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Osaka J. Math.
2 (1965), 351-356

NOTE ON QUASI-INJECTIVE MODULES

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(Received June 25, 1965)

Let R be a ring with identity element and M be a unitary left R-module. M is called quasi-injective if every element in $\mathrm{Hom}_R(N, M)$ for any R-module N in M is extened to an element in $\mathrm{Hom}_R(M, M)$. M is an essential extension of N if $M' \cap N \neq (0)$ for any non-zero R-submodule M' of M and we call in this case that N is an essential submodule in M.

In Goldie [2] and Johnson, Wong [4] they have defined an R-submodule in M for R-submodule N as follows: $\mathrm{cl}N = \{m \in M | (N; m)$ is an essential left ideal in $R\}$. If $\mathrm{cl}N = N$, then N is called closed. We call $\mathrm{cl}(0)$ the singular submodule of M.

Johnson and Wong studied structures of closed submodules of a quasi-injective R-module with zero singular submodule and Goldie has also considered rings with zero singular ideal in [3], [4] and [2], respectively.

In this short note we shall prove the following theorem:

Let M be a quasi-injective R-module. Then M is a direct-sum of $Z_2(M) = \mathrm{clcl}(0)$ and any maximal submodule M_0 with zero singular submodule: $M = M_0 \oplus Z_2(M)$. Furthermore, every closed submodule in M corresponds uniquely to a direct summand of M_0, which is closed in M_0.

From this result we know some results in [3], [4] are valid without assumption $\mathrm{cl}(0) = (0)$.

In § 2 we shall study all types of quasi-injective modules in a case where either R is a Dedekind domain or an algebra over a field with finite dimension.

We always assume that R is a ring with identity and M a unitary left R-module.

1. Closed submodules.

We shall denote the singular submodule $\mathrm{cl}(0)$ of M by $Z(M)$ and $\mathrm{clcl}(0)$ by $Z_2(M)$ following to [2]. We also call $Z_2(M)$ the torsion submodule of M and M is torsion free if $Z_2(M) = (0)$. If R is a commutative integral domain, then they coincide with the usual torsion submodules and torsion-free modules.

We note that if M is an essential extension of N, then the left ideal $(N:m)$ is essential in R for any element m in M.

From [2], Lemma 2, 2 we have

Lemma 1.1. *$Z_2(M)$ is a closed submodule in M.*

From the definition of closed module we have

Lemma 1.2. *Every closed submodule of M contains $Z_2(M)$.*

Lemma 1.3. *For submodules N_1, N_2 of M we have*

$$\mathrm{cl}(N_1\cap N_2) = \mathrm{cl}N_1\cap \mathrm{cl}N_2\,.$$

Let N be a submodule of M. If a submodule B of M is a maximal one with property $N\cap B=(0)$, then we call B a complement of N in M. We denote it by N^c.

Lemma 1.4. *Let N be a submodule of M and B a complement of N. Then M is an essential extension of $B\oplus N$. Hence, $\mathrm{cl}(B\oplus N)=M$.*

Proposition 1.5. *Let M be a quasi-injective R-module. Then every closed submodule N is a direct summand of M, namely $M=N\oplus N^c$* (cf. [4], Proposition 1.5).

Proof. Let N be a closed submodule and B a complement of N in M. Put $M_0=B\oplus N$. Let p be a projection of M_0 to N. Then there exists an element $g\in\mathrm{Hom}_R(M, M)$ such that $g|M_0=p$. Since $g^{-1}(0)\supseteq B$ and $g^{-1}(0)\cap N=(0)$, $g^{-1}(0)=B$. Furthermore, since $\mathrm{cl}M_0=M$ by Lemma 1.4, there exists an essential left ideal L for any element m in M such that $Lm\subseteq M_0$. Therefore, $Lg(m)=g(Lm)\subseteq N$. Since $\mathrm{cl}N=N$, $g(m)\in N$. Hence, $g(M)=N$. Therefore, $M=g^{-1}(0)+g(M)=B\oplus N$.

Corollary. *Let M be a quasi-injective. If N is closed, then N is quasi-injective* (cf. [3], Theorem 1.6).

Proof. Since it is clear that a direct summand of a quasi-injective module is quasi-injective, we have the corollary from Proposition 1.5.

If we consider R as a left R-module, we have from the definition

Lemma 1.6. *Let $M\supseteqq N$ be R-modules. Then* 1) *$Z(R)M\subseteqq Z(M)$,* 2) *$Z_2(R)M\subseteqq Z_2(M)$,* 3) *$Z(N)=N\cap Z(M)$ and* 4) *$Z_2(N)=N\cap Z_2(M)$.*

Theorem 1.7. *Let M be a quasi-injective R-module and M_0 a submodule which is a maximal one with $Z(M_0)=(0)$. Then $M=M_0\oplus Z_2(M)$. A submodule N of M is closed if and only if N contains $Z_2(M)$ and $M_0\cap N$ is a direct summand of M_0.*

Proof. From Lemma 1.6 we obtain that $M_0\cap Z_2(M)=(0)$ and M_0 is a complement of $Z_2(M)$. Hence, $M=M_0\oplus Z_2(M)$ by Proposition 1.5, since $Z_2(M)$ is closed. If N is a closed submodule of M, then $N\supseteq Z_2(M)$ by Lemma 1.1 and $N=M_0\cap N\oplus Z_2(M)$. Since $N\cap M_0$ is closed in M_0, $N\cap M_0$ is a direct summand of M_0 by Proposition 1.5. Conversely, we assume that $N\supseteq Z_2(M)$ and $N\cap M_0$ is a direct summand of M_0; $M_0=N\cap M_0\oplus N_1$. Considering in M_0, $M_0=\mathrm{cl}M_0=\mathrm{cl}(N\cap M_0)+\mathrm{cl}N_1$. Since $\mathrm{cl}(N\cap M_0)\cap\mathrm{cl}N_1=\mathrm{cl}(N\cap M_0\cap N_1)=\mathrm{cl}(0)=(0)$ by Lemma 1.2, $N\cap M_0$ is closed in M_0. Let $x\in\mathrm{cl}N$: $x=m_0+y$, where $m_0\in M_0$, $y\in Z_2(M)$. Since $Lx\subseteq N$ for an essential left ideal L, $Lm_0\subseteq N\cap M_0$. Hence, $m_0\in N\cap M_0$. Therefore, $x\in N$.

Corollary. *Let M be a quasi-injective. If N_1, N_2 are closed in M, then N_1+N_2 is closed. Hence,* $\mathrm{clcl}(S_1+S_2)=\mathrm{clcl}(S_1)+\mathrm{clcl}(S_2)$ *for any submodules S_1 and S_2* (cf. [3], Theorem 1.4 and [4], Theorem 1.2).

Proof. Since N_i is closed, N_i contains $Z_2(M)$. Hence, it is sufficient to show that $N_1\cap M_0+N_2\cap M_0$ is a direct summand of M_0 by Theorem 1.7, where $M=M_0\oplus Z_2(M)$. Thus, we may assume $Z(M)=(0)$. $N_1\cap N_2$ is closed by Lemma 1.2. Hence, $M=N_1\oplus N_1'=(N_1\cap N_2)\oplus M'$. Since $N_2\cap(N_1\cap M')=(0)$, there exists a submodule N_2' such that $N_2'\supseteqq N_1\cap M'$ and $M=N_2\oplus N_2'$. Furthermore, $N_2'=(N_1\cap M')\oplus N_3'$. Therefore, $M=N_2\oplus(N_1\cap M')\oplus N_3'$. On the other hand $N_1=(N_1\cap N_2)\oplus N_1\cap M'$. Hence, $N_1+N_2=N_2+(N_1\cap M')$. Therefore, $M=(N_1+N_2)\oplus N_3'$. The second half is clear from the first.

Proposition 1.8. *Let M be quasi-injective. Then the set of closed submodules coincides with the set of complement submodules containing $Z(M)$. Especially, if we assume $Z(M)=(0)$, then every complement of a submodule N is isomorphic to each other and N^{cc} containing N coincides with* $\mathrm{cl}N$.

Proof. Let $N=N_1^c\supseteqq Z(M)$. For any element $n\in N_1\cap\mathrm{cl}N$ we have $Ln\subseteqq N_1\cap N=(0)$, where L is an essential left ideal. Hence, $n\in Z(M)\cap N_1=(0)$. Therefore, $\mathrm{cl}N=N$. The converse is clear from Proposition 1.5. We assume $Z(M)=(0)$. In this case we note that $\mathrm{clcl}N=\mathrm{cl}N$. By Lemmas 1.3 and 1.4 and Corollary to Theorem 1.7 we have $M=\mathrm{cl}(N\oplus N^c)=\mathrm{cl}N\oplus N^c$ for any submodule N. Hence, $N^c\approx M/\mathrm{cl}N$. Furthermore, we obtain $M=N^c\oplus N^{cc}=\mathrm{cl}N\oplus N^c$ by Proposition 1.5. If $N^{cc}\supseteq N$, then $N^{cc}\supseteq\mathrm{cl}N$. Hence, $N^{cc}=\mathrm{cl}N$.

2. Special cases.

First we consider some relations between a quasi-injective module M and its injective envelope $E(M)$.

Proposition 2.1. *Let M be an R-module. Then $E(M)=E(Z_2(M))\oplus E(B)$ and $Z_2(E(M))=E(Z_2(M))$, where B is a maximal torsion-free submodule in M.*

Proof. We assume $Z_2(M)=(0)$ and $E=E(M)$. Then $E=E_0\oplus Z_2(E)$ by Theorem 1.7. Let p be a projection of E to E_0. If $p(m)=0$ for $m\in M$, then $m\in M\cap Z_2(E)=Z_2(M)=(0)$ by Lemma 1.6. Hence, M is monomorphic to E_0. Therefore, $Z_2(E)=(0)$. If $Z_2(M)=M$, then $M\subseteq E(M)\subseteq Z_2(E)$. Hence, $Z_2(E)=E$. Since M is an essential extension of $B\oplus Z_2(M)$, $E=E(B)\oplus E(Z_2(M))$.

Lemma 2.2. *Let M be an R-module and $K=\mathrm{Hom}_R(E(M), E(M))$. M is quasi-injective if and only if M is a K-module.* (See [3], Theorem 1.1.)

Proposition 2.3. *Let M be quasi-injective. If $E(M)=N_1\oplus N_2$, then $M=M\cap N_1\oplus M\cap N_2$, and $N_i=E(M\cap N_i)$.*

Proof. Let p be a projection of $E(M)$ to N. Since $p\in K$, $p(M)\subseteq M$ by Lemma 2.2. Hence, $M=M\cap N_1\oplus M\cap N_2$.

Corollary. *Let R be a commutative integral domain. Then every injective module is a direct sum of the torsion submodule and a maximal torsion-free submodule. An injective envelope of torsion (resp. torsion-free) module is torsion (resp. torsion-free).*

Proposition 2.4. *Let M_1, M_2 be quasi-injective such that $E(M_1)\overset{\varphi}{\approx}E(M_2)$. Then $M_1\oplus M_2$ is quasi-injective if and only if $M_1\approx M_2$.*

Proof. $E(M_1\oplus M_2)=E(M_1)\oplus E(M_2)$. If $M_1\approx M_2$, $M=M_1\oplus M_2$ is a $\mathrm{Hom}_R(E(M), E(M))$-module, and hence M is quasi-injective by Lemma 2.2. Conversely, we assume that M is quasi-injective. Let f be an element in $K=\mathrm{Hom}_R(E(M), E(M))$ such that $f|E(M_2)\equiv 0$, $f|E(M_1)$ induces the isomorphism φ. Then $f(M)=f(M_1)\subseteqq M\cap E(M_2)=M_2$ by Proposition 2.3. If we consider the same argument on M_2 for φ^{-1}, we can find $g\in K$ such that $gf|E(M_1)\equiv \text{identity}\,|E(M_1)$. Hence, $M_1=gf(M_1)\subseteqq g(M_2)\subseteqq M_1$. Therefore, $M_1\approx M_2$.

Proposition 2.5. *Let R be a left noetherian ring and M a quasi-injective R-module. Then M is a direct-sum of indecomposable quasi-injective R-modules. Furthermore, this decomposition is unique up to isomorphism,* ([10], Theorem 4.5).

Proof. $E(M)=\sum\oplus\bar{M}_\alpha$ by [6], Theorem 2.5, where $\bar{M}_\alpha$ is an indecomposable injective R-module. Put $M_\alpha=M\cap\bar{M}_\alpha$. Then $M=\sum\oplus M_\alpha$ by Lemma 2.2. Since M_α is a direct summand of M, M_α is quasi-injective. It is clear that M_α is indecomposable. We assume $M=\sum\oplus M_{\alpha'}$

is a second decomposition. Since R is left noetherian, $E(M)=\sum\oplus E(M_{\alpha'})$ and $E(M_{\alpha'})$ is indecomposable by Proposition 2.3. Then there exists an automorphism φ in K such that φ: $\bar{M}_\alpha\approx E(M_{\rho(\alpha)'})$ for all α by [5], Proposition 2.7, where ρ is a permutation of indices α. Since $\varphi(M)\subseteqq M$, $\varphi(M_\alpha)=\varphi(M\cap\bar{M}_\alpha)\subseteq M\cap E(M_{\rho(\alpha)'})=M_{\rho(\alpha)'}$. Taking φ^{-1}, we obtain $M_\alpha\approx M_{\rho(\alpha)'}$.

Now we assume that R is a commutative noetherian ring. Then we know by [6], Proposition 3.1 that every indecomposable injective R-module is isomorphic to $E(R/P)$, where P is a prime ideal in R.

Proposition 2.6. *Let M be an indecompoable quasi-injective R-module. If M is torsion-free, then M is injective.*

Proof. Since $E(M)=E(R/P)$ is torsion-free by Proposition 2.1, $Z(R/P)=(0)$. Hence, P is not essential in R. There exists a non-zero ideal Q such that $P\cap Q=(0)$. Therefore, P is minimal prime and $(0)_P=PR_P$. From [6], Theorem 3.6 $E(R/P)$ is R_P-injective. Since R_P is the quotient field K of R/P, $E(M)=Km$. Furthermore, $K\subseteq\mathrm{Hom}_R(E(R/P), E(R/P))$. Hence, $M=KM=E(M)$.

Corollary. *Let R be a Dedekind domain and M a quasi-injective R-module. Then M is either injective or a direct-sum of R-modules $E(R/P_i)$ and $R/S_j^{n_j}$. Conversely, such a module is quasi-injective, where $\{P_i, S_j\}$ is a set of non-zero distinct primes in R.*

Proof. $M=M_0\oplus Z_2(M)$. If $M_0\neq(0)$, then $M_0\approx\sum\oplus Q$ by Propositions 2.5 and 2.6, where Q is the quotient field of R. Since $Z_2(M)$ is torsion, $Z_2(M)=\sum_{i\in I}\oplus(E(R/P_i))^{\alpha_i}\oplus\sum_{j\in J}(R/(S_j^{n_j})^{\beta_j}$[1] by Proposition 2.5 and [5], Theorem 10. However if $M_0\neq(0)$ then there exist natural epimorphisms of M_0 to $E(R/S_j)$. Hence, $J=\phi$ by Lemma 2.2, which means M is injective. If $M_0=(0)$, then $M\approx\sum\oplus E(R/P_i)^{\alpha_i}\oplus\sum(R/(S_j^{n_j}))^{\beta_j}$ and $\{P_i, S_j\}$ is a set of non-zero distinct primes by Proposition 2.4. The converse is clear.

Next, we consider a case of algebra A over a field K with finite dimension. Then we know from [7] that every indecomposable A-injective module M is isomorphic to $(eA)^*=\mathrm{Hom}_K(eA, K)$, where e is a primitive idempotent in A. Hence, there exists a non-degenerated bilinear mapping (,) of $eA\underset{K}{\otimes}M$ to K with respect to A. It is clear that the adjoint elements of $\mathrm{Hom}_A(M, M)=B$ is equal to eAe. Hence for an A-submodule N of M, ann $N=\{x\mid\in eA, (x, N)=0\}$ is an eAe-module if and only if N is a B-module. Thus from Lemma 2.2 we have

1) M^α means a direcsum of α-copies of M.

Proposition 2.7. *Let A be an algebra over a field K such that $[A:K]<\infty$. Then every quasi-injective A-module is a direct-sum of modules $(e_iA/e_iR_i)^*$, where e_i is a primitive idempotents and R_i is a right ideal in A such that $e_iAe_iR_i \subseteqq R_i$.*

Corollary. *Let A be as above. If A is a generalized uniserial ring, then every sub-module of indecomposable injective module is quasi-injective.*

Remark. The converse of Proposition 2.7 is, in general, not true. Finally, we consider the singular ideal of quasi-Frobenius ring.

Proposition 2.8. *Let R be quasi-Frobenius. Then $Z(R)$ is eqal to the radical N of R and R is a direct sum of semi-simple subring and a quasi-Frobenius ring R_1 such that $Z_2(R_1)=R_1$.*

Proof. Let S be a left scole of R, namely the sum of minimal left ideals in R. Then S is a unique minimal essential left ideal of R. Hence, $Z(R)=$ the right annihilator S_r of S in R. Since $N_r=N_l$ by [8] and $S=N_r$, $Z(R)=S_r=N_{rr}=N_{lr}=N$ by [8]. Furthermore, R is left R-injective by [1]. Hence, $R=L\oplus Z_2(R)$ as a left R-module by Theorem 1.7. Since $Z_2(R)\supseteqq N$, $NL\subseteqq N\cap L=(0)$. Hence, $L\subseteqq S$. $S=S\cap N\oplus L'$, where $L'\supseteqq L$ is a direct-sum of non-nilpotent minimal left ideals. Since $SN=(0)$, $S^2=L'$. It is clear that $Z_2(R)=(S^2)_r$. Therefore, $L'Z_2(R)=(0)$, which means that L is a two-sided ideal. Since L is completely reducible, L is semi-simple.

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