-\infty}^{\infty}xf(x)\left\{\binom{c-2}{k-2}[F(x)]^{k-2}[1-F(x)]^{c-k}\right.\\&\left.\left.-\binom{c-2}{c-k-1}[F(x)]^{k-1}[1-F(x)]^{c-k-1}\right\}dF(x)\right)+o\left(\frac{1}{\sqrt{N}}\right)\\&=\frac{1}{r+1}+\frac{1}{s+1}+\frac{\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)}{\sigma\sqrt{N}(c-1)}\\&\times\int_{-\infty}^{\infty}xf(x)\left\{r[F(x)]^{r-1}-s[1-F(x)]^{s-1}\right\}dF(x)+o\left(\frac{1}{\sqrt{N}}\right).\end{aligned} \tag{6.11}$$

It follows from Theorem 2.1 that under the sequence of alternatives $K_N$ defined by (6.1), the statistic (5.4) is distributed asymptotically normal as $N\to\infty$, with the mean vector $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_c)$ and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$ where

$$\begin{aligned}\mu_i&=\frac{\sum_{\alpha=1}^{c}(\theta_i-\theta_\alpha)}{\sigma(c-1)}\int_{-\infty}^{\infty}xf(x)\left\{r[F(x)]^{r-1}-s[1-F(x)]^{s-1}\right\}dF(x)\\\sigma_{ij}&=\frac{L(r,s)}{(c-1)^2}\left[\sum_{\alpha=1}^{c}\frac{1}{\rho_\alpha}-\frac{c}{\rho_i}-\frac{c}{\rho_j}+\frac{c^2\delta_{ij}}{\rho_i}\right].\end{aligned} \tag{6.12}$$

By Lemma 4.2, Lemma 3.2 and (3.24) where $K=L(r,s)/(c-1)^2$, we can obtain Theorem 6.1.

From Theorem 5.1 and Theorem 6.1, the asymptotically uniformly most powerful critical region for the $D_{rs}$-statistic is given by $D_{rs}>D_0$, where $D_0$ is a preassigned constant. This test will be called $D_{rs}$-test.

Since the asymptotic relative efficiency is given by the ratio of noncentrality parameters in the limiting distributions under the sequence of alternatives $K_N$, we immediately have the following Theorem 6.2 from Theorem 6.1 and its Corollary 1.

**Theorem 6.2.** *Under the same assumptions as in Theorem* 6.1, *the asymptotic relative efficiency of the $D_{rs}$-test against the $D_{22}$-test is given by*

$$(6.13)\qquad \varepsilon_{D_{rs}/D_{22}}=\frac{\left(\int_{-\infty}^{\infty} xf(x)\left\{r[F(x)]^{r-1}-s[1-F(x)]^{s-1}\right\}dF(x)\right)^2}{180L(r,s)\left(\int_{-\infty}^{\infty} xf(x)[2F(x)-1]dF(x)\right)^2}.$$

In particular $\varepsilon_{D_{21}/D_{22}}=\varepsilon_{D_{12}/D_{22}}=\varepsilon_{D_{33}/D_{22}}=1$. $D_{22}$, $D_{12}$, $D_{21}$, $D_{33}$-test are equivalent, since their statistics are equivalent. Let us now specify the distribution function $F(x)$ as follows.

EXAMPLE 1. Uniform distribution: $f(x)=1$ for $0<x<1$ and zero otherwise.

$$(6.14)\qquad \varepsilon_{D_{rs}/D_{22}}=\begin{cases}\left(\dfrac{r}{r+1}-\dfrac{1}{s+1}\right)^2/5L(r,s) & \text{if } s\geqq 1\\ (2r+1)/5 & \text{if } s=0.\end{cases}$$

EXAMPLE 2. Exponential distribution: $f(x)=e^{-x}$ for $x>0$ and zero otherwise.

$$(6.15)\qquad \varepsilon_{D_{rs}/D_{22}}=\begin{cases}\dfrac{36}{5L(r,s)}\left[\dfrac{1}{r+1}\left(\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{r+1}\right)-\dfrac{s}{(s+1)^2}\right]^2 & \text{for } r\geqq 1\\ \dfrac{36}{5}\dfrac{2s+1}{(s+1)^2} & \text{for } r=0.\end{cases}$$

EXAMPLE 3. Normal distribution: $f(x)=(2\pi)^{-1/2}e^{-x^2/2}$ for $-\infty<x<\infty$.

$$(6.16)\qquad \varepsilon_{D_{rs}/D_{22}}=\frac{\pi^2}{15L(r,s)}\left\{\frac{1}{r+1}\left[E(X^2_{r+1/r+1})-1\right]+\frac{1}{s+1}\left[E(X^2_{s+1/s+1})-1\right]\right\}^2.$$

EXAMPLE 4. Double exponential distribution: $f(x)=\dfrac{1}{2}e^{-|x|}$ for $-\infty<x<\infty$.

$$(6.17)\qquad \varepsilon_{D_{rs}/D_{22}}=\frac{144}{125L(r,s)}\left[\frac{r}{4}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^j}{2^j(j+2)^2}-\frac{r}{(r+1)^2 2^{r+1}}\right.$$

$$+\frac{s}{4}\sum_{j=0}^{s-1}\binom{s-1}{j}\frac{(-1)^j}{2^j(j+2)^2}-\frac{s}{(s+1)^2 2^{s+1}}\Bigg]^2 .$$

EXAMPLE 5. Logistic distribution: $f(x)=e^{-x}/(1+e^{-x})^2$ for $-\infty<x<\infty$.

$$(6.18)\quad \varepsilon_{D_{rs}/D_{22}}=\frac{4}{5L(r,s)}\Bigg[\frac{r}{(r+1)(r+2)}\Big(\sum_{j=1}^{r}\frac{1}{j}-1\Big) + \frac{s}{(s+1)(s+2)}\Big(\sum_{j=1}^{s}\frac{1}{j}-1\Big)\Bigg]^2 .$$

From these formulas we get the following Tables of $\varepsilon_{D_{rs}/D_{22}}$, which is symmetric with respect to $r$ and $s$, if the distribution $F(x)$ is symmetric at the origin as is the case with Example 3, 4 and 5.

Table 6. Asymptotic relative efficiency $\varepsilon_{D_{rs}/D_{22}}$ for the uniform distribution

| $r$ \ $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | — | .600 | .250 | .156 | .113 | .088 | .072 | .061 | .053 | .047 | .042 |
| 1 | .600 | — | 1.000 | .913 | .853 | .811 | .780 | .757 | .739 | .725 | .713 |
| 2 | 1.000 | | 1.000 | .967 | .953 | .947 | .944 | .944 | .945 | .946 | .948 |
| 3 | 1.400 | | | 1.000 | 1.032 | 1.061 | 1.087 | 1.110 | 1.129 | 1.147 | 1.162 |
| 4 | 1.800 | | | | 1.101 | 1.160 | 1.210 | 1.254 | 1.292 | 1.325 | 1.355 |
| 5 | 2.200 | | | | | 1.244 | 1.317 | 1.380 | 1.435 | 1.484 | 1.526 |
| 6 | 2.600 | | | | | | 1.410 | 1.491 | 1.561 | 1.624 | 1.680 |
| 7 | 3.000 | | | | | | | 1.587 | 1.673 | 1.749 | 1.817 |
| 8 | 3.400 | | | | | | | | 1.772 | 1.861 | 1.940 |
| 9 | 3.800 | | | | | | | | | 1.961 | 2.052 |
| 10 | 4.200 | | | | | | | | | | 2.153 |

Table 7. Asymptotic relative efficiency $\varepsilon_{D_r/D_{22}}$ for the exponential distribution

| $r$ \ $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | — | 5.400 | 4.000 | 3.150 | 2.592 | 2.200 | 1.910 | 1.688 | 1.511 | 1.368 | 1.250 |
| 1 | 5.400 | — | 1.000 | 2.054 | 2.763 | 3.242 | 3.582 | 3.832 | 4.024 | 4.175 | 4.297 |
| 2 | 6.250 | 1.000 | 1.000 | 1.634 | 2.185 | 2.630 | 2.990 | 3.285 | 3.532 | 3.740 | 3.918 |
| 3 | 6.572 | .228 | .474 | 1.000 | 1.509 | 1.955 | 2.337 | 2.666 | 2.949 | 3.195 | 3.411 |
| 4 | 6.670 | .015 | .187 | .588 | 1.029 | 1.444 | 1.818 | 2.150 | 2.444 | 2.705 | 2.937 |
| 5 | 6.661 | .018 | .052 | .329 | .694 | 1.065 | 1.414 | 1.733 | 2.023 | 2.285 | 2.521 |
| 6 | 6.597 | .111 | .004 | .170 | .460 | .782 | 1.099 | 1.398 | 1.675 | 1.930 | 2.163 |
| 7 | 6.504 | .241 | .006 | .076 | .297 | .570 | .853 | 1.128 | 1.388 | 1.631 | 1.856 |
| 8 | 6.398 | .385 | .040 | .025 | .183 | .410 | .659 | .909 | 1.151 | 1.380 | 1.595 |
| 9 | 6.284 | .532 | .091 | .003 | .105 | .290 | .506 | .731 | .954 | 1.168 | 1.372 |
| 10 | 6.169 | .676 | .153 | .001 | .054 | .200 | .385 | .587 | .790 | .989 | 1.181 |

Table 8. Asymptotic relative efficiency $\varepsilon_{D_{rs}/D_{22}}$ for the normal distribution

| r \ s | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | — | 0 | .063 | .156 | .237 | .302 | .354 | .395 | .427 | .452 | .473 |
| 1 | | — | 1.000 | .913 | .798 | .696 | .611 | .542 | .485 | .437 | .397 |
| 2 | | | 1.000 | .967 | .913 | .856 | .803 | .754 | .710 | .671 | .636 |
| 3 | | | | 1.000 | .996 | .974 | .945 | .914 | .883 | .853 | .824 |
| 4 | | | | | 1.030 | 1.038 | 1.031 | 1.016 | .997 | .976 | .955 |
| 5 | | | | | | 1.068 | 1.079 | 1.079 | 1.071 | 1.058 | 1.044 |
| 6 | | | | | | | 1.105 | 1.116 | 1.117 | 1.113 | 1.104 |
| 7 | | | | | | | | 1.137 | 1.146 | 1.148 | 1.145 |
| 8 | | | | | | | | | 1.162 | 1.170 | 1.171 |
| 9 | | | | | | | | | | 1.182 | 1.188 |
| 10 | | | | | | | | | | | 1.197 |

Table 9. Asymptotic relative efficiency $\varepsilon_{D_{rs}/D_{22}}$ for the double exponential distribution

| r \ s | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | — | 0 | .062 | .156 | .234 | .294 | .338 |
| 1 | | — | 1.000 | .913 | .789 | .677 | .584 |
| 2 | | | 1.000 | .967 | .906 | .841 | .779 |
| 3 | | | | 1.000 | .990 | .959 | .922 |
| 4 | | | | | 1.018 | 1.017 | 1.001 |
| 5 | | | | | | 1.039 | 1.041 |
| 6 | | | | | | | 1.056 |

Table 10. Asymptotic relative efficiency $\varepsilon_{D_{rs}/D_{22}}$ for the logistic distribution

| r \ s | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | — | 0 | .063 | .156 | .235 | .296 | .342 |
| 1 | | — | 1.000 | .913 | .791 | .681 | .590 |
| 2 | | | 1.000 | .967 | .908 | .844 | .784 |
| 3 | | | | 1.000 | .991 | .963 | .927 |
| 4 | | | | | 1.021 | 1.022 | 1.008 |
| 5 | | | | | | 1.046 | 1.049 |
| 6 | | | | | | | 1.067 |

From these Tables we can see that in the range $(r, s)=(4, 4)\sim(10, 10)$ the asymptotic efficiency of $D_{rs}$-test against the $D_{2,2}$-test is larger than one for each of the uniform, exponential and the normal distributions. Further the asymptotic efficiency of the $D_{4,4}\sim D_{6,6}$-test is larger than one for the double exponential and the logistic distribution. This situation is considerably different from that for the problem of location discussed in section 4.

## 7. A generalization of Tamura's $Q$-test

In the following two sections we shall investigate another type of nonparametric several-sample test for scale which is a natural extension of Tamura's $Q$-test [29] for the two-sample problem. Let again $X_{ij}$ $(j=1, 2, \cdots, n_i)$ be a random sample from the continuous univariate distribution $F_i(x)$ $(i=1, 2, \cdots, c)$. From these samples we want to test the hypothesis $H$: $F_1=\cdots=F_c$ against the scale alternative $K$: $F_i(x)=F(x/\sigma_i)$ where $\sigma_i$ is some constant (not all $\sigma_i$'s are equal) and the functional form of $F$ is unknown. Put

$$
(7.1)\qquad
\begin{aligned}
U^{(i)} &= \left[\binom{n_1}{2}\cdots\binom{n_c}{2}\right]^{-1}\sum_{\alpha_1<\beta_1}\cdots\sum_{\alpha_c<\beta_c}\phi^{(i)}(X_{1\alpha_1}, X_{1\beta_1}; \cdots; X_{c\alpha_c}, X_{c\beta_c})\\
\phi^{(i)}(X_{11}, X_{12}; \cdots; X_{c1}, X_{c2}) &= \begin{cases} 1 & \text{if } X_{i1}<X_{kl}<X_{i2} \text{ or } X_{i2}<X_{kl}<X_{i1} \\ & \text{for all } k\neq i \text{ and } l=1, 2 \\ 0 & \text{otherwise,}\end{cases}
\end{aligned}
$$

where $\sum_{\alpha_1<\beta_1}\cdots\sum_{\alpha_c<\beta_c}$ means the summation extending on all possible pairs $(\alpha_i, \beta_i)$ such that $1\leqq\alpha_i<\beta_i\leqq n_i$ for $i=1, 2, \cdots, c$. Then $U^{(i)}$ is a generalized $U$ statistic stated in section 2 and under the hypothesis $H$

$$
(7.2)\qquad E[U^{(i)}] = 1/c(2c-1)\,.
$$

Putting $n_i=\rho_i N$ as before, we can conclude from Theorem 2.1 that under the hypothesis $H$, the limiting distribution of the statistic

$$
(7.3)\qquad \sqrt{N}(U^{(1)}-[c(2c-1)]^{-1}, \cdots, U^{(c)}-[c(2c-1)]^{-1})
$$

is normal as $N\to\infty$, with the mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{ij})$;

$$
(7.4)\qquad \sigma_{ij} = \frac{4}{\rho_1}\zeta^{(i,j)}_{10,\cdots,0}+\cdots+\frac{4}{\rho_c}\zeta^{(i,j)}_{00,\cdots,1}\,,
$$

where

(i) $\zeta^{(i,i)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $i$-th place)

$$= E[\phi^{(i)}(X_{11}, X_{12}; \cdots; X_{c1}, X_{c2})\phi^{(i)}(X'_{11}, X'_{12}; \cdots; X_{i1}, X'_{i2}; \cdots; X'_{c1}, X'_{c2})] - [c(2c-1)]^{-2}$$

$$= P(X_{i1} < X_{\alpha\beta} < X_{i2} \text{ and } X_{i1} < X'_{\alpha\beta} < X'_{i2} \text{ for all } \alpha \neq i \text{ and } \beta = 1, 2)$$

$$+ 2P(X_{i1} < X_{\alpha\beta} < X_{i2} \text{ and } X'_{i2} < X'_{\alpha\beta} < X_{i1} \text{ for all } \alpha \neq i \text{ and } \beta = 1, 2)$$

(7.5) $$+ P(X_{i2} < X_{\alpha\beta} < X_{i1} \text{ and } X'_{i2} < X'_{\alpha\beta} < X_{i1} \text{ for all } \alpha \neq i \text{ and } \beta = 1, 2)$$

$$-[c(2c-1)]^{-2}$$

$$= \int_0^1 \left[\frac{(1-F)^{2c-1}}{2c-1}\right]^2 dF + 2\int_0^1 \frac{(1-F)^{2c-1}}{2c-1} \frac{F^{2c-1}}{2c-1} dF$$

$$+ \int_0^1 \left[\frac{F^{2c-1}}{2c-1}\right]^2 dF - \left[\frac{1}{c(2c-1)}\right]^2$$

$$= \frac{2}{(2c-1)^2(4c-1)}\left[\frac{2c^2-4c+1}{2c^2} + \binom{4c-2}{2c-1}^{-1}\right],$$

and an analogous calculation leads us to the following:

(ii) $\zeta^{(i,j)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $k$-th place, where $k \neq i, j$)

(7.6) $$= L(2c-1, 2c-1)/(c-1)^2(2c-1)^2$$

(iii) $\zeta^{(i,j)}_{0,\cdots,1,\cdots,0}$ (1 lies at the $i$-th place, where $i \neq j$)

(7.7) $$= -L(2c-1, 2c-1)/(c-1)(2c-1)^2,$$

where $L(2c-1, 2c-1)$ was already defined by (5.6), that is,

(7.8) $$L(2c-1, 2c-1) = \frac{2}{4c-1}\left[\frac{2c^2-4c+1}{2c^2} + \binom{4c-2}{2c-1}^{-1}\right].$$

Hence we have

(7.9) $$\sigma_{ij} = \frac{4L(2c-1, 2c-1)}{(c-1)^2(2c-1)^2}\left[\sum_{\alpha=1}^{c} \frac{1}{\rho_\alpha} - \frac{c}{\rho_i} - \frac{c}{\rho_j} + \frac{c^2\delta_{ij}}{\rho_i}\right].$$

From Lemma 3.1 and Lemma 3.2 follows Theorem 7.1.

**Theorem 7.1.** *Put $n_i = \rho_i N$ where $\rho_i > 0$ is independent of $N$. Then under the hypothesis $H$: $F_1 = \cdots = F_c$, the statistic defined by*

(7.10) $$Q = \frac{(2c-1)^2(c-1)^2}{4c^2L(2c-1, 2c-1)} \sum_{i=1}^{c} n_i(U^{(i)} - \tilde{U})^2$$

$$\tilde{U} = \textstyle\sum_{i=1}^{c} n_i U^{(i)} / \sum_{i=1}^{c} n_i$$

*where $U^{(i)}$ is given by (7.1) and $L(2c-1, 2c-1)$ is given by (7.8), is distributed asymptotically as $\chi^2$ with $c-1$ degrees of freedom as $N \to \infty$.*

## 8. Asymptotic efficiency of the $D_{22}$-test and $Q$-test against the asymptotically UMP invariant test

We shall first consider the limiting distribution of the statistic $Q$ given by (7.10) under the sequence of alternatives $K_N$: $F_i(x)=F(x/(\sigma+N^{-1/2}\theta_i))$, where $\sigma$ and $\theta_i$ are some constants (not all $\theta_i'$s are equal).

**Theorem 8.1.** *Under the same assumptions as in Theorem* 6.1 *the limiting distribution of $Q$ is noncentral $\chi^2$ with $c-1$ degrees of freedom and the noncentrality parameter*;

$$\lambda_Q^2 = \frac{(2c-1)^2\sum_{i=1}^c \rho_i(\theta_i-\tilde{\theta})^2}{\sigma^2 L(2c-1,\, 2c-1)} \times\left(\int_{-\infty}^{\infty} xf(x)\{[F(x)]^{2c-2}-[1-F(x)]^{2c-2}\}\,dF(x)\right)^2, \tag{8.1}$$

*where $L(2c-1, 2c-1)$ is given by* (7.8) *and* $\tilde{\theta}=\sum_{i=1}^c \rho_i\theta_i/\sum_{i=1}^c \rho_i$.

Proof. By the definition of $\phi^{(i)}$ in (7.1) and Lemma 6.1 it follows that

$$\begin{aligned} E[U^{(i)}\mid K_N] &= 2\iint_{x<y}\prod_{j\neq i}[F_j(y)-F_j(x)]^2 dF_i(x)dF_i(y) \\ &= 2\iint_{x<y}[F(y)-F(x)]^{2c-2}dF(x)dF(y)+\frac{2\sum_{\alpha=1}^c(\theta_i-\theta_\alpha)}{\sigma\sqrt{N}} \\ &\quad \times\iint_{x<y}[yf(y)-xf(x)][F(y)-F(x)]^{2c-3}dF(x)dF(y)+o\left(\frac{1}{\sqrt{N}}\right) \\ &= \frac{1}{c(2c-1)}+\frac{2\sum_{\alpha=1}^c(\theta_i-\theta_\alpha)}{\sigma\sqrt{N}(c-1)}\int_{-\infty}^{\infty} xf(x) \\ &\quad \times\{[F(x)]^{2c-2}-[1-F(x)]^{2c-2}\}\,dF(x)+o\left(\frac{1}{\sqrt{N}}\right). \end{aligned} \tag{8.2}$$

The proof left is the same as in Theorem 6.1.

From the same reason as for the $V_{rs}$-test and the $D_{rs}$-test, asymptotically best critical region for $Q$-staistic is given by $Q>Q_0$ where $Q_0$ is a preassigned constant. This test will be called $Q$-test.
It is interesting to note that the noncentrality parameter given by (8.1) is formally the same as $\lambda^2_{2c-1,2c-1}$ in (6.3), though $D_{2c-1,2c-1}$-test cannot be defined.

If the distribution function $F(x)$ is normal with the mean zero, the asymptotically UMP invariant test of the hypothesis $H$: $\sigma_1=\cdots=\sigma_c$ proposed by Lehmann [20, p. 275] is to reject the hypothesis $H$ if the statistic

$$L=\sum_{i=1}^{c}\frac{1}{a_i^2}\left[Z_i-\frac{\sum_{j=1}^{c}Z_j/a_j^2}{\sum_{j=1}^{c}1/a_j^2}\right]^2 \tag{8.3}$$

yields the observed value larger than the preassigned constant, where $Z_i=\log(\sum_{\alpha=1}^{n_i}X_{i\alpha}^2/n_i)$ and $a_i^2=2/n_i$. The limiting distribution of $L$ is $\chi^2$ with $c-1$ degrees of freedom undr the hypothesis and noncentral $\chi^2$ with $c-1$ degrees of freedom under the sequence of alternatives $K_N$ as $N\to\infty$, whose noncentrality parameter is given by

$$\frac{2}{\sigma^2}\sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2. \tag{8.4}$$

Hence from (8.1) and (8.4) the relative efficiency of the $Q$-test against the asymptotically UMP invariant test for normal alternative is equal to

$$\begin{aligned}\varepsilon_{Q/L}&=\frac{2(2c-1)^2}{L(2c-1,\,2c-1)}\left(\int_{-\infty}^{\infty}x[f(x)]^2[F(x)]^{2c-2}dx\right)^2\\&=\frac{1}{2c^2L(2c-1,\,2c-1)}[E(X_{2c/2c}^2)-1]^2,\end{aligned} \tag{8.5}$$

where $X_{2c|2c}$ means the maximum among the random sample of size $2c$ from the standard normal distribution. Some numerical values are shown below.

| $c$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon_{Q/L}$ | .760 | .812 | .864 | .898 | .918 | .928 | .931 | .929 | .925 |

From (6.5) and (8.4) we have

$$\varepsilon_{D_{22}/L}=90\left(\int_{-\infty}^{\infty}xf(x)[2F(x)-1]dF(x)\right)^2=15/2\pi^2\quad(=.760) \tag{8.6}$$

which is equal to the asymptotic efficiency of Mood's square rank test for dispersion [24] as well as Tamura's $Q$-test [29] against the variance ratio $F$-test for normal distribution.

If the distribution function $F(x)$ is double exponential, it is shown that the asymptotically UMP invariant test exists, which is given in fact by (8.3), where $Z_i=\log(\sum_{j=1}^{n_i}|X_{ij}|/n_i)$ and $a_i^2=1/n_i$. The limiting distribution of $L$ as $N\to\infty$, is $\chi^2$ with $c-1$ degrees of freedom under the hypothesis, whereas it is noncentral $\chi^2$ with $c-1$ degrees of freedom under the sequence of alternatives $K_N$, with the noncentrality parameter $\sigma^{-2}\sum_{i=1}^{c}\rho_i(\theta_i-\tilde{\theta})^2$. Hence we have from (8.1)

$$\varepsilon_{Q/L}=\frac{(2c-1)^2}{L(2c-1,\,2c-1)}\left[\frac{1}{2}\sum_{j=0}^{2c-2}\binom{2c-2}{j}\frac{(-1)^j}{2^j(j+2)^2}-\frac{1}{2^{2c+1}c^2}\right]^2. \tag{8.7}$$

Some numerical values are shown below.

| $c$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon_{Q/L}$ | .868 | .902 | .928 | .936 | .933 | .924 | .910 | .895 | .879 |

From (6. 5) we have

$$(8.8)\qquad \varepsilon_{D_{22}/L} = 180\left\{\int_{-\infty}^{\infty} xf(x)[2F(x)-1]dF(x)\right\}^2 = 125/144 \qquad (= .868).$$

Finally if the distribution function $F(x)$ is exponential, the asymptotically UMP invariant test is given by (8. 3), where $Z_i=\log(\sum_{j=1}^{n_i} X_{ij}/n_i)$ and $a_i^2=1/n_i$. The limiting distribution of $L$ is the same as in the double exponential distribution. Hence from (8. 1) we have

$$(8.9)\qquad \varepsilon_{Q/L} = \frac{1}{4c^2L(2c-1,\, 2c-1)}\left[\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2c}\right)-\frac{2c-1}{2c}\right]^2.$$

Some numerical values are shown below.

| $c$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon_{Q/L}$ | .139 | .148 | .157 | .162 | .165 | .166 | .166 | .166 | .164 |

From (6. 5) we have $\varepsilon_{D_{22}/L}=5/36$ $(=.139)$.

From these values we can see that our $Q$-test is more efficient against $D_{22}$-test for normal, double exponential and exponential distributions at least $3\leqq c\leqq 10$, and further for normal and double exponential distributions the efficiency of our $Q$-test against the parametric test is considerably high.

## 9. A multivariate Wilcoxon test

Since the famous Wilcoxon test was introduced by Wilcoxon and Mann and Whitney, many generalizations to the multisample case have been done, but as far as the author is aware generalization to the multivariate case seems not to be attempted. Recently Bickel [7] considered one sample problem in multivariate case that is to test the hypothesis $H$; $\theta=0$ against the alternative $K$: $\theta\neq 0$ when a sample $x_1, x_2, \cdots, x_n$ is drawn from the $p$-variate distribution $F(x-\theta)$. He proposed two tests due to the statistics $M_n$ and $W_n$ where

$$(9.1)\qquad \begin{aligned} M_n &= (\underset{1\leqq\alpha\leqq n}{\text{Median}}\, X_{1\alpha}, \cdots, \underset{1\leqq\alpha\leqq n}{\text{Median}}\, X_{p\alpha})' \\ W_n &= \left(\underset{1\leqq\alpha\leqq\beta\leqq n}{\text{Median}}\, \frac{X_{1\alpha}+X_{1\beta}}{2}, \cdots, \underset{1\leqq\alpha\leqq\beta\leqq n}{\text{Median}}\, \frac{X_{p\alpha}+X_{p\beta}}{2}\right)', \end{aligned}$$

and $x_\alpha=(X_{1\alpha}, X_{2\alpha}, \cdots, X_{p\alpha})'$.

These tests are shown by Bickel to be asymptotically equivalent to those $\hat{M}_n$ and $\hat{W}_n$ constructed respectively from

$$(9.2)\qquad U_M^{(i)} = \frac{1}{n}\sum_{\alpha=1}^{n}\phi^{(i)}(x_\alpha)$$
$$\phi^{(i)}(x) = \begin{cases} 1 & \text{if the } i\text{-th component of } x>0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(9.3)\qquad U_W^{(i)} = \frac{2}{n(n+1)}\sum_{1\leq\alpha\leq\beta\leq n}\phi^{(i)}(x_\alpha+x_\beta)\,.$$

In this section we shall consider the multivariate nonparametric two-sample problem of location and propose a Wilcoxon analogue corresponding to (9.2). The asymptotic efficiency of this test is shown to be equal to that of Bickel's $\hat{W}_n$-test [7] for the one-sample problem. Let $p$-variate column vectors $x_1, \cdots, x_{n_1}$; $y_1, \cdots, y_{n_2}$ constitute two independent random samples from the absolutely continuous distrbution $F(x)$ and $F(x-\theta)$ respectively. Based on these samples we want to test the hypothesis $H$: $\vartheta=0$ against the alternative $K$: $\vartheta\neq 0$. Put

$$(9.4)\qquad U^{(i)} = \frac{1}{n_1 n_2}\sum_{\alpha=1}^{n_1}\sum_{\beta=1}^{n_2}\phi^{(i)}(x_\alpha, y_\beta)$$
$$\phi^{(i)}(x, y) = \begin{cases} 1 & \text{if the } i\text{-th component of } x \text{ is larger} \\ & \text{than the } i\text{-th component of } y \\ 0 & \text{otherwise.} \end{cases}$$

Then the statistic $U^{(i)}$ is a multivariate generalized $U$ statistic stated in section 2. So under the hypothesis $H$

$$(9.5)\qquad E[U^{(i)}] = 1/2\,,$$

and by Theorem 2.1 the statistic $\sqrt{N}\left(U^{(1)}-\frac{1}{2}, \cdots, U^{(p)}-\frac{1}{2}\right)$ where $n_i=\rho_i N$ for some fixed $\rho_i$ is distributed asymptotically normal as $N\to\infty$ with the mean vector $0$ and the covariance matrix $T=(t_{ij})$;

$$(9.6)\qquad t_{ij} = \frac{\zeta_{10}^{(i,j)}}{\rho_1}+\frac{\zeta_{01}^{(i,j)}}{\rho_2}\,,$$

where, denoting the $i$-th component of $x$ and $y$ by $X_i$ and $Y_i$,

$$\text{(i)}\qquad \zeta_{01}^{(i,i)} = \zeta_{10}^{(i,i)} = \mathrm{Cov}[\phi^{(i)}(x, y), \phi^{(i)}(x, y')]$$
$$= P(X_i > Y_i \text{ and } Y_i') - \frac{1}{4}$$
$$= 1/12$$

(ii)
$$\zeta_{01}^{(i,j)}(i \neq j) = \zeta_{10}^{(i,j)}(i \neq j) = \mathrm{Cov}[\phi^{(i)}(\boldsymbol{x}, \boldsymbol{y}), \phi^{(j)}(\boldsymbol{x}, \boldsymbol{y}')]$$
$$= P(X_i > Y_i \text{ and } X_j > Y_j') - \frac{1}{4}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(x_i)F_j(x_j)dF_{ij}(x_i, x_j) - \frac{1}{4};$$

$F_i(x)$, $F_j(x)$ and $F_{ij}(x, y)$ are the marginal distribution functions of $F(\boldsymbol{x})$ with respect to $X_i$, $X_j$ and $(X_i, X_j)$, respectively. Hence from (9.6) we have

$$t_{ij} = \frac{1}{12}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\tau_{ij} \tag{9.7}$$

where

$$\tau_{ij} = \begin{cases} 1 & \text{if } i = j \\ 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(x)F_j(y)dF_{ij}(x, y) - 3 & \text{if } i \neq j. \end{cases}$$

If we substitute for $F_i(x_i)$, $F_j(x_j)$ and $F_{ij}(x_i, x_j)$ by its empirical distribution $F_i^{(N)}(x_i)$, $F_j^{(N)}(x_j)$ and $F_{ij}^{(N)}(x_i, x_j)$ constructed from the combined sample $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n_1}$, and $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{n_2}$ and put

$$\hat{\tau}_{ij} = \begin{cases} 1 & \text{if } i = j \\ 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i^{(N)}(x_i)F_j^{(N)}(x_j)dF_{ij}^{(N)}(x_i, x_j) - 3 & \text{if } i \neq j, \end{cases} \tag{9.8}$$

then the statistic $\hat{\tau}_{ij}$ converges in probability to $\tau_{ij}$ as $N \to \infty$. If further we wish to obtain an unbiased estimate of $\tau_{ij}$, we can use as Bickel [7],

$$\hat{\tau}_{ij} = \begin{cases} 1 & \text{if } i = j \\ \dfrac{12}{(n_1+n_2)_3}\sum_{\alpha \neq \beta \neq \gamma} I(Z_\alpha^{(i)} - Z_\beta^{(i)}; Z_\alpha^{(j)} - Z_\gamma^{(j)}) - 3 & \text{if } i \neq j \end{cases} \tag{9.9}$$

where $Z_\alpha^{(i)} = i$-th component of $\boldsymbol{x}_\alpha$ $\alpha = 1, 2, \cdots, n_1$ and $Z_{n_1+\alpha} = i$-th component of $\boldsymbol{y}_\alpha$ $\alpha = 1, 2, \cdots, n_2$. From Lemma 3.2 we easily have

**Lemma 9.1.** *If the random vector $\boldsymbol{x}_N$ converges in law to the random vector $\boldsymbol{x}$, and futher the random vector $\boldsymbol{y}_N$ converges in probability to $\boldsymbol{c}$, then $g(\boldsymbol{x}_N, \boldsymbol{y}_N)$ converges in law to $g(\boldsymbol{x}, \boldsymbol{c})$ for any continuous function $g(\boldsymbol{x}, \boldsymbol{y})$.*

Applying Lemma 9.1 with $\boldsymbol{x}_N = \sqrt{N}\left(U^{(1)} - \frac{1}{2}, \cdots, U^{(p)} - \frac{1}{2}\right)$ and $\boldsymbol{y}_N = \hat{\tau}_{ij}$, we obtain the following theorem.

**Theorem 9.1.** *Put $n_i = \rho_i N$ where $\rho_i > 0$ is independent of $N$, and* *assume that* $\det(\tau_{ij}) \neq 0$. *Then under the hypothesis* $H: \boldsymbol{\theta}=\mathbf{0}$, *the statistic*

$$W = \frac{12n_1n_2}{n_1+n_2} \sum_{i,j=1}^{p} \hat{\tau}^{ij} \left(U^{(i)} - \frac{1}{2}\right)\left(U^{(j)} - \frac{1}{2}\right) \tag{9.10}$$

*is distributed asymptotically as* $\chi^2$ *with* $p$ *degrees of freedom as* $N \to \infty$, *where* $(\hat{\tau}^{ij})$ *is the inverse matrix of* $(\hat{\tau}_{ij})$ *given by* (9.8) *or* (9.9).

Exact covariance of $U^{(i)}$ and $U^{(j)}$ is also obtained from Theorem 2.1.

$$\mathrm{Cov}\,[U^{(i)}, U^{(j)}] = \begin{cases} (n_1+n_2+1)/12n_1n_2 & \text{if } i=j \\ [(n_1+n_2-2)\tau_{ij}+\varepsilon_{ij}]/12n_1n_2 & \text{if } i \neq j, \end{cases} \tag{9.11}$$

where

$$\varepsilon_{ij} = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{ij}(x_i, x_j)\,dF_{ij}(x_i, x_j) - 3\,.$$

To investigate the asymptotic relative efficiency we shall consider the distribution of $W$ under the following sequence of alternatives $K_N$ that the sample $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{n_2}$ is drawn from the distribution $G(\boldsymbol{x})$ where

$$K_N: G(\boldsymbol{x}) = F(\boldsymbol{x} - N^{-1/2}\boldsymbol{\theta})\,, \tag{9.12}$$

and $\theta$ is some constant vector different from zero.

**Theorem 9.2.** *Suppose that each one-dimensional marginal distribution* $F_i(x)$ *of* $F(\boldsymbol{x})$ *satisfies the assumption of Theorem* 4.1. *Then under the sequence of alternatives* $K_N$, *the limiting distribution of* $W$ *in Theorem* 9.1 *is noncentral* $\chi^2$ *with* $p$ *degrees of feedom and the noncentrality parameter*

$$\lambda_W^2 = \frac{12\rho_1\rho_2}{\rho_1+\rho_2} \sum_{i,j=1}^{p} \tau^{ij}\theta_i\theta_j \int_{-\infty}^{\infty} f_i(x)\,dF_i(x) \int_{-\infty}^{\infty} f_j(x)\,dF_j(x) \tag{9.13}$$

*as* $N \to \infty$, *where* $n_i = \rho_i N$ *for fixed* $\rho_i > 0$ *and* $\theta = (\theta_1, \cdots, \theta_p)'$. *The matrix* $(\tau^{ij}) = (\tau_{ij})^{-1}$ *given by* (9.7) *is assumed to exist.*

Proof. It is clear from the assumption that

$$E[U^{(i)} | K_N] = \frac{1}{2} - \frac{\theta_i}{\sqrt{N}} \int_{-\infty}^{\infty} f_i(x)\,dF_i(x) + o\left(\frac{1}{\sqrt{N}}\right). \tag{9.14}$$

Also by Theorem 2.1 we easily see that

$$N \cdot \mathrm{Cov}\,[U^{(i)}, U^{(j)} | K_N] = \frac{1}{12}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\tau_{ij} + O\left(\frac{1}{\sqrt{N}}\right). \tag{9.15}$$

The estimater $\hat{\tau}_{ij}$ converges in probability to $\tau_{ij}$ even under $K_N$, since $E[\hat{\tau}_{ij} | K_N] = E[\hat{\tau}_{ij} | H] + O(N^{-1/2})$ and $\mathrm{Var}[\hat{\tau}_{ij} | K_N] = O(N^{-1})$. Using the asymptotic normality of the statistic $\sqrt{N}\left(U^{(1)}-\frac{1}{2}, \cdots, U^{(p)}-\frac{1}{2}\right)$ under $K_N$, we get the conclusion.

From Theorem 9.1 and Theorem 9.2 the hypothesis may be considered as $\lambda_W^2=0$ and the alternatives as $\lambda_W^2\neq 0$ in the limiting distribution of $W$. Hence the asymptotically best critical region for $W$ is given by $W>W_0$ where $W_0$ is a preassigned constant. This test will be called $W$-test. In case $p=1$ this test reduces to the ordinary two-sided Wilcoxon test.

If the distribution $F(x)$ is known to be normal, the standard test for the hypothesis $H$: $\theta=0$ against $K$: $\theta\neq 0$ is Hotelling's two-sample $T^2$-test: Putting

$$(9.16)\qquad \begin{aligned} T^2 &= \frac{n_1 n_2}{n_1+n_2}(\bar{x}-\bar{y})' S^{-1}(\bar{x}-\bar{y}) \\ S &= \frac{1}{n_1+n_2-2}\Big[\sum_{\alpha=1}^{n_1}(x_\alpha-\bar{x})(x_\alpha-\bar{x})'+\sum_{\beta=1}^{n_2}(y_\beta-\bar{y})(y_\beta-\bar{y})'\Big], \end{aligned}$$

we reject the hypothesis when the observed value of $T^2$ is larger than a preassigned constant. (See for example Anderson [1, p. 109]). By the central limit theorem, the statistic $T^2$ is distributed asymptotically as noncentral $\chi^2$ with $p$-degrees of freedom and noncentrality parameter

$$(9.17)\qquad \lambda^2_{T^2} = \frac{\rho_1\rho_2}{\rho_1+\rho_2}\sum_{i,j=1}^{p}\sigma^{ij}\theta_i\theta_j$$

as $N\to\infty$, even if the distribution $F(x)$ is not normal. The matrix $(\sigma^{ij})$ is the inverse of the population covariance matrix. From Theorem 9.2 the efficiency of the $W$-test against the $T^2$-test is given by

$$(9.18)\qquad \mathcal{E}_{W/T^2} = 12\sum_{i,j=1}^{p}\tau^{ij}\theta_i\theta_j\int_{-\infty}^{\infty} f_i(x)\,dF_i(x)\int_{-\infty}^{\infty} f_j(x)\,dF_j(x)\Big/\sum_{i,j=1}^{p}\sigma^{ij}\theta_i\theta_j\,.$$

This expression is the same as the asymptotic relative efficiency of Bickel's $\hat{W}_n$-test [7] against Hotelling's $T^2$-test for the one-sample problem.

If further the distribution $F(x)$ is normal with the mean $\mu$ and the covariance matrix $(\sigma_i\sigma_j\rho_{ij})$,

$$(9.19)\qquad \mathcal{E}_{W/T^2} = \frac{3}{\pi}\sum_{i,j=1}^{p}\tau^{ij}\frac{\theta_i\theta_j}{\sigma_i\sigma_j}\Big/\sum_{i,j=1}^{p}\rho^{ij}\frac{\theta_i\theta_j}{\sigma_i\sigma_j}\,,$$

where $(\rho^{ij})=(\rho_{ij})^{-1}$ and $\tau_{ij}$ is given by (9.7). By Bickel [6] and Kendall [15, p. 351] we have

$$P(X_i > Y_i \text{ and } X_j > Y_j') = P(X_i - Y_i > 0 \text{ and } X_j - Y_j' > 0)$$

(9.20)
$$= \frac{1}{4} + \frac{1}{2\pi} \operatorname{Sin}^{-1} \frac{\rho_{ij}}{2}$$

which yields

(9.21)
$$\tau_{ij} = \frac{6}{\pi} \operatorname{Sin}^{-1} \frac{\rho_{ij}}{2}.$$

If we search for the maximum or the minimum value of $\frac{\pi}{3} \times \mathcal{E}_{W/T^2}$ with respect to $\sigma_i$ and $\theta_j$ for all $i, j = 1, 2, \cdots, p$, it is given by the root of the following determinantal equation,

(9.22)
$$\begin{vmatrix} 1-\lambda, & \rho_{12}-\lambda\tau_{12}, & \cdots & \rho_{1p}-\lambda\tau_{1p} \\ \rho_{21}-\lambda\tau_{21}, & 1-\lambda, & \cdots & \rho_{2p}-\lambda\tau_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{p1}-\lambda\tau_{p1}, & \rho_{p2}-\lambda\tau_{p2}, & \cdots & 1-\lambda \end{vmatrix} = 0.$$

In case of $\rho_{ij} = \rho_{12}$ for all $i, j = 1, 2, \cdots, p$ $(i \neq j)$, the solution of the above equation is given by

(9.23)
$$\lambda = \frac{1-\rho_{12}}{1-\tau_{12}} \text{ or } \frac{1+(p-1)\rho_{12}}{1+(p-1)\tau_{12}},$$

and the extreme value of $\mathcal{E}_{W/T^2}$ is given by

(9.24)
$$\frac{1-\rho_{12}}{(\pi/3) - 2\operatorname{Sin}^{-1}(\rho_{12}/2)} \quad \text{or} \quad \frac{1+(p-1)\rho_{12}}{(\pi/3) + 2(p-1)\operatorname{Sin}^{-1}(\rho_{12}/2)}.$$

For the simplest multivariate case of $p=2$ we get

(9.25)
$$\frac{1-\rho_{12}}{(\pi/3) - 2\operatorname{Sin}^{-1}(\rho_{12}/2)} \quad \text{or} \quad \frac{1+\rho_{12}}{(\pi/3) + 2\operatorname{Sin}^{-1}(\rho_{12}/2)},$$

which coincides with Bickel's result though it contains some misprints and (9.25) seems to be simpler. The property of the extreme value of $\mathcal{E}_{W/T^2}$ given by (9.25) considered as a function of $\rho_{12}$ is investigated by Bickel [7], and he showed that $\max_{\theta,\sigma} \mathcal{E}_{W/T^2} \geqq 3/\pi$ and $\min_{\theta,\sigma} \mathcal{E}_{W/T^2} \geqq \sqrt{3}/2$. So we shall show some numerical values.

| $\rho_{12}$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\max_{\theta,\sigma} \mathcal{E}_{W/T^2}$ | .955 | .959 | .962 | .964 | .966 | .966 | .966 | .965 | .962 | .959 | .955 |
| $\min_{\theta,\sigma} \mathcal{E}_{W/T^2}$ | .955 | .950 | .945 | .938 | .931 | .923 | .914 | .903 | .892 | .880 | .866 |

For $\rho_{12} < 0$, the maximum or the minimum of $\mathcal{E}_{W/T^2}$ is the same as for $\rho_{12} > 0$.

## 10. Multivariate $V_{rs}$-test

In the following two sections we shall propose some multivariate nonparametric several-sample tests for location and derive the asymptotic distributions of their test statistics. In the problem of two-samples these test reduces to the multivariate Wilcoxon test discussed in section 9. We shall first generalize the $V_{rs}$-test discussed in section 3 and 4 to the multivariate case.

Let $p\times 1$ vectors $\boldsymbol{x}_{\alpha 1},\cdots,\boldsymbol{x}_{\alpha n_\alpha}$ be a random sample from a continuous distribution $F_\alpha(\boldsymbol{x})$ $(\alpha=1, 2, \cdots, c)$. The problem is to test the hypothesis $H: F_1=F_2=\cdots=F_c$ against the location alternatives $K: F_\alpha(\boldsymbol{x})=(\boldsymbol{x}-\boldsymbol{\theta}_\alpha)$ $\alpha=1, 2, \cdots, c$ where $\boldsymbol{\theta}_\alpha$ is some constant vectors (not all $\boldsymbol{\theta}$'s are equal). Put for $\alpha=1, 2, \cdots, c$ and $i=1, 2, \cdots, p$

(10.1)
$$U_\alpha^{(i)} = \frac{1}{n_1\cdots n_c}\sum_{\alpha_1=1}^{n_1}\cdots\sum_{\alpha_c=1}^{n_c}\phi_\alpha^{(i)}(\boldsymbol{x}_{1\alpha_1},\cdots,\boldsymbol{x}_{c\alpha_c})$$
$$\phi_\alpha^{(i)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_c) = \frac{(j-1)_r}{(c-1)_r}-\frac{(c-j)_s}{(c-1)_s}$$
if the $i$-th component of $\boldsymbol{x}_\alpha$ is the $j$-th smallest among the $i$-th components of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_c$.

Then the expectation of $\boldsymbol{U}^{(i)}=(U_1^{(i)},\cdots, U_c^{(i)})$ under the hypothesis is the same as (3.5), that is

$$E[\boldsymbol{U}^{(i)}] = \left(\frac{1}{r+1}-\frac{1}{s+1}, \cdots, \frac{1}{r+1}-\frac{1}{s+1}\right). \tag{10.2}$$

By Theorem 2.1 the statistic $\sqrt{N}(\boldsymbol{U}-E[\boldsymbol{U}])=\sqrt{N}(\boldsymbol{U}^{(1)}-E[\boldsymbol{U}^{(1)}], \cdots, \boldsymbol{U}^{(p)}-E[\boldsymbol{U}^{(p)}])$ is distributed asymptotically normal as $N\to\infty$ under the hypothesis with the mean 0 and the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{\alpha\beta}^{(i,j)})$ where $\sigma_{\alpha\beta}^{(i,j)}$ means the covariance of $U_\alpha^{(i)}$ and $U_\beta^{(j)}$. The covariance $\sigma_{\alpha\beta}^{(i,i)}$ is the same as (3.20), that is

$$\sigma_{\alpha\beta}^{(i,i)} = \frac{K(r, s)}{(c-1)^2}\left[\sum_{l=1}^{c}\frac{1}{\rho_l}-\frac{c}{\rho_\alpha}-\frac{c}{\rho_\beta}+\frac{c^2\delta_{\alpha\beta}}{\rho_\alpha}\right]. \tag{10.3}$$

In case $i\neq j$ the covariance $\sigma_{\alpha\beta}^{(i,j)}$ is given by

$$\sigma_{\alpha\beta}^{(i,j)} = \frac{1}{\rho_1}\zeta_{10,\cdots,0}^{(\alpha,\beta)}+\cdots+\frac{1}{\rho_c}\zeta_{00,\cdots,1}^{(\alpha,\beta)}, \tag{10.4}$$

where $\zeta_{0,\cdots,1,\cdots,0}^{(\alpha,\beta)}$ (1 lies at the $\gamma$-th place) means the covariance of $\phi_\alpha^{(i)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_c)$ and $\phi_\beta^{(j)}(\boldsymbol{x}_1', \cdots, \boldsymbol{x}_{\gamma-1}', \boldsymbol{x}_\gamma, \boldsymbol{x}_{\gamma+1}', \cdots, \boldsymbol{x}_c')$. We shall calculate $\zeta_{0,\cdots,1,\cdots,0}^{(\alpha,\beta)}$ by considering the following three cases. The functions $F^{(i)}$, $F^{(j)}$ and $F^{(i,j)}$ mean the marginal distributions of $F=F_1=\cdots=F_c$ with respect to the components of their superscripts.

$$(10.5)\quad \begin{aligned} &(\mathrm{i})\ \zeta^{(\alpha,\alpha)}_{0,\cdots,1,\cdots,0} && (1 \text{ lies at the } \alpha\text{-th place}) = a_{ij} \\ &(\mathrm{ii})\ \zeta^{(\alpha,\beta)}_{0,\cdots,1,\cdots,0} && (1 \text{ lies at the } \gamma\text{-th place, where } \gamma \neq \alpha,\ \beta) \\ & && = \frac{a_{ij}}{(c-1)^2} \\ &(\mathrm{iii})\ \zeta^{(\alpha,\beta)}_{0,\cdots,1,\cdots,0} && (1 \text{ lies at the } \alpha\text{-th place, where } \alpha \neq \beta) \\ & && = -\frac{a_{ij}}{c-1}, \end{aligned}$$

where

$$(10.6)\quad \begin{aligned} a_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\left\{[F^{(i)}(x)]^r - [1-F^{(i)}(x)]^s\right\} \\ &\times \left\{[F^{(j)}(y)]^r - [1-F^{(j)}(y)]^s\right\} dF^{(i,j)}(x,y) - \left(\frac{1}{r+1} - \frac{1}{s+1}\right)^2. \end{aligned}$$

In case $i=j$, these results reduces to (3.17), (3.18) and (3.19) respectively.
Substituting these results of (i), (ii) and (iii) into (10.4), we get

$$(10.7)\quad \sigma^{(i,j)}_{\alpha\beta} = \frac{a_{ij}}{(c-1)^2}\left[\sum_{l=1}^{c}\frac{1}{\rho_l} - \frac{c}{\rho_\alpha} - \frac{c}{\rho_\beta} + \frac{c^2\delta_{\alpha\beta}}{\rho_\alpha}\right].$$

The above equation holds true also in case $i=j$, if we define $a_{ii}=K(r,s)$. It is easy to see that an example of the consistent and unbiased estimate of $a_{ij}$ is given by

$$(10.8)\quad \hat{a}_{ij} = \begin{cases} K(r,s) & i=j \\ \hat{a}^{(1)}_{ij} - \hat{a}^{(2)}_{ij} - \hat{a}^{(3)}_{ij} + \hat{a}^{(4)}_{ij} - \left(\dfrac{1}{r+1} - \dfrac{1}{s+1}\right)^2 & i \neq j \end{cases}$$

where $\hat{a}^{(1)}_{ij}$, $\hat{a}^{(2)}_{ij}$, $\hat{a}^{(3)}_{ij}$ and $\hat{a}^{(4)}_{ij}$ are the estimates of $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F^{(i)}(x)]^r[F^{(j)}(y)]^r dF^{(i,j)}(x,y)$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F^{(i)}(x)]^r[1-F^{(j)}(y)]^s dF^{(i,j)}(x,y)$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1-F^{(i)}(x)]^s[F^{(j)}(y)]^r dF^{(i,j)}(x,y)$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1-F^{(i)}(x)]^s[1-F^{(j)}(y)]^s dF^{(i,j)}(x,y)$ respectively. They are defined by

$$(10.9)\quad \begin{aligned} \hat{a}^{(1)}_{ij} &= [(\textstyle\sum_{l=1}^{c} n_l)_{2r+1}]^{-1}\sum_{(\alpha,\beta,\gamma)} I(Z^{(i)}_\alpha - Z^{(i)}_{\beta_1}, \cdots, \\ &\qquad Z^{(i)}_\alpha - Z^{(i)}_{\beta_r};\ Z^{(j)}_\alpha - Z^{(j)}_{\gamma_1}, \cdots, Z^{(j)}_\alpha - Z^{(j)}_{\gamma_r}) \\ \hat{a}^{(2)}_{ij} &= [(\textstyle\sum_{l=1}^{c} n_l)_{r+s+1}]^{-1}\sum_{(\alpha,\beta,\gamma)} I(Z^{(i)}_\alpha - Z^{(i)}_{\beta_1}, \cdots, \\ &\qquad Z^{(i)}_\alpha - Z^{(i)}_{\beta_r};\ Z^{(j)}_{\gamma_1} - Z^{(j)}_\alpha, \cdots, Z^{(j)}_{\gamma_s} - Z^{(j)}_\alpha) \\ \hat{a}^{(3)}_{ij} &= [(\textstyle\sum_{l=1}^{c} n_l)_{r+s+1}]^{-1}\sum_{(\alpha,\beta,\gamma)} I(Z^{(i)}_{\beta_1} - Z^{(i)}_\alpha, \cdots, \\ &\qquad Z^{(i)}_{\beta_s} - Z^{(i)}_\alpha;\ Z^{(j)}_\alpha - Z^{(j)}_{\gamma_1}, \cdots, Z^{(j)}_\alpha - Z^{(j)}_{\gamma_r}) \\ \hat{a}^{(4)}_{ij} &= [(\textstyle\sum_{l=1}^{c} n_l)_{2s+1}]^{-1}\sum_{(\alpha,\beta,\gamma)} I(Z^{(i)}_{\beta_1} - Z^{(i)}_\alpha, \cdots, \\ &\qquad Z^{(i)}_{\beta_s} - Z^{(i)}_\alpha;\ Z^{(j)}_{\gamma_1} - Z^{(j)}_\alpha, \cdots, Z^{(j)}_{\gamma_s} - Z^{(j)}_\alpha), \end{aligned}$$

where $Z_\alpha^{(i)}$, $\alpha=1, 2, \cdots, \sum_{l=1}^c n_l$ means the $i$-th component of $x_{\beta\gamma}$, $\gamma=1, 2, \cdots, n_\beta$ and $\beta=1, 2, \cdots, c$; $I(X_1, \cdots, X_k; Y_1, \cdots, Y_l)=1$ if all $X$'s and $Y$'s are positive and $=0$ otherwise. The summation on $\sum_{(\alpha,\beta,\gamma)}$ in $\hat{a}_{ij}^{(1)}$ extends over all possible sets of integers $(\alpha, \beta_1, \cdots, \beta_r, \gamma_1, \cdots, \gamma_r)$ such that $1 \leqq \alpha, \beta_1, \cdots, \beta_r, \gamma_1, \cdots, \gamma_r \leqq \sum_{l=1}^c n_l$ aud any two elements of them are different from each other. It follows that $\mathrm{Var}[\hat{a}_{ij}^{(\alpha)}]=O(N^{-1})$ under the hypothesis $H$, so $\hat{a}_{ij}^{(\alpha)}$ is the consistent and unbiased estimate of $E[\hat{a}_{ij}^{(\alpha)} | H]$ for $\alpha=1, 2, 3, 4$.

From (10.7) we can express the covariance matrix in the form $\boldsymbol{\Sigma}=\boldsymbol{A}_0 \otimes \boldsymbol{\Sigma}_0$ where $\boldsymbol{A}_0=(a_{ij})$, $\boldsymbol{\Sigma}_0=[\sum_{l=1}^c \rho_l^{-1}-c\rho_\alpha^{-1}-c\rho_\beta^{-1}+c^2\delta_{\alpha\beta}\rho_\alpha^{-1}]/(c-1)^2$ and $\otimes$ means the Kronecker product. If we assume the $p\times p$ matrix $\boldsymbol{A}_0$ is nonsingular, the rank of $\boldsymbol{\Sigma}$ is equal to $r(\boldsymbol{A}_0)\times r(\boldsymbol{\Sigma}_0)=p(c-1)$.

**Lemma 10.1.** *If the covariance matrix $\boldsymbol{\Sigma}=\boldsymbol{A}_0\otimes\boldsymbol{\Sigma}_0$ is given by* (10.7) *and $\boldsymbol{A}_0=(a_{ij})$ is nonsingular, then the solution in Lemma* 4.2 *is given by $\boldsymbol{\Lambda}=\boldsymbol{A}_0^{-1}\otimes\boldsymbol{\Lambda}_0$ where*

$$\boldsymbol{\Lambda}_0 = \frac{(c-1)^2}{c^2}\left[\delta_{\alpha\beta}\rho_\alpha - \frac{\rho_\alpha\rho_\beta}{\sum_{\alpha=1}^c \rho_\alpha}\right]. \tag{10.10}$$

Proof. In this case the projection $\boldsymbol{B}$ in Lemma 4.2 is given by $\boldsymbol{B}=\boldsymbol{I}_p\otimes\boldsymbol{B}_0$ where $\boldsymbol{I}_p$ means the $p\times p$ unit matrix and $\boldsymbol{B}_0$ is given by (4.13). If we apply Lemma 4.2 for the covariance matrix $\boldsymbol{\Sigma}_0$, the projection is given by $\boldsymbol{B}_0$ and the solution of (4.12) is given by $\boldsymbol{\Lambda}_0$. Hence $\boldsymbol{\Sigma}_0\boldsymbol{\Lambda}_0=\boldsymbol{I}_c-\boldsymbol{B}_0$ and $\boldsymbol{B}_0\boldsymbol{\Lambda}_0=\boldsymbol{0}$. Using the formala $(\boldsymbol{A}_1\otimes\boldsymbol{A}_2)\cdot(\boldsymbol{B}_1\otimes\boldsymbol{B}_2)=(\boldsymbol{A}_1\cdot\boldsymbol{B}_1)\otimes(\boldsymbol{A}_2\cdot\boldsymbol{B}_2)$, we easily see that $\boldsymbol{\Sigma\Lambda}=\boldsymbol{I}_{pc}-\boldsymbol{B}$ and $\boldsymbol{B\Lambda}=\boldsymbol{0}$. Hence $\boldsymbol{\Lambda}=\boldsymbol{A}_0^{-1}\otimes\boldsymbol{\Lambda}_0$ is the unique solution of the equation (4.12).

Calculating the quadratic form $N(\boldsymbol{U}^{(1)}-E[\boldsymbol{U}^{(1)}], \cdots, \boldsymbol{U}^{(p)}-E[\boldsymbol{U}^{(p)}])$ $\boldsymbol{\Lambda}(\boldsymbol{U}^{(1)}-E[\boldsymbol{U}^{(1)}], \cdots, \boldsymbol{U}^{(p)}-E[\boldsymbol{U}^{(p)}])'$, in view of Lemma 4.2 we have the following theorem from Lemma 3.2.

**Theorem 10.1.** *Put $n_i=\rho_i N$ where $\rho_i>0$ is independent of $N$ and suppose that $\det(\boldsymbol{A})\neq 0$ where the $p\times p$ matrix $\boldsymbol{A}=(a_{ij})$ is given by* (10.6), *then under the hypothesis $H: F_1=\cdots=F_c=F$, the statistic $\tilde{V}_{rs}$ defined by*

$$\begin{aligned} \tilde{V}_{rs} &= \frac{(c-1)^2}{c^2}\sum_{i,j=1}^c \hat{a}^{ij}\cdot\sum_{\alpha=1}^c n_\alpha(U_\alpha^{(i)}-\tilde{U}^{(i)})(U_\alpha^{(j)}-\tilde{U}^{(j)}) \\ \tilde{U}^{(i)} &= \textstyle\sum_{l=1}^c n_l U_l^{(i)}/\sum_{l=1}^c n_l, \end{aligned} \tag{10.11}$$

*where $(\hat{a}^{ij})=(\hat{a}_{ij})^{-1}$ and $\hat{a}_{ij}$, given by* (10.8), *being the consistent and unbiased estimate of $a_{ij}$, is distributed asymptotically as $\chi^2$ with $p(c-1)$ degrees of freedom as $N\to\infty$.*

In case $p=1$ the statistic $\tilde{V}_{rs}$ is the same as $V_{rs}$ in Theorem 3.1, The limiting distribution of $\tilde{V}_{rs}$ under the sequence of alternatives $K_N$: $F_\alpha(\boldsymbol{x})=F(\boldsymbol{x}-N^{-1/2}\boldsymbol{\theta}_\alpha)$, where the constant vectors $\theta_\alpha$, $\alpha=1, 2, \cdots, c$ are not all equal, is given by the following theorem.

**Theorem 10. 2.** *Suppose that every marginal distribution $F^{(i)}(x)$ of $F(\boldsymbol{x})$ satisfies the assumptions of Theorem 4.1 and det $(\boldsymbol{A})\neq 0$. Then under the sequence of alternatives $K_N$ the limiting distribution of $\tilde{V}_{rs}$ is noncentral $\chi^2$ with $p(c-1)$ degrees of freedom and noncentrality parameter*

$$
\begin{aligned}
\lambda_{rs}^2 &= \sum_{i,j=1}^{p} a^{ij}\cdot\sum_{\alpha=1}^{c}\rho_\alpha(\theta_\alpha^{(i)}-\tilde{\theta}^{(i)})(\theta_\alpha^{(j)}-\tilde{\theta}^{(j)}) \\
&\times\int_{-\infty}^{\infty}\{r[F^{(i)}(x)]^{r-1}+s[1-F^{(i)}(x)]^{s-1}\}f_i(x)dF^{(i)}(x) \\
&\times\int_{-\infty}^{\infty}\{r[F^{(j)}(x)]^{r-1}+s[1-F^{(j)}(x)]^{s-1}\}f_j(x)dF^{(j)}(x),
\end{aligned}
\tag{10.12}
$$

*where $\boldsymbol{\theta}_\alpha=(\theta_\alpha^{(1)},\cdots,\theta_\alpha^{(p)})'$, $\tilde{\theta}^{(i)}=\sum_{\alpha=1}^{c}\rho_\alpha\theta_\alpha^{(i)}/\sum_{\alpha=1}^{c}\rho_\alpha$, $a_{ij}$ being given by* (10. 6) *and $f_i(x)$ is the derivative of $F^{(i)}(x)$.*

Proof. Corresponding to (4. 10), we can express

$$
\begin{aligned}
E[U_\alpha^{(i)}\,|\,K_N] &= \frac{1}{r+1}-\frac{1}{s+1}+\frac{\sum_{l=1}^{c}(\theta_\alpha^{(i)}-\theta_l^{(i)})}{(c-1)\sqrt{N}} \\
&\times\int_{-\infty}^{\infty}\{r[F^{(i)}(x)]^{r-1}+s[1-F^{(i)}(x)]^{s-1}\}f_i(x)dF^{(i)}(x)+o\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}
\tag{10.13}
$$

We can also remark that $\hat{a}_{ij}$ converges in probability to $a_{ij}$ even under $K_N$, since $E[\hat{a}_{ij}^{(\alpha)}\,|\,K_N]=E[\hat{a}_{ij}^{(\alpha)}\,|\,H]+O(N^{-1/2})$ and $\mathrm{Var}[\hat{a}_{ij}^{(\alpha)}\,|\,K_N]=O(N^{-1})$ for $\alpha=1,2,3,4$. Using Lemma 4. 2, Lemma 10. 1 and Lemma 3. 2, we easily get the desired conclusion.

As a generalization of $V_{rs}$-test, we can propose a test due to the statistic $\tilde{V}_{rs}$ which rejects the hypothesis if the observed value of $\tilde{V}_{rs}$ is larger than a preassigned constant. This test will be called $\tilde{V}_{rs}$-test.

## 11. Multivariate Kruskal and Wallis' test

In this section we shall generalize Kruskal and Wallis' $H$-test [18] to the multivariate case and derive its asymptotic distribution. Analogously as in Andrews [2], we shall put

$$
U_\alpha^{(i)} = \frac{1}{n_1\cdots n_c}\sum_{\beta_1=1}^{n_1}\cdots\sum_{\beta_c=1}^{n_c}\phi_\alpha^{(i)}(\boldsymbol{x}_{1\beta_1},\cdots,\boldsymbol{x}_{c\beta_c})
$$

$$\phi_\alpha^{(i)}(x_1, \cdots, x_c) = \sum_{\gamma=1}^{c} \frac{n_\gamma}{n_\alpha} \delta^{(i)}(x_\gamma, x_\alpha) \tag{11.1}$$

$$\delta^{(i)}(x, y) = \begin{cases} 1 & \text{if the } i\text{-th component of } x \text{ is smaller than the } i\text{-th component of } y \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that under the hypothesis $H$

$$E[U_\alpha^{(i)}] = \frac{1}{2} \sum_{\gamma \neq \alpha} \frac{n_\gamma}{n_\alpha}$$
$$n_\alpha U_\alpha^{(i)} = \bar{R}_\alpha^{(i)} - \frac{n_\alpha + 1}{2}, \tag{11.2}$$

where $n_\alpha \bar{R}_\alpha^{(i)}$ means the sum of over-all ranks corresponding to the $i$-th component of $x_{\alpha 1}, \cdots, x_{\alpha n_\alpha}$. Put $U^{(i)} = (U_1^{(i)}, \cdots, U_c^{(i)})$, then from Theorem 2.1 the limiting distribution of the statistic $\sqrt{N}(U^{(1)} - E[U^{(1)}]), \cdots, \sqrt{N}(U^{(p)} - E[U^{(p)}])$ is normal under the hypothesis $H$ with the mean $0$ and the covariance matrix $\Sigma = (\sigma_{\alpha\beta}^{(i,j)})$, where $\sigma_{\alpha\beta}^{(i,j)}$ means the asymptotic covariance of $U_\alpha^{(i)}$ and $U_\beta^{(j)}$. In case $i = j$ it was already obtained by Andrews [2], that is

$$\sigma_{\alpha\beta}^{(i,i)} = \frac{\sum_{l=1}^{c} \rho_l}{12} \left[ \frac{\delta_{\alpha\beta}}{\rho_\alpha^3} \sum_{l=1}^{c} \rho_l - \frac{1}{\rho_\alpha \rho_\beta} \right]. \tag{11.3}$$

In case $i \neq j$, $\sigma_{\alpha\beta}^{(i,j)}$ is given by (10.4) and the straightforward calculation yields;

( i ) $\zeta_{0,\cdots,1,\cdots,0}^{(\alpha,\alpha)}$ (1 lies at the $\alpha$-th place) $= \dfrac{1}{12}\left(\sum_{l \neq \alpha} \dfrac{\rho_l}{\rho_\alpha}\right)^2 \tau_{ij}$

(ii) $\zeta_{0,\cdots,1,\cdots,0}^{(\alpha,\beta)}$ (1 lies at the $\gamma$-th place, where $\gamma \neq \alpha, \beta$) $= \dfrac{\rho_\gamma^2}{12 \rho_\alpha \rho_\beta} \tau_{ij}$

(iii) $\zeta_{0,\cdots,1,\cdots,0}^{(\alpha,\beta)}$ (1 lies at the $\alpha$-th place, where $\alpha \neq \beta$) $= -\sum_{l \neq \alpha} \dfrac{\rho_l}{12 \rho_\beta} \tau_{ij}$,

where $\tau_{ij}$ is given by

$$\tau_{ij} = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{(i)}(x) F^{(j)}(y)\, dF^{(i,j)}(x, y) - 3. \tag{11.4}$$

Hence we have

$$\sigma_{\alpha\beta}^{(i,j)} = \frac{\tau_{ij}}{12} \left( \sum_{l=1}^{c} \rho_l \right) \left[ \delta_{\alpha\beta} \frac{\sum_{l=1}^{c} \rho_l}{\rho_\alpha^3} - \frac{1}{\rho_\alpha \rho_\beta} \right]. \tag{11.5}$$

This formula holds true also in case $i = j$, if we define $\tau_{ii} = 1$. A consistent and unbiased estimate of $\tau_{ij}$ is given by (9.9). From (11.5) we can express $\boldsymbol{\Sigma}=\boldsymbol{A}_0\otimes\boldsymbol{\Sigma}_0$ by using Kronecker product, where $\boldsymbol{A}_0=(\tau_{ij})$ and $\boldsymbol{\Sigma}_0=(1/12)(\sum_{l=1}^c \rho_l)[\delta_{\alpha\beta}\rho_\alpha^{-3}\sum_{l=1}^c \rho_l-\rho_\alpha^{-1}\rho_\beta^{-1}]$. If the $p\times p$ matrix $\boldsymbol{A}_0$ is nonsingular, the rank of $\boldsymbol{\Sigma}$ is equal to $p(c-1)$, since the rank of $\boldsymbol{\Sigma}_0$ is shown to be equal to $c-1$. The solution of (4.12) for the covariance matrix $\boldsymbol{\Sigma}=(\sigma_{\alpha\beta}^{(i,j)})$ defined by (11.5) is given by $\boldsymbol{\Lambda}=\boldsymbol{A}_0^{-1}\otimes\boldsymbol{\Lambda}_0$ where

$$\text{(11.6)} \qquad \boldsymbol{\Lambda}_0=\frac{12}{(\sum_{l=1}^c \rho_l)^2}\left[\delta_{\alpha\beta}\rho_\alpha^3-\frac{\rho_\alpha^2\rho_\beta^2(\rho_\alpha^3+\rho_\beta^3)}{\sum_{l=1}^c \rho_l^4}+\frac{\rho_\alpha^2\rho_\beta^2\sum_{l=1}^c \rho_l^7}{(\sum_{l=1}^c \rho_l^4)^2}\right].$$

The above $\boldsymbol{\Lambda}_0$ is the solution of (4.12) for the covariance matrix $\boldsymbol{\Sigma}_0$ and is already obtained by Sugiura [26]. Hence we obtain the following theorem from Lemma 4.2, Lemma 3.2 and $\sum_{\alpha=1}^c n_\alpha^2[U_\alpha^{(i)}-2^{-1}\sum_{l\neq\alpha}(n_l/n_\alpha)]=0$.

**Theorem 11.1.** *Under the hypothesis H, the limiting distribution of the statistic*

$$\text{(11.7)} \quad \tilde{H}=\sum_{i,j=1}^p \hat{\tau}^{ij}\cdot\frac{12}{(\sum_{l=1}^c n_l)^2}\sum_{\alpha=1}^c n_\alpha\left(\bar{R}_\alpha^{(i)}-\frac{\sum_{l=1}^c n_l+1}{2}\right)\left(\bar{R}_\alpha^{(j)}-\frac{\sum_{l=1}^c n_l+1}{2}\right)$$

*is* $\chi^2$ *with* $p(c-1)$ *degrees of freedom, where* $(\hat{\tau}^{ij})=(\hat{\tau}_{ij})^{-1}$ *and* $\hat{\tau}_{ij}$ *is given by* (9.9) *which is a consistent and unbiased estimate of* $\tau_{ij}$ *given by* (11.4). *The matrix* $(\tau_{ij})$ *is assumed to be nonsingular.*

**Theorem 11.2.** *Under the same conditions as in Theorem* 10.2, *the limiting distribution of the statistic* $\tilde{H}$ *is noncentral* $\chi^2$ *with* $p(c-1)$ *degrees of freedom and noncentrality parameter*

$$\text{(11.8)} \quad \lambda^2_{\tilde{H}}=12\sum_{i,j=1}^p \tau^{ij}\cdot\sum_{\alpha=1}^c \rho_\alpha(\theta_\alpha^{(i)}-\tilde{\theta}^{(i)})(\theta_\alpha^{(j)}-\tilde{\theta}^{(j)})\int_{-\infty}^{\infty} f_i(x)\,dF^{(i)}(x) \times\int_{-\infty}^{\infty} f_j(x)\,dF^{(j)}(x)\,.$$

Proof. Under the assumption of Theorem 10.2 we can express

$$\text{(11.9)} \qquad E[U^{(i)}\mid K_N]=\frac{1}{2}\sum_{l\neq\alpha}\frac{\rho_l}{\rho_\alpha}+\frac{\sum_{l=1}^c \rho_l}{\sqrt{N}\rho_\alpha}(\theta_\alpha^{(i)}-\tilde{\theta}^{(i)}) \times\int_{-\infty}^{\infty} f_i(x)\,dF^{(i)}(x)+o\left(\frac{1}{\sqrt{N}}\right).$$

Calculating the noncentrality parameter $\lambda^2_{\tilde{H}}=\boldsymbol{\mu}(\boldsymbol{A}_0^{-1}\otimes\boldsymbol{\Lambda}_0)\boldsymbol{\mu}'$ by Lemma 4.2 where $\boldsymbol{\mu}=(\boldsymbol{\mu}^{(1)},\cdots,\boldsymbol{\mu}^{(p)})$ and $\boldsymbol{\mu}^{(i)}=(\mu_1^{(i)},\cdots,\mu_c^{(i)})$

$$\text{(11.10)} \qquad \mu_\alpha^{(i)}=\frac{\sum_{l=1}^c \rho_l}{\rho_\alpha}(\theta_\alpha^{(i)}-\tilde{\theta}^{(i)})\int_{-\infty}^{\infty} f_i(x)\,dF^{(i)}(x)\,,$$

we have the desired conclusion.

From Theorem 10.2 and Theorem 11.2 we can easily see the following theorem, which is a generalization of Theorem 4.2 to the multivariate case.

**Theorem 11.3.** *Under the sequence of alternatives $K_N$, the limiting distribution of the statistic $\tilde{V}_{11}$ is the same as that of $\tilde{H}$ under the condition of Theorem* 10.2.

As a generalization of Kruskal and Wallis' $H$-test, we can propose a multivariate nonparametric several-sample test due to the statistic $\tilde{H}$ which rejects the hypothesis if the observed value of $\tilde{H}$ is larger than a preassigned constant. This test will be called multivariate Kruskal and Wallis' test. In case $p=1$, the statistic $\tilde{H}$ defined by (11.7) reduces to Kruskal and Wallis' $H$-statistic.

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