

## ON SEMI-PRIMARY PP-RINGS

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This paper is a supplement to the author [2]. Let  $\Lambda$  be a ring with identity. If every principal left ideal in  $\Lambda$  is  $\Lambda$ -projective, then Hattori and Nakano called  $\Lambda$  a left PP-ring in [3], [4]. Nakano and Chase ([1], [4]) showed that if  $\Lambda$  is a semi-primary left PP-ring, then  $\Lambda$  is a generalized triangular matrix ring. The author has defined a generalized triangular matrix ring over semi-simple rings with bi-linear mappings  $\varphi_{i,k}^j$  in [2] and found a criterion of semi-primary hereditary ring.

In §1 we shall use the same argument and give a similar criterion of a semi-primary left PP-ring. Using this criterion we shall show that  $\Lambda$  is a left PP-ring if and only if  $\Lambda$  is a right PP-ring, provided  $\Lambda$  is semi-primary.

As we see in [2], some results were obtained from monomorphic mapping  $\varphi_{i,k}^j$  in a hereditary ring. Thus, in §2, we define a partially PP-ring, which is a ring with property that  $\varphi_{i,k}^j$  is monomorphic and show that if  $\Lambda$  is a semi-primary partially PP-ring with nilpotency  $n$ , then  $\Lambda$  is isomorphic to a generalized triangular matrix ring over semi-simple rings with degree  $n$  and each component of it is uniquely determined up to isomorphism. From this fact we note that some results in [2] are generalized in a case of partially PP-ring.

In this paper we only consider semi-primary rings and semi-simple rings with minimum conditions.

### 1. PP-rings

We recall the definition of a *generalized triangular matrix ring* (briefly *g.t.a. matrix ring*).

Let  $\{R_1, R_2, \dots, R_n\}$  be a set of semi-simple rings and  $\{M_{i,j} \text{ for } i > j\}$  a set of  $R_i, R_j$ -modules. With a bi-linear mapping  $\varphi_{i,k}^j : M_{i,j} \otimes_{R_j} M_{j,k} \rightarrow M_{i,k}$  we define a *g.t.a. matrix ring* by the usual way. We denote it by  $T_n(R_i; M_{i,j})$  and  $n$  is called the *degree* of it:

$$\Lambda = \begin{pmatrix} R_1 & 0 & \dots & \dots \\ M_{2,1} & R_2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ M_{n,n-1} & M_{n,n-2} & \dots & R_n \end{pmatrix} = T_n(R_i; M_{i,j}),$$

(see [2], § 2).

The following lemmas were given in [1] and [4]. We shall give here simple proofs, one of which is the same as in [1], Theorem 4.2 and will be used later.

**Lemma 1.** *Let  $\Lambda$  be a ring and  $M$  a left  $\Lambda$ -module. If  $\Lambda m$  is  $\Lambda$ -projective and  $em = m$  for  $m \in M$  and an idempotent  $e$  in  $\Lambda$ , then there exists an idempotent  $f$  in  $\Lambda$  such that  $ef = fe = f$ ,  $\Lambda f \approx \Lambda m$  and  $fm = m$ . Especially,  $e\Lambda em$  is  $e\Lambda e$ -projective.*

Proof. Since  $\Lambda e \rightarrow \Lambda m \rightarrow (0)$  splits, we obtain  $\Lambda e = \Lambda f \oplus \Lambda f'$  and  $\Lambda f \approx \Lambda m$ ,  $fm = m$ . Hence,  $e\Lambda em = e\Lambda m \approx e\Lambda f = e\Lambda efe$  is a direct summand of  $e\Lambda e$ .

**Lemma 2.** 1) *Every semi-primary left PP-ring is a g.t.a.matrix ring.*

2) *Let  $\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$  be a g.t.a.matrix ring, where  $R$  is semi-simple,  $S$  is a semi-primary left PP-ring and  $M$  is a  $S, R$ -module. If every principal left  $S$ -module in  $M$  is  $S$ -projective, then  $\Lambda$  is a left PP-ring.*

Proof. 1) Let  $N = N(\Lambda)$  be the radical of  $\Lambda$ . We assume  $N^s \neq (0)$ ,  $N^{s+1} = (0)$ . Then  $N^s$  is completely reducible and hence,  $N^s$  is  $\Lambda$ -projective by the assumption;  $N^s \approx \sum \oplus \Lambda e_i$ , where  $e_i$  is a primitive idempotent. Then  $(0) = \sum \oplus N e_i$ . Let  $\{e_i\}$  be a complete set of mutually orthogonal primitive idempotents such that  $1 = \sum_{i=1}^n e_i$  and  $N e_i = (0)$  for  $i \leq t \leq n$  and  $N e_i \neq (0)$  for  $j < i$ . Let  $E = e_1 + \dots + e_{i-1}$ . Since  $e_j \perp e_i$  for  $j < i \leq t$ ,  $E\Lambda(1-E) = EN(1-E) = (0)$ . It is clear that  $\Lambda = T_2(E\Lambda E, (1-E)\Lambda(1-E); (1-E)\Lambda E)$  and that  $(1-E)\Lambda(1-E)$  is semi-simple and  $E\Lambda E$  is a left PP-ring by Lemma 1. Hence, we can prove 1) by induction on number of idempotents  $n$ .

2) Let  $\Lambda = T_2(R, S; M)$  and  $m \in M, r \in R$ . We put  $Rr = Re, e^2 = e$ . Then  $Mr + Sm = Me \oplus Sm(1-e)$ . Let  $x = T_2(r, s; m)$ . Then  $\Lambda x = T_2(Rr, Rs; Mr + Sm) = \Lambda T_2(e, 0; 0) \oplus T_2((0), (0); Sm(1-e)) \oplus T_2((0), (0); Ss)$ . Since  $Sm(1-e), Ss$  are  $S$ -projective, the last two modules are  $\Lambda$ -projective by [2] Lemma 4. Hence  $\Lambda x$  is  $\Lambda$ -projective.

**Proposition 1.**  *$\Lambda$  is a semi-primary left PP-ring if and only if  $\Lambda$*

is a g.t.a.matrix ring  $T_n(R_i; M_{i,j})$  over semi-simple rings  $R_i$  with the following conditions;

1)  $\varphi_{i,k}^j: M_{i,j} \otimes_{R_j} R_j x_{j,k} \rightarrow M_{i,j} x_{j,k}$  is monomorphic for all  $i > j > k$  and  $x_{j,k} \in M_{j,k}$ .

2) For any system  $\{x_{j+1,j}, \dots, x_{i,j}; x_{k,j} \in M_{k,j}, i > j\}$   $M_{i,j+1}C_{(j+1,j)} + \dots + M_{i,i-1}C_{(i-1,j)}$  is a direct sum in  $M_{i,j}$ , where  $R_k x_{k,j} = C(x_{k,j}) \oplus (\sum_{s=j+1}^{k-1} M_{k,s} x_{s,j}) \cap R_k x_{k,j}$  as a left  $R_k$ -module.

Proof. We use the same argument as in the proof of [2], Theorem 1. By induction argument and Lemma 2 it is sufficient to show that every principal submodule of

$$\begin{pmatrix} 0 \\ M_{2,1} \\ \vdots \\ M_{n,1} \end{pmatrix}$$

is  $\Gamma (= T_{n-1}(R_2, \dots, R_n; M_{i,j}, j \neq 1))$ -projective.

Let

$$x = \begin{pmatrix} x_{2,1} \\ \vdots \\ x_{n,1} \end{pmatrix}.$$

Then

$$\Gamma x = \begin{pmatrix} R_2 x_{2,1} \\ M_{3,2} x_{2,1} + R_3 x_{3,1} \\ \dots \\ \dots \\ M_{n,2} x_{2,1} + \dots + R_n x_{n,1} \end{pmatrix} \supset N'x = \begin{pmatrix} 0 \\ 0 \\ M_{3,2} x_{2,1} \\ \dots \\ \dots \\ M_{n,2} x_{2,1} + \dots + M_{n,n-1} x_{n-1,1} \end{pmatrix}$$

and

$$x\Gamma/N'x = \begin{pmatrix} C(x_{2,1}) \\ C(x_{3,1}) \\ \vdots \\ C(x_{n,1}) \end{pmatrix},$$

where  $N' = N(\Gamma)$ ,  $C(x_{2,1}) = R_2 x_{2,1}$ . Hence,  $\Gamma x$  is  $\Gamma$ -projective if and only if

$$\Gamma x \approx \Gamma \begin{pmatrix} x_{2,1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \Gamma \begin{pmatrix} 0 \\ C(x_{3,1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \oplus \Gamma \begin{pmatrix} 0 \\ 0 \\ \vdots \\ C(x_{n,1}) \end{pmatrix}$$

and

$$\Gamma \otimes \begin{pmatrix} 0 \\ 0 \\ \vdots \\ C(x_{i,1}) \\ \vdots \\ 0 \end{pmatrix} \rightarrow \Gamma$$

is monomorphic. Which is equivalent to 2) and 1')  $\varphi_{e,1}^j M_{i,j} \otimes C(x_{j,1}) \rightarrow M_{i,1}$  is monomorphic. However if we replace  $\{x_{2,1}, \dots, x_{n,1}\}$  by  $\{0, 0, \dots, x_{i,1}, \dots, x_{n,1}\}$ , the  $C(x_{i,1}) = R_i x_{i,1}$ . Hence, we have 1).

REMARK 1. Let  $e$  be a sum of the set of non-isomorphic primitive idempotents in  $\Lambda$ . From [3], Corollary 1 we know that  $\Lambda$  is hereditary if and only if so is  $e\Lambda e$ . However, it is not true for a left PP-ring as we see in the following example. (Only if part is true by Lemma 1).

EXAMPLE. Let  $K$  be the field of real numbers.  $M, N$  and  $L$  be  $K$ -vector spaces with basis  $(u, v)$ ,  $(a, b)$  and  $(t, s)$ , respectively. We define a bi-linear mapping  $\varphi: M \otimes N \rightarrow L$ .  $M \otimes N = u \otimes (Ka + Kb) \oplus v \otimes (Ka + Kb)$ .  $\varphi(u(ax + by) + v(ax' + by')) = t(x + y') + s(y - x')$ . Then we can easily check that  $\varphi$  is monomorphic on  $M \otimes Kn$  for any  $n \neq 0$  in  $N$ . However,  $\varphi(u(a + b) + v(a - b)) = 0$  and  $u(a + b) + v(a - b) \neq 0$ .

$$\text{Let } \Lambda = \begin{pmatrix} K & 0 & 0 \\ N & K_2 & 0 \\ L & (M, M) & K \end{pmatrix} \text{ and } e = \begin{pmatrix} 1 & 0 \\ e_{1,1} \\ 0 & 1 \end{pmatrix}. \text{ Then } e\Lambda e = \begin{pmatrix} K & 0 & 0 \\ N & K & 0 \\ L & M & K \end{pmatrix}.$$

From Proposition 1 and the above observation we know that  $e\Lambda e$  is a left PP-ring. Let

$$x = \begin{pmatrix} 0 & 0 & 0 \\ \begin{pmatrix} a \\ b \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in  $\Lambda$ .  $(M, M) \otimes_{K_2} K_2 \begin{pmatrix} a \\ b \end{pmatrix} = (M, M) \otimes \begin{pmatrix} N \\ N \end{pmatrix}$ . Since  $\varphi$  is not monomorphic on  $M \otimes N$ ,  $\tilde{\varphi}: (M, M) \otimes_{K_2} K_2 \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow L$  is not monomorphic, and hence,  $\Lambda$  is not a left PP-ring.

REMARK 2. The above example shows that the endomorphism ring of finitely generated projective module over a PP-ring is not, in general, a PP-ring, (cf. [2], Theorem 7). Because  $\Lambda = \text{Hom}_{e\Lambda e}^r(\Lambda e, \Lambda e)$  and  $\Lambda e$  is a finitely generated  $e\Lambda e$ -projective module.

Lemma 2. Let  $\Lambda$  be a g.t.a.matrix ring with property 1) and  $M_{i,j} \ni y$ . We assume  $R_i y \approx R_i e$  by a correspondence  $y \leftarrow e$  and  $e^2 = e$ . If  $xy = 0$  for  $x \in M_{k,i}$ , then  $xe = 0$ .

Proof.  $M_{k,i} \otimes_{R_i} R_i y \approx M_{k,i} \otimes_{R_i} R_i e$  is a direct summand of  $M_{k,i} \approx M_{k,i} \otimes_{R_i} R_i$ . Hence,  $x \otimes y \approx x \otimes e = xe \otimes 1 \approx xe$ . Since  $M_{k,i} \otimes_{R_i} R_i y \rightarrow M_{k,j}$  is monomorphic,  $xe = 0$ .

From the example given in [1] we know that a left PP-ring is not necessarily to be a right PP-ring. However, we have

**Theorem 1.** *Let  $\Lambda$  be semi-primary. If  $\Lambda$  is a left PP-ring, then  $\Lambda$  is a right PP-ring.*

Proof. We shall show that  $\Lambda$  satisfies the following conditions 1') and 2') which are replaced left by right in Proposition 1.

1')  $\varphi_{i,k}^j: x_{i,j} R_j \otimes_{R_j} M_{j,k} \rightarrow M_{i,k}$  is monomorphic.

2') For any system  $(x_{j,i}, x_{j,i+1}, \dots, x_{j,j-1})$   $D(x_{j,j-1})M_{j-1,i} + D(x_{j,j-2})M_{j-2,i} + \dots + D(x_{j,i+1})M_{i+1,i}$  is a direct sum in  $M_{j,i}$ , where  $x_{j,k} R_k = D(x_{j,k}) \oplus (\sum_{t=k+1}^{j-1} x_{j,t} M_{t,k}) \cap x_{j,k} R_k$  as a right  $R_k$ -module.

1'): We assume  $\varphi_{j,s}^k: x_{j,k} R_k \otimes M_{k,s} \rightarrow x_{j,k} M_{k,s}$  is not monomorphic. Then there were elements  $x = x_{j,k}$  and  $m \in M_{k,s}$  such that  $x \otimes m \neq 0$  in  $x_{j,k} R_k \otimes M_{k,s}$  and  $xm = 0$ .  $M_{k,s} = Rm \oplus N$  and  $M_{j,k} = xR_k \oplus N'$  as left and right  $R_k$ -module, respectively. Then  $M_{j,k} \otimes M_{k,s} = xR_k \otimes R_k m \oplus N' \otimes R_k m \oplus xR_k \otimes N \oplus N' \otimes N$ . Hence  $x \otimes m \neq 0$  in  $M_{j,k} \otimes R_k m$ , which contradicts 1) of Proposition 1. Next, we shall show that  $\Lambda$  satisfies 2'). Let  $\Lambda = T_n(R_i; M_{i,j}); R_i$  simple ring. We prove 2') by induction on degree  $n$ . If  $n = 1$ , then  $\Lambda$  is simple, and hence, it is clear. From Lemma 1 and the induction hypothesis  $\Gamma_i = T_{n-1}(R_1, \dots, R_n; M_{k,j}, k \neq i, j \neq i)$  satisfies 2'). Hence it is sufficient to consider a system

$$\{x_{n,2}, x_{n,3}, \dots, x_{n,n-1}; x_{n,j} \in M_{n,j}\}.$$

We shall show a sum  $x_{n,n-1} M_{n-1,1} + D(x_{n,n-2}) M_{n-2,1} \dots + D(x_{n,2}) M_{2,1}$  is a direct sum. If  $D(x_{n,i}) = 0$  for some  $i$ , then for  $\{x_{n,2}, \dots, x_{n,n-1}\}$  in  $\Gamma_i$   $x_{n,n-1} M_{n-1,1} \oplus D'(x_{n,n-2}) M_{n-2,1} \oplus \dots \oplus D'(x_{n,2}) M_{2,1}$ , where  $D'(x_{n,j})$  is a direct summand in  $M_{n,j}$  as in 2'). Since  $D'(x_{n,j}) \supseteq D(x_{n,j})$ , we obtain 2'). Hence, we assume  $D(x_{n,i}) \neq (0)$  for all  $i$ . If the above sum were not a direct sum then there were an element  $0 = x'_{n,n-1} m_{n-1,1} + \dots + x'_{n,2} m_{2,1}$  such that some  $x'_{n,j} m_{j,1} \neq 0$ , where  $x'_{n,j} \in D(x_{n,j})$ . By the same reason as above, we may assume  $x'_{n,j} m_{j,1} \neq 0$  for all  $j$ .

Let  $x'_{n,i} R_i \approx e_i R_i$  by a correspondence  $x'_{n,i} \leftrightarrow e_i$ . Then  $x'_{n,i} e_i = x'_{n,i}$ . Hence, we may assume  $0 \neq m_{i,1} = e_i m_{i,1}$ . We have, from Lemma 1, an idempotent  $f_i$  such that  $R_i m_{i,1} \approx R_i f_i$  and  $f_i e_i f_i = f_i e_i, f_i m_{i,1} = m_{i,1}$ . Since  $x'_{n,i} R_i \approx e_i R_i, x'_{n,1} f_i R_i \approx f_i R_i$ . Thus, we may assume that  $0 \neq x'_{n,i} f_i = x'_{n,i}, f_i m_{i,1} = m_{i,1}$  and  $x'_{n,i} R_i \approx f_i R_i, R_i m_{i,1} \approx R_i f_i$ . Hence, the right annihilator

$r(x'_{n,i})$  of  $x'_{n,i}$  in  $R_i$  is equal to  $(1-f_i)R_i$ . We consider a system  $\{m_{2,1}, m_{3,1}, \dots, m_{n-1,1}\}$  as in 2). If  $C(m_{i,1})=(0)$ , then there exist elements  $m_{i,j} \in M_{i,j}$  such that  $m_{i,1} = m_{i,2}m_{2,1} + \dots + m_{i,i-1}m_{i-1,1}$  ( $m_{i,j} = m_{i,j}f_j$ ). Hence,

$$(*) \quad 0 = x'_{n,n-1}m_{n-1,1} + \dots + x'_{n,i+1}m_{i+1,1} + (x'_{n,i}m_{i,i-1} + x'_{n,i-1})m_{i-1,1} + (x'_{n,i}m_{i,2} + x'_{n,2})m_{2,1}.$$

Again we consider a system  $\{x'_{n,n-1}, \dots, x'_{n,i-1}, (x'_{n,i-1} + x'_{n,i}m_{i,i-1}), \dots, (x'_{n,2} + x'_{n,i}m_{i,2})\}$  in  $\Gamma_i$ . Since  $x'_{n,t}R_t \cap (\sum_{k=t+1}^{n-1} x'_{n,k}M_{k,t}) \subseteq x'_{n,t}R_t \cap (\sum_{k=t+1}^{n-1} x_{n,k}M_{k,t}) = (0)$  for  $t > i$ ,  $D(x'_{n,t}) = x'_{n,t}R_t$  for  $t > i$ . For  $(x'_{n,t} + x'_{n,i}m_{i,t})r \in (x'_{n,t} + x'_{n,i}m_{i,t})R_t \cap ((x'_{n,t+1} + x'_{n,i}m_{i,t+1})M_{t+1,t} + \dots + (x'_{n,i-1} + x'_{n,i}m_{i,i-1})M_{i-1,t} + x'_{n,i+1}M_{i+1,t} + \dots + x'_{n,n-1}M_{n-1,t})$  ( $t < i$ ), we have  $x'_{n,t}r \in D(x_{n,t}) \cap (\sum_{k=t+1}^{n-1} x_{n,k}M_{k,t}) = (0)$ . Hence,  $r \in r(x'_{n,t}) = (1-f_t)R_t$ . Therefore,  $x'_{n,i}m_{i,t}r = 0$ , which means  $D(x'_{n,t} + x'_{n,i}m_{i,t}) = (x'_{n,t} + x'_{n,i}m_{i,t})R_t$ . From the induction hypothesis we know that  $(*)$  is a direct sum. Hence,  $x'_{n,n-1}m_{n-1,1} = 0$  or  $(x'_{n,i}m_{i,2} + x'_{n,2})m_{2,1} = 0$ . From the latter and Lemma 3 we obtain  $0 = x'_{n,i}m_{i,2}f_2 + x'_{n,2}f_2 = x'_{n,i}m_{i,2} + x'_{n,2}$ , which contradicts  $D(x'_{n,2}) \neq (0)$ . In either case we have a contradiction and hence,  $C(m_{i,1}) \neq (0)$  for all  $i$ . Therefore, we have from 2)

$$M_{n,2}m_{2,1} + M_{n,3}C(m_{3,1}) + \dots + M_{n,n-1}C(m_{n-1,1})$$

is a direct sum. Hence,

$$\begin{aligned} 0 &= x'_{n,2}m_{2,1} + \dots + x'_{n,n-1}m_{n-1,1} \\ &= (x'_{n,2} + x'_{n,n-1}t_{n-1,2} + \dots + x'_{n,3}t_{3,2})m_{2,1} \\ &\quad + (x'_{n,3} + \dots)m'_{3,1} \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots \\ &\quad + x'_{n,n-1}m'_{n-1,1}, \end{aligned}$$

where  $m_{i,1} = m'_{i,1} + t_{i,2}m_{2,1} + \dots + t_{i,i-1}m_{i-1,1}$  and  $t_{i,j}e_j = t_{i,j}$ ,  $m'_{i,1} \in C(m_{i,1})$ . Hence,  $x'_{n,2} = -(x'_{n,n-1}t_{n-1,2} + \dots + x'_{n,3}t_{3,2})$ , which contradicts the fact  $D(x'_{n,2}) \neq (0)$ . We have proved the theorem.

**2. Partially PP-rings**

We found a criterion of semi-primary PP-rings in Proposition 1. However, we need only the condition 1) in this section.

Let  $\Lambda$  be a semi-primary ring such that  $\Lambda/N = \sum \oplus S_i$ ;  $S_i$  is a simple ring. Let  $1 = \sum_i E_i$ ,  $E_i^2 = E_i$  and  $E_i$  is the identity in  $S_i$  modulo  $N$ . We assume for idempotents  $E'_i, E'_j$  that  $E_i \approx E'_i, E_j \approx E'_j$ . Let  $x$  be in  $E_i \Lambda E_j$ .

Since  $\Lambda E_j \approx \Lambda E'_j$ , there exists  $y'$  in  $E_i \Lambda E'_j$  such that  $\Lambda x \approx \Lambda y'$ . Furthermore, since  $E_i \Lambda \approx E'_i \Lambda$ , we have  $t \in E'_i \Lambda E_i$ ,  $u \in E_i \Lambda E'_i$  such that  $ut = E_i$ . If we put  $y = ty' \in E'_i \Lambda E'_j$ , then  $\Lambda y = \Lambda ty' = \Lambda E_i y' = \Lambda y' \approx \Lambda x$ , since  $\Lambda t \supseteq \Lambda E_i \supseteq \Lambda t$ . Hence, if  $\Lambda x$  is  $\Lambda$ -projective for every  $x$  in  $E_i \Lambda E_j$ , then  $\Lambda y$  is  $\Lambda$ -projective for every  $y$  in  $E'_i \Lambda E'_j$ .

Thus, we can define a partially PP-ring as follows:

Let  $\Lambda$  and  $E_i$  be as above. If  $\Lambda x$  is  $\Lambda$ -projective for all  $x \in E_i \Lambda E_j$  ( $i, j = 1, \dots, n$ ), then we call  $\Lambda$  a *partially PP-ring*.

From Lemma 1, we obtain

**Lemma 4.** *Let  $\Lambda$  be a partially PP-ring and  $e$  an idempotent. Then  $e\Lambda e$  is a partially PP-ring.*

**Proposition 2.**  *$\Lambda$  is a partially PP-ring if and only if  $\Lambda$  is a g.t.a. matrix ring  $T_n(S_i; M_{i,j})$  over simple rings  $S_i$  with property 1) in Proposition 1).*

Proof. First we shall show that a partially PP-ring is a g.t.a. matrix ring. Let  $1 = \sum_{i,j=1}^{n,p_i} e_{i,j}$ , where  $\{e_{i,j}\}$  is a complete set of primitive idempotents. Let  $N^n = (0)$ ,  $N^{n-1} \neq (0)$ . Then there exist primitive idempotents  $e, f$  such that  $(0) \neq eN^{n-1}f \subset E_i \Lambda E_j$ , where  $e \in E_i, f \in E_j$ . Hence,  $\Lambda x$  is  $\Lambda$ -projective for  $0 \neq x \in eN^{n-1}f$ . Since  $\Lambda x \approx \sum \oplus \Lambda e_{\kappa,1}$  and  $Nx = (0)$ , there exists a primitive idempotent  $e_{\kappa,1}$  such that  $Ne_{\kappa,1} = (0)$ . Then we can prove similarly to the proof of Lemma 2 that  $\Lambda \approx T(R_i; M_{i,j}); R_i$  semi-simple. Hence, the proposition is an immediate consequence from the next lemma.

**Lemma 5.** *Let  $\Lambda$  be a g.t.a. matrix ring.  $T_k(S_i; M_{i,j})$  over simple rings  $S_i$ .  $\Lambda$  is a partially PP-ring if and only if  $\varphi_{i,j,k} : M_{i,j} \otimes S_j x_{j,k}$  is monomorphic for every  $i, j, k$  and  $x \in M_{j,k}$ .*

Proof. It is clear from the proof of Proposition 1.

REMARK 3. From the first half of the proof of Theorem 1,  $\Lambda$  is a partially PP-ring if  $x\Lambda$  is  $\Lambda$ -projective for every  $x \in E_i \Lambda E_j$ .

REMARK 4. We can show by examples that the set of semi-primary hereditary ring  $\subset$  that of PP-rings  $\subset$  that of partially PP-rings.

Let  $\Lambda$  be a partially PP-ring and  $1 = \sum_{i,j=1}^{n,p_i} e_{i,j}$  as in the proof of Proposition 2. Since we can find an idempotent  $e_{n,1}$  such that  $Ne_{n,1} = (0)$ , we may assume  $Ne_{p_1,1} = Ne_{p_1+1,1} = \dots = Ne_{n,1} = (0)$  and  $Ne_{i,1} \neq (0)$  for  $i < p$ . Then  $\Lambda$  is isomorphic to  $\begin{pmatrix} S_1 & 0 \\ M_1 & R_1 \end{pmatrix}$  as in the proof of Lemma 2.  $S_1$  is a

partially PP-ring by Lemma 4. After rearranging primitive idempotents  $e_{p_2,1}, \dots, e_{p_1-1,1}$  such that  $N(S_1)e_{p_2,1} = \dots = N(S_1)e_{p_1-1,1} = (0)$  and  $N(S_1)e_{i,1} \neq (0)$  for  $i < p_2$ , we have

$$\Lambda = \begin{pmatrix} S_2 & & 0 \\ M_1 & R_2 & \\ M_2 & M_3 & R_1 \end{pmatrix}; S_2 \text{ is a partially PP-ring and } R_1, R_2 \text{ are semi-simple.}$$

Furthermore,  $M_2 f \neq (0)$  for any primitive idempotent  $f$  in  $R_2$ . Repeating this argument we know that  $\Lambda \approx T_{n'}(R_i; M_{i,j})$  over semi-simple rings  $R_i$  and  $M_{i+1,i} f_i \neq (0)$  for any primitive idempotent  $f_i$  in  $R_i$ .

The following theorem and corollary are generalizations of [2], Theorem 4''' and Proposition 5.

**Theorem 2.** *Let  $\Lambda$  be a semi-primary partially PP-ring and  $N^{n-1} \neq (0)$ ,  $N^n = (0)$ . Then  $\Lambda$  is isomorphic to a g.t.a.matrix ring  $T_n(R_i, M_{i,j})$  over semi-simple rings  $R_i$  with degree  $n$ . Furthermore,  $M_{i,j} \supseteq M_{i,i-1} M_{i-1,i-2} \dots M_{j+1,j} f_j \neq (0)$  for any idempotent  $f_j$  in  $R_j$  and for all  $i$ .*

*Proof.* From the above argument we have  $\Lambda = T_m(R_i; M_{i,j})$  and  $M_{i+1,i} f_i \neq (0)$  for all primitive idempotent  $f_i$  in  $R_i$ . Let  $L = M_{i,i-1} \dots M_{j+1,j}$ . We assume that  $L f_j \neq (0)$  for any primitive idempotents  $f_j$  in  $R_j$ . There exist  $m \in M_{j,j-1}$  and  $f_j$  such that  $f_j m f_{j-1} \neq (0)$  and  $R_j f_j \approx R_j f_j m f_{j-1}$ . Since  $L f_j \neq (0)$ ,  $LM_{j,j-1} \supseteq L f_j m f_{j-1} \neq (0)$  by Lemma 3. Thus, we can prove by induction  $M_{i',i'-1} \dots M_{j,j-1} f_{j-1} \neq (0)$  for all  $i'$ . Therefore,  $(0) \neq M_{m,m-1} \dots M_{2,1} \subseteq N^{m-1}$ . Hence,  $m-1 < n$ . Since  $N^n = (0)$ ,  $m \geq n$ . Hence,  $n = m$ .

**Corollary.** *Let  $\Lambda$  be as above. Then*

$$n = \text{gl.dim } (\Lambda/N^2) = l(\Lambda),$$

where  $l(\Lambda)$  is the maximal length of connected sequence of primitive idempotents (see [2]).

*Proof.* From [2], Proposition 4 we know  $n \leq l(\Lambda) = \text{gl.dim } (\Lambda/N^2)$ . On the other hand  $\Lambda \approx T_n(R_i; M_{i,j})$  by Theorem 2. Hence  $n \geq l(\Lambda)$ .

**REMARK 5.** We know from Theorem 2 that  $n(f_i) = n - i + 1$ , where  $n(f)$  is an integer  $m$  such that  $N^{m-1} f \neq (0)$ ,  $N^m f = (0)$ , (see [2]).

In the expression of  $\Lambda$  as a g.t.a.matrix ring in Theorem 2 the set of primitive idempotents in  $R_j$  consists of those  $f_i$  such that  $n(f_i) = n - i + 1$ . Hence,  $R_i, M_{k,j}$  in  $T_n(R_i, M_{k,j})$  are uniquely determined up to isomorphism.

By making use of the same argument as in the proof of [2], Proposition 8 we have



**Proposition 3.** *Let  $\Lambda$  be an indecomposable semi-primary partially PP-ring. Then the center  $K$  of  $\Lambda$  is a field. If  $\Lambda \otimes_K L$  is a semi-primary partially PP-ring for every extension field  $L$  of  $K$ , then  $\Lambda/N$  is separable over  $K$ .*

REMARK 6. The converse is not true in general. In the example after Proposition 1 we obtain  $\Lambda/N$  is separable over  $K$ . However if  $C$  is the field of complex numbers, then  $\varphi$  is not monomorphic on  $(M \otimes C) \otimes_C (a + bi)$ .

REMARK 7. Proposition 10 in [2] is valid for a semi-primary PP-ring from Theorem 2. Furthermore, all results in [2], §5 are true for a semi-primary PP-ring by a slight change of proof.

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