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SUPPLEMENTARY RESULTS ON GALOIS EXTENSION

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Recently, Galois theory of commutative ring has been developed by Auslander, Chase, Goldman, Harrison and Rosenberg in [1], [2] and many important results in it has been generalized for separable R-algebra by DeMeyer, Kanzaki and Takeuchi in [3], [8], [9], [10] and [11]. There are some equivalent definitions of a Galois extension, which will be given in §1.

Let Λ be an R-algebra which is a finitely generated R-module and G a finite group of R-automorphisms of Λ.

In §2 we consider some relations between Λ^T and $V_\Lambda(\Lambda^T)$ for a Galois extension Λ with G and for a subgroup T, which contains some results in [3], [10] and [11]. We give a sufficient condition of Λ being a central Galois, which is a converse of [10], Theorem 2.

In §3 we study a criterion of the center C of Λ being a Galois extension. Let e be a primitive idempotent in C and $T=\{\sigma|\in G, \sigma(e)=e\}$ and $H=\{\sigma|\in T, \sigma$ induces the identity mapping on the center of $\Lambda e\}$. Under an assumption that R is indecomposable, we obtain that C is a Galois extension of R if and only if H is a normal subgroup in G, which is a generalization of [10], Proposition 10.

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1. Definitions and notations.

Let R be a commutative ring with identity and Λ an R-algebra. We always assume that every R-algebra is finitely generated R-module. A group G means a finite group of R-algebra automorphisms of Λ. We recall the definition of Galois extension.

Let Γ be the fixed ring of G, which we denote by Λ^G. If Λ satisfies one of the following conditions, then we call Λ a *Galois extension of* Γ *with* G:

I. Λ is a finitely generated projective right Γ-module and $\Delta(\Lambda, G)$ is isomorphic to $\mathrm{Hom}_\Gamma(\Lambda, \Lambda)$ by defining $(\lambda u_\sigma)\cdot\mu=\lambda\sigma(\mu)$, where $\Delta(\Lambda, G)$ is a trivial crossed product with basis u_σ and $\lambda, \mu\in\Lambda$.

II. There exist elements x_i, y_i in Λ such that $\sum_i x_i y_i = 1$, $\sum_i x_i \sigma(y_i) = 0$ if $\sigma \neq 1$.

We call such $\{x_i, y_i\}$ a Galois generator.

We know that this definition is symmetric with respect to right and left. By $c(\Lambda)$ we menas the center of Λ and we dente it by C sometimes.

We note that many results in [2] for a commutative case are valid with a slight modification for a non-commutative case. So we quote them without proof.

2. Central Galois extension

Let G be a finite group of automorphism of Λ as an R-algebra. We put $J_\sigma = \{x \mid \in \Lambda,\ xy = \sigma(y)x$ for all y in $\Lambda\}$ for $\sigma \in G$. Then it is clear that J_σ is a C-module, $J_\sigma J_\tau \subseteq J_{\sigma\tau}$ and $J_1 = C$, where C is the center of Λ. First, we give a converse of [10], Theorem 2.

Theorem 1. *Let Λ be a separable R-algebra with automorphism group G. If $\Lambda = \sum_{\sigma \in G} \oplus J_\sigma$ and $J_\sigma J_{\sigma^{-1}} = C$ for any σ in G, then Λ is a Galois extension of the center C with G.*

Proof.[1] Since Λ is R-separable, Λ is central separable by [1], Theorem 2.3. Hence, $\mathrm{Hom}_C(\Lambda, \Lambda) \approx \Lambda_r \underset{C}{\otimes} \Lambda_l$ by [1] Theorem 2.1, where Λ_r (resp. Λ_l) means the set of multiplications of elements in Λ from the right (resp. left) side. $\Lambda_r = \sum \oplus (J_\sigma)_r$. It is clear that $(J_\sigma)_r = (J_\sigma)_l \sigma^{-1}$. Hence, $\Lambda_r \underset{C}{\otimes} \Lambda_l = \sum \oplus (J_\sigma)_l \Lambda_l \sigma^{-1} = \sum \oplus (J_\sigma \Lambda)_l \sigma^{-1} = \sum \Lambda_l \sigma = \Delta(\Lambda, G)$ since $J_\sigma \Lambda = \Lambda$ by the assumption.

The converse of the following corollary was given in [3], [13] and we shall consider it later.

Corollary. *Let Λ and C be as above. We assume that G consists of inner-automorphisms which is induced by unit elements u_σ. If $\Lambda = \sum \oplus C u_\sigma$, then Λ is a Galois extension of C with G.*

Proof. It is clear that $J_\sigma = C u_\sigma$ and $J_\sigma J_{\sigma^{-1}} = C$.

Proposition 2. *Let Λ be a separable R-algebra with automorphism group G. If $R = \Lambda^G$ and $J_\sigma = 0$ for $\sigma \neq 1$, then Λ is a Galois extension of R and Λ is a commutative ring. The converse is also true.*

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1) The first proof was a little longer and Prof. Kanzaki pointed out this proof to the author.

Proof. We define an automorphism ψ of $\Lambda \underset{R}{\otimes} \Lambda$ by setting $\psi(\lambda\otimes\mu)=\lambda\otimes\sigma(\mu)$. Since Λ is separable over R, then there exist elements x_i, y_i in Λ such that $\sum x_i y_i=1$ and $x(\sum x_i\otimes y_i)=(\sum x_i\otimes y_i)x$ for all $x\in\Lambda$. Hence, $x\sum x_i\otimes\sigma(y_i)=\sum x_i\otimes\sigma(y_i)\sigma(x)$. Therefore, $x\sum x_i\sigma(y_i)=\sum x_i\sigma(y_i)\sigma(x)$, which means $\sum x_i\sigma(y_i)\in J_{\sigma^{-1}}=(0)$ if $\sigma\neq 1$. Thus, we can find a Galois generator $\{x_i, y_i\}$. Hence, Λ is a Galois extension of R and $\Lambda=V_\Lambda(\Lambda^G)=C\oplus J_\sigma\oplus\cdots=C$ by [10], Proposition 1.

Proposition 3. *Let Λ be a Galois extension of R with G and $C=\Lambda^H$ for a subgroup H. Then $H\cap T=(1)$ for a subgroup T of G if and only if $V_\Lambda(\Lambda^T)=C$. In this case we have $\Lambda^S=C^S\underset{C^T}{\otimes}\Lambda^T$ for $S\subseteq T$.*

Proof. We assume $H\cap T=(1)$. Then T is isomorphic to the induced automorphism group $T|C$. Hence, $\Lambda=C\Lambda^T$ by [3], Lemma 2. Therefore, $V_\Lambda(\Lambda^T)=V_\Lambda(C\Lambda^T)=C$. If $V_\Lambda(\Lambda^T)=C$, then $\mathfrak{c}(\Lambda^T)=C\cap\Lambda^T=C^T$. C and Λ are Galois extensions of C^T and Λ^T, respectively. Hence, $C\underset{C^T}{\otimes}\Lambda^T$ is a Galois extension of Λ^T. Since $\mathfrak{c}(C\otimes\Lambda^T)=C$ and $C\otimes\Lambda^T$ is separable, $C\otimes\Lambda^T\approx C\Lambda^T$. Hence, $C\Lambda^T$ and Λ are Galois extensions of Λ^T with T. Therefore, $C\Lambda^T=\Lambda$ by [2], Theorem 3.4, which means $H\cap T=(1)$. The last part is clear from [3], Lemma 2.

For a twised group ring the following lemma is well known.

Lemma 4. *Let $\Delta(\Lambda, G)$ be a trivial crossd product of R-algebra Λ and $H=\{\sigma|\in G, \sigma|\mathfrak{c}(\Lambda)=I_{\mathfrak{c}(\Lambda)}\}$. If Δ is R-separable then the order $|H|$ of H is a unit in R and $\mathrm{Tr}_G(\Lambda)=\Lambda^G$.*

Proof. Let $\Delta=\Delta(\Lambda, G)=\Lambda\oplus\Lambda\sigma\oplus\cdots$. Since Δ is R-separable, there exists an element θ in $\Delta\underset{R}{\otimes}\Delta$ such that $\varphi(\theta)=1$ and $\delta\theta=\theta\delta$ for $\delta\in\Delta$, where φ is a natural homomorphism of $\Delta\otimes\Delta$ to Δ. Let $\theta=\sum\oplus a_{\sigma,\tau}(\sigma\otimes\tau)$, where $a_{\sigma,\tau}=\sum_t\lambda_t(\sigma, \tau)\otimes\mu_t(\sigma, \tau)\in\Lambda\underset{R}{\otimes}\Lambda$. Since $\lambda\theta=\theta\lambda$ for $\lambda\in\Lambda$, we can easily see that $\varphi(a_{1,1})$ is in C. Furthermore, by the standard argument we has

$$\varphi(a_{1,1}) = \sum_t\rho(\lambda_t(\rho^{-1}, \rho)\cdot\mu_t(\rho^{-1}, \rho)) \quad \text{for} \quad \rho\in G\,. \tag{1}$$

Let $G=H+H\tau_2+\cdots+H\tau_s$. Replacing ρ^{-1} by $\xi\tau_i$ in (1) we have $\tau_i^{-1}\varphi(a_{1,1})=\sum\lambda_t(\sigma^{-1}, \sigma)\sigma^{-1}(\mu_t(\sigma^{-1}, \sigma)$, where $\sigma=\xi\tau_i$. Therefore, $1=\varphi(\theta)=|H|(\varphi(a_{1,1})+\tau_2^{-1}(\varphi(a_{1\,1}))+\cdots+\tau_s^{-1}(\varphi(a_{1,1}))=|H|\mathrm{Tr}_{G/H}(\varphi(a_{1,1}))$. Hence $|H|$ is a unit in R and $\mathrm{Tr}_G(\Lambda)=\Lambda^G$.

The following is a slight generalization of [10], Proposition 5.

Proposition 5. *Let Λ be a Galois extension of Λ^G with G and $H=$*

$\{\sigma | \in G, \sigma | C = I_C\}$. *We assume* Λ^G *is R-separable. Then* $|H|$ *is a unit in R and* $\mathrm{Tr}_G(\Lambda) = \Lambda^G$.

Proof. By [8], Proposition 4 we know that $\Delta(\Lambda, G)$ is R-separable.

Proposition 6. *Let* Λ *be a Galois extension of an R-separable algebra* Λ^G *with G. Let H be a subgroup of G. Then the center of* Λ^H *is equal to C if and only if* $V_\Lambda(\Lambda^H)$ *is a Galois extension of C with H.*

Proof. We put $\Gamma = \Lambda^H$ and $\Omega = V_\Lambda(\Lambda^H)$. If the center of Γ is equal to C, then $\Lambda = \Gamma \underset{C}{\otimes} \Omega$ by [1], Theorem 3.3. We note that H induces an automorphism group on Ω. $\mathrm{Hom}_C(\Omega, \Omega) \underset{C}{\otimes} \Gamma \approx \mathrm{Hom}_\Gamma(\Omega \underset{C}{\otimes} \Gamma, \Omega \underset{C}{\otimes} \Gamma) \approx \Delta(\Omega \otimes \Gamma, H)$ since Λ is a Galois extension of Γ by [11], Theorem 1 and Ω is C-projective. Furthermore, $\Delta(\Omega \underset{C}{\otimes} \Gamma, H) \approx \Delta(\Omega, H) \underset{C}{\otimes} \Gamma$. In the above isomorphism we can easily check that $\Delta(\Omega, H)$ is isomorphic to $\mathrm{Hom}_C(\Omega, \Omega)$ by the natural mapping. Hence, $C = \mathfrak{c}(\mathrm{Hom}_C(\Omega, \Omega)) = V_{\mathrm{Hom}_C(\Omega, \Omega)}(\Delta(\Omega, H)) = \Omega^H$. Therefore, Ω is a Galois extension of C with H. Conversely, it is clear that $\mathfrak{c}(\Gamma) = \mathfrak{c}(\Omega)$. Let x be in that center. Then $x \in V_\Lambda(\Omega) \cap \Omega = \Lambda^H \cap \Omega = \Omega^H = C$ by [8], Theorem 2. Hence, C is the center of Γ.

Corollary. *Let* Λ *be as above and* Γ *a separable C-subalgebra. Let* $H = \{\sigma | \in G, \sigma(x) = x$ *for* $x \in V_\Lambda(\Gamma)\}$. *Then H induces an automorphism group of* Γ. *If* $\Gamma^H = C$, *then* Λ *and* Γ *are Galois extensions of* $V_\Lambda(\Gamma)$ *and C with H, respectively.*

Proof. Since $V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$, the first part is clear. Let x be in the center of Λ^H. Since $\Lambda^H \supseteq V_\Lambda(\Gamma)$, $x \in \Gamma \cap \Lambda^H = \Gamma^H = C$. Hence $V_\Lambda(\Lambda^H)$ is a Galois extension of C with H by Proposition 6. Furthermore, $V_\Lambda(\Lambda^H) \subseteq \Gamma$. If we apply [2], Theorem 3.4 to the inclusion map, we have $\Gamma = V_\Lambda(\Lambda^H)$. Hence, $V_\Lambda(\Gamma) = \Lambda^H$.

Corollary. *Let* Λ *be a Galois extension of* Λ^G *with G. We assume* Λ^G *is R-separable. Let H be an inner-subgroup of G which is induced by unit elements* u_σ *and* $A = \sum C u_\sigma$. *Then the following conditions are equivalent.*

1) $\mathfrak{c}(\Lambda^H) = C$.
2) $\Lambda = A \underset{C}{\otimes} \Lambda^H$.
3) *A is a Galois extension of C with H.*

In this cases $A = \Sigma \oplus C u_\sigma$.

Proof. It is clear that A is C-algebra and $V_\Lambda(A) = \Lambda^H$. Since $|H|$ is a unit by Lemma 4, A is C-separable by [5], Lemma 4. Hence,

$V_\Lambda(\Lambda^H)=A$ by [8], Theorem 2. Therefore, we have the first part by Proposition 6, since $\mathfrak{c}(A)=\mathfrak{c}(\Lambda^H)$. Since $A_r\otimes\Lambda_l\approx\Delta(\Lambda, H)$ $A=\sum\oplus Cu_\sigma$.

3. Central subgroup.

Let Λ be a Glois extension of R with G and C the center of Λ. Let $H=\{\sigma\mid\in G, \sigma|C=I_C\}$. We call H the *central subgroup* of G.

In this section we study a criterion of C being the fixed ring of H.

First, we consider a criterion of a separable subalgebra Γ of Λ being a fixed subalgebra of a subgroup T.

In commutative case we know a condition "*strongly distinct*" (see [2], p. 16) and it was shown in [2] that this condition gives a criterion of the above. We shall generalize this condition to a case of non-commutative ring.

Let f, g be homomorphisms of R-algebra Γ to Λ $(f\neq g)$. We consider a condition:

(*) *For any $x\neq 0$ in Λ we can find y in Γ such that*

$$x\Lambda(f(y)-g(y))\neq(0).$$

If Λ and Γ are commutative and (*) is satisfied, then f and g are strongly distinct. Conversely, we assume $\Lambda=\Gamma$ and Λ is R-separable. If f, g are strongly distinct two automorphisms of Λ, then there exist elements x_i, y_i in Λ such that $\sum_i x_iy_i=1$ and $\sum x_if^{-1}g(y_i)=0$ by [2], Lemma 1.2. Hence, $1=\sum x_i(y_i-f^{-1}g(y_i))=\sum f(x_i)(f(y_i)-g(y_i))$. Therefore, (*) is satisfied.

We note that if Λ is a Galois extension of R with G and T a subgroup, then Λ^T is R-separable (see, [9]).

Theorem 7. *Let Λ be a Galois extension of R with G and Γ a separable R-subalgebra of Λ. We put $T=\{\sigma\mid\in G, \sigma|\Gamma=I_\Gamma\}$. Then $\Gamma=\Lambda^T$ if and only if distinct two elements σ, τ in G satisfy the above (*) on Γ condition.*

Proof. We assume $\Gamma=\Lambda^T$. Then there exist x_i, y_i in Γ such that $\sum x_iy_i=1$ and $\sum x_i\tau(y_i)=0$ if $\tau\notin T$ by [2], p. 23. Hence, Γ satisfies (*). Conversely, we assume Γ satisfies (*). $\mathfrak{a}=\mathrm{Hom}_R(\Lambda, \Lambda)\approx\Delta(\Lambda, G)=\sum\oplus\Lambda_l\cdot\tau$ and $\mathrm{Hom}_\Gamma^r(\Lambda, \Lambda)=V_{\mathfrak{a}}(\Gamma_r)$. Let x be in $V_\Lambda(\Gamma_r)$. $x=x(1)_l+x(\sigma)_l\sigma+\cdots$. Since $\gamma_rx=x\gamma_r$ for any $\gamma\in\Gamma$, we have $\gamma_rx(\sigma)_l=x(\sigma)_l\gamma_r=x(\sigma)_l\sigma(\gamma)_r$. Hence, $x(\sigma)\Lambda(\gamma-\sigma(\gamma))=(0)$. Since Γ satisfies (*), if $\sigma\notin T$, $x(\sigma)=0$. Therefore, $\mathrm{Hom}_\Gamma(\Lambda, \Lambda)=\sum_{\tau\in T}\oplus\Lambda_l=\Delta(\Lambda, T)$. Since $c(\mathfrak{a})=R$ and

Γ_r is sepapable R-algebra, $\Gamma_r = V_\Lambda(V_\Lambda(\Gamma_r)) = V_\Lambda(\Delta(\Lambda, T)) = (\Lambda^T)_r$ by [8], Theorem 2.

Corollary. *Let* Λ, Γ, G *and* T *be as above. We assume that* R *is a hereditary ring. Then* $\Gamma = \Lambda^T$ *for a subgroup* T *if and only if* Γ *is* G*-strong, (see the definition in* [2], *p.* 22).

Proof. It is sufficient to show that the condition "*G-strong*" implies the above condition (*), if R is hereditary. We note that if $x\Lambda$ is Λ-projective, then $x\Lambda \approx e\Lambda$ for an idempotent e. Hence, if $x\Lambda(y-\sigma(y))=0$ for all $y \in \Gamma$ and $\sigma \notin T$, then $e\Lambda(y-\sigma(y))=(0)$. Hence, $e=0$ and $x=0$. Thus, we may show that Λ is right hereditary. Since $\Delta(\Lambda_r, G) \approx \mathrm{Hom}_R^l(\Lambda, \Lambda)$ and Λ is R-projective, $\Delta(\Lambda_r, G)$ is right hereditary by [5], Lemma 1. 2. Hence, Λ_r is right hereditary by [4], Proposition 10.

Remark. We can easily extend the same argument of [2], §2 to a non-commutative case except Theorem 2. 2 in [2]. Theorem and its corllary are concerned with that theorem.

Finally, we consider the problem mentioned in the beginning of this section. Namely we consider the case in which T is the central group and $\Gamma = C$ in Theorem 7.

We always assume that C is a direct sum of indecomposable ideals (e.g. if R is indecomposable, then C is as above, see [7], Theorem 7).

Let $e_1, e_2, \cdots, e_n$ be the set of primitive idempotents in C. Let $T_i = \{\sigma \mid \in G, \sigma(e_i)=e_i\}$. We call T_i a *decomposition group* of e_i. We can classify e_i by a relation $e_i \equiv e_j$ if $e_j = \sigma(e_i)$ for some $\sigma \in G$. Let $\mathfrak{e}_i$ be one of the classes. Then $E_i = \sum_{e_t \in \mathfrak{e}_1} e_t$ is an idempotent and $E_i \in \Lambda^G = R$. In this case $\Lambda = \sum_i \oplus \Lambda E_i$ and we can easily see that if Λ is a Galois extension of R with G, then each ΛE_i is a Galois extension of RE_i with G, which implies that each element of G operates faithfully on each ΛE_i. The converse is clear.

Lemma 8. *Let* C *be a commutative separable* R*-algebra as above, and* $R = C^G$. C *is a Galois extension of* R *with* G *if and only if for* $\sigma \neq 1$ *in* G *there exist no idempotents* e *such that* $\sigma \mid Ce = I_{Ce}$.

Proof. Let $\sigma \neq 1$. If there exists e such that $\sigma \mid Ce = I_{Ce}$, then $\sigma(C(1-e)) = C(1-e)$. Hence, σ is not strongly distinct from 1. Conversely, if σ is not strongly distinct from 1, then there exists an idempotent e such that $\sigma(x)e = xe$ for all $x \in C$. We may assume that e is primitive. Then $\sigma(e)=e$. Hence, $\sigma \mid Ce = I_{Ce}$.

Proposition 9. *Let* Λ *be a Galois extension of* R *with* G. *We assume*

$R=Re_1 \oplus Re_2$. Then Λe_i is a Galois extension of Re_i with group G and $\Lambda=\Lambda e_1 \oplus \Lambda e_2$. Let H, H_i be central subgroup of G in Λ and Λ_i, respectively. Then $C=\Lambda^H$ if and only if $(e_i\Lambda)^{H_i}=Ce_i$ and $H=H_1=H_2$.

Proof. We note tnat $C=\Lambda^H$ if and only if C is a Galois extension of R with G/H (see [3], and [11]). It is clear that $H \subseteqq H_1 \cap H_2$. We assume $C=\Lambda^H$. Then if $\sigma \notin H$, $\sigma | Ce_i \neq I_{Ce_i}$ by Lemma 8. Hence, $H \supseteqq H_i$ and $H=H_1=H_2$. $(\Lambda e_i)^{H_i} \subseteqq C$. Therefore, $(\Lambda e_i)^{H_i}=Ce_i$. The converse is clear.

Thus, we may assume that R is indecomposable and there exists only a finite number of primitive idempotents e_i in C. Since $\sigma(e_i)$ is also primitive, Λe_i is isomorphic to each other and a decomposition group T_i induces the group of automorphism of Λe_i. It is clear that Λe_i is a Galois extension of Re_i with T_i (cf. [3]).

Theorem 10. *Let R be indecomposable and Λ a Galois extension of R with G. Then $\Lambda^H=C$ if and only if the central group H_1 in T_1 is a normal subgroup in G.*

Proof. As above we have $\Lambda=\sum \oplus \Lambda e_i$ and Λe_i is a Galois extension of Re_i with T_i. Since Ce_i is indecomposable, $Ce_i=(\Lambda e_i)^{H_i}$. by [11], Theorem 2. Let $G=T_1+\tau_2 T_1+\cdots+\tau_s T_1$. Then it is clear that $T_i=\tau_i T_1 \tau_i^{-1}$ by rearranging e_i and $H_i=\tau_i H_1 \tau_i^{-1}$. By the same argument as the above proof we have $\Lambda^H=C$ if and only if $H=H_1=\tau_2 H_1 \tau_2^{-1}=\cdots$.

We conclude this section with the following.

Proposition 11. *Let Λ be a separable R-algebra with group G. We assume that $R=\Lambda^G$ is indecomposable. Then Λ is a Galois extension of R with G if and only if an indecomposable component Λe_i of Λ is a Galois extension of Re_i with T_i, where T_i is the decomposition group of e_i.*

Proof. The "only if" part is clear. We assume that Λe_1 is a Galois extension of Re_1 with T_1. Let $\Lambda=\Lambda e_1 \oplus \cdots \oplus \Lambda e_t$. We use the same argument as in [6], p. 70. Let T_i be the decomposition group of e_i and $G=T_i+\sigma_{i,2}T_i+\cdots+\sigma_{i,t}T_i=T_i+T_i\sigma_{i,2}+\cdots+T_i\sigma_{i,t}$. We may assume $\sum \oplus \Lambda u_\sigma=\Delta(\Lambda, G) \supseteqq \Delta(\Lambda_i, T_i)$, where $\Lambda_i=\Lambda e_i$.

$$
\begin{aligned}
\Delta(\Lambda, G) = {} & \Delta(\Lambda_1, T_1) \oplus \Delta(\Lambda_1, T_1) u_{\sigma_{1,2}} \oplus \cdots \oplus \Delta(\Lambda_1, T_1) u_{\sigma_{1,t}} \\
& \oplus \Delta(\Lambda_2, T_2) \oplus \Delta(\Lambda_2, T_2) u_{\sigma_{2,2}} \oplus \cdots \oplus \Delta(\Lambda_2, T_2) u_{\sigma_{2,t}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \oplus \Delta(\Lambda_t, T_t) \oplus \Delta(\Lambda_t, T_t) u_{\sigma_{t,2}} \oplus \cdots \oplus \Delta(\Lambda_t, T_t) u_{\sigma_{t,t}}
\end{aligned}
$$

as a module.

From the assumption we have $\Delta(\Lambda_1, T_1) \approx \mathrm{Hom}_R(\Lambda_1, \Lambda_1)$. We have a natural homomorphism φ of $\Delta(\Lambda, G)$ to $\mathrm{Hom}_R(\Lambda, \Lambda)=\mathrm{Hom}_R(\Lambda e_1 \oplus \cdots \oplus \Lambda e_t, \Lambda e_1 \oplus \cdots \oplus \Lambda e_t)$. We consider an operation of $\varphi(\Delta(\Lambda_i, T_i)u_{\sigma_{i,j}})$ on Λe_k. If $\sigma_{i,j}(e_k)=e_i$, $\varphi(\Delta(\Lambda_i, T_i)u_{\sigma_{i,j}})(\Lambda e_k)=\varphi(\Delta(\Lambda_i, T_i))(\Lambda e_i)$. Since Λ_i is a Galois extension of Re_i $\varphi(\Delta(\Lambda_i, T_i))\approx\Delta(\Lambda_i, T_i)=\mathrm{Hom}_R(\Lambda e_i, \Lambda e_i)$. Hence, $\Delta(\Lambda_i, T_i)u_{\sigma_{i,j}}\approx\varphi(\Delta(\Lambda_i, T_i)u_{\sigma_{i,j}})=\mathrm{Hom}_R(\Lambda e_k, \Lambda e_i)$ and $\varphi(\Delta(\Lambda_i, T_i u_{\sigma_{i,j}})(\Lambda e_{k'})=(0)$ if $k \neq k'$. Conversely, we can find, for $\mathrm{Hom}_R(\Lambda e_i, \Lambda e_j)$, $\Delta(\Lambda_s, T_s)u_\sigma$ such that $\varphi(\Delta(\Lambda_s, T_s)u_\sigma)\approx\mathrm{Hom}_R(\Lambda e_i, \Lambda e_j)$. Hence, φ is isomprphic. Therefore, Λ is a Galois extension of R with G.

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