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MULTIPLICATIVE STRUCTURES IN MOD q COHOMOLOGY THEORIES I.

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Introduction. In the course of developments of algebraic topology, one of the advantages to use the cohomology theory (in the ordinary sense) rather than the homology theory was the existence of multiplicative structures, i. e., cup products. Cup products exist not only in the integral cohomology but also in the cohomology with any coefficients whenever the coefficient domain has a multiplicative structure. Recently it is known that for any general cohomology theory one can associate cohomology theories with coefficients. Since there are important general cohomology theories with multiplicative structures such as K-theories, one can expect to introduce multiplicative structures in their associated cohomology theories with coefficients. The present work is directed to introduce and to study multiplicative structures in cohomology theories with coefficients. However, since coefficients are limited to finitely generated groups at the present time, the most important cases are those with coefficients in some finite cyclic groups, i. e., Z_q, $q>1$. Henceforth our research is limited to mod q cohomology theories to avoid complexity of discussions.

To introduce multiplications in mod q cohomology theories it is important to check a connection with the multiplication in the original cohomology theory. Since there is the notion of "reduction mod q" also in general cohomology theories, we postulate this connection as the compatibility through reduction mod q, postulation (Λ_1). In the ordinary mod q cohomology theory, the mod q Bockstein homomorphism works as a derivation. Since this property has been proved to be much useful, we postulate a corresponding property also in general mod q cohomology theories, postulation (Λ_2). Proof of associativity of multiplications in mod q cohomology theories are very round about even if it is possible. But, to get some uniqueness type theorems, it is sufficient to postulate a weaker form of associativity, which we call "quasi-associativity," postulation (Λ_3). We have an example of mod q

cohomology theory which have a multiplication satisfying (Λ_1), (Λ_2) and (Λ_3) but no commutative one, i.e., complex K-theory mod 2. So that we postulate nothing about commutativity. A multiplication in a mod q cohomology theory satisfying (Λ_1), (Λ_2) and (Λ_3) is called "an admissible multiplication."

If $q \not\equiv 2$ (mod 4), then there exist an admissible multiplication always whenever the multiplication in the original cohomology theory is given and associative (Theorem 5.9). But, if $q \equiv 2$ (mod 4), some condition is necessary for the existence of admissible multiplications. A sufficient condition for this is obtained (Theorem 5.9). In case $q \equiv 2$ (mod 4), a necessary and sufficient condition for the existence of a multiplication satisfying only (Λ_1) is obtained (Corollary 5.6). There are examples which do not satisfy this condition, e.g., KO-cohomology theory of real vector bundles.

Not only the existence but also a uniqueness-type theorem of admissible multiplications is discussed, which states that admissible multiplications are in a one-to-one correspondence with elements of a group which is specific to the considered cohomology theory under the assumption that the original multiplication is commutative and associative (Corollary 3.10). In case of ordinary cohomology this group consists only of zero, hence true uniqueness holds. In case of complex K-theory, this group is Z_q, hence there are q different admissible multiplications in mod q K-theory. In case of the KO-theory, this group consists only of zero if q is odd, hence true uniqueness holds for odd q.

In §1 we summarize some known basic properties of stable homotopy groups of two CW-complexes, which is necessary as preliminaries. In §2 we exhibit some elements of additive mod q cohomology theories: reduction mod q, Bockstein homomorphisms, universal coefficient sequences, cohomology maps induced by coefficient homomorphisms, etc. In §3 we develop an axiomatic approach to multiplications in mod q cohomology theories, and arrive to the notion of admissible multiplications. Uniqueness-type theorems and deviations from the commutativities are discussed. In §4 we compute some stable homotopy groups and make preparations for the existence theorem of admissible multiplications from the homotopy theoretical point of view. §5 is devoted to the proof of the existence theorem of admissible multiplications by constructing a multiplication.

In a subsequent paper with the same title we will discuss associativities, commutativities and multiplications in Bockstein spectral sequences, at least for the K-theory.

1. Preliminaries

1.1. First we shall fix some notations:

$X\wedge Y$ the reduced join of two spaces X and Y with base points,

$f\wedge g$ the reduced join of two base-point-preserving maps f and g,

$SX=X\wedge S^1$ the (reduced) suspension of X,

1_A (or simply 1) $: A\to A$ an identity map of A into itself,

$T(A, B)$ (or simply T) $: A\wedge B\to B\wedge A$ a map switching factors,

$S^nA=S(S^{n-1}A)=A\wedge S^{n-1}\wedge S^1=A\wedge S^n$ an n-fold suspension of A,

$S^nf=f\wedge 1_{S^n}$ an n-fold suspension of a map f,

$q: S^1\to S^1$ for an integer q to denote a map of degree q given by $q\{t\}=\{qt\}$ for $\{t \bmod 1\}\in S^1$.

Denote by $\{A, B\}$ the stable homotopy groups of CW-complexes A and B with base point, i.e., the limit of the sets of the base-point-preserving homotopy classes of maps $: S^nA\to S^nB$ with respect to suspensions, endowed with the usual structure of an abelian group. Obviously we have

$$\{A, B\}\cong\{S^nA, S^nB\}, \qquad n=0, 1, 2, \cdots. \tag{1.1}$$

For a map $f: S^nA\to S^nB$, $n\geqq 0$, we denote by the same letter f the stable homotopy class represented by f, i.e., $f\in\{A, B\}$ when there arises no confusion. For example,

$$1=1_A\in\{A, A\} \quad\text{and}\quad T=T(A, B)\in\{A\wedge B, B\wedge A\}$$

for the classes of the identity map and the switching map,

$$\eta\in\{S^2, S^1\} \quad\text{and}\quad \nu\in\{S^4, S^1\}$$

for the classes of the Hopf maps $\eta: S^3\to S^2$ and $\nu: S^7\to S^4$.

1.2. The *composition* of $\alpha\in\{A, B\}$ and $\beta\in\{B, C\}$, denoted by

$$\beta\circ\alpha \quad\text{or simply by}\quad \beta\alpha\in\{A, C\},$$

is defined as the class of $S^mg\circ S^nf$, where $f: S^mA\to S^mB$ and $g: S^nB\to S^nC$ are respectively representatives of α and β. Tnis definition is independent of the choices of f and g.

For $\alpha\in\{A, B\}$ and $\alpha'\in\{A', B'\}$, their *reduced join*

$$\alpha\wedge\alpha'\in\{A\wedge A', B\wedge B'\}$$

is defined as follows: choosing representatives $f: S^m A \to S^m B$ and $g: S^n A' \to S^n B'$ of α and α' respectively, $\alpha \wedge \alpha'$ is the class of the composition

$$(1\wedge T\wedge 1)\circ(f\wedge g)\circ(1\wedge T\wedge 1): A\wedge A'\wedge S^{m+n} = A\wedge A'\wedge S^m\wedge S^n$$
$$\to A\wedge S^m\wedge A'\wedge S^n \to B\wedge S^m\wedge B'\wedge S^n \to B\wedge B'\wedge S^{m+n}.$$

This definition is also independent of the particular choices of f and g.

We have obviously

(1.2). *The composition and the reduced join are bilinear and the relations*

$$(\beta\alpha)\wedge(\beta'\alpha') = (\beta\wedge\beta')(\alpha\wedge\alpha')$$

and $$(\alpha\wedge\alpha')\,T(A', A) = T(B', B)(\alpha'\wedge\alpha)$$

hold for $\alpha\in\{A, B\}$, $\alpha'\in\{A', B'\}$, $\beta\in\{B, C\}$ *and* $\beta'\in\{B', C'\}$.

We regard that, for any $\alpha\in\{A, B\}$ and the class $1_n\in\{S^n, S^n\}$ of the identity map, the relation

(1.3) $$S^n\alpha = \alpha\wedge 1_n = \alpha$$

holds *via* the identification (1.1). In general $1_n\wedge\alpha$ differs from $\alpha\wedge 1_n$, but, if $\alpha\in\{S^{p+s}, S^p\}$ and $\beta\in\{S^{q+t}, S^q\}$, then we see from (1.2) that

(1.4) $$\alpha\wedge\beta = (-1)^{(p+s)t}\alpha\beta = (-1)^{pt}\beta\alpha,$$

in particular $$1_n\wedge\alpha = (-1)^{ns}\alpha.$$

By the bilinearity of compositions, $\{A, A\}$ (or $\sum_n \{S^n A, A\}$) forms a ring (or a graded ring) with the composition as the multiplication.

Also the formula

$$\beta_*(\alpha) = \alpha^*(\beta) = \beta\alpha, \quad \alpha\in\{A, B\}, \ \beta\in\{B, C\},$$

defines homomorphisms

$$\beta_*: \{A, B\} \to \{A, C\} \quad\text{and}\quad \alpha^*: \{B, C\} \to \{A, C\}.$$

1. 3. Let $f: A \to B$ be a map of finite CW-complexes and let

$$C_f = B \cup_f CA$$

be the mapping cone of f. We have Puppe's exact sequence [7]

(1. 5).
$$\cdots \xrightarrow{S^{n+1}f^*} \{S^{n+1}A, X\} \xrightarrow{S^n\pi^*} \{S^nC_f, X\} \xrightarrow{S^ni^*} \{S^nB, X\} \xrightarrow{S^nf^*} \{S^nA, X\} \to \cdots$$

for any X, where $\pi : C_f \to SA$ is the map collapsing B to a point and $i : B \to C_f$ the canonical inclusion. As a dual of (1. 5) we have the following exact sequuence for finite dimensional CW-complex Y [4]:

(1. 5′)
$$\cdots \to \{Y, S^{n-1}C_f\} \xrightarrow{S^{n-1}\pi_*} \{Y, S^nA\} \xrightarrow{S^nf_*} \to \{Y, S^nB\} \xrightarrow{S^ni_*} \{Y, S^nC_f\} \xrightarrow{S^n\pi_*} \cdots .$$

Both sequences will be called the exact sequences associated with the cofibration

$$B \xrightarrow{i} C_f \xrightarrow{\pi} SA .$$

1. 4. Let M_q be a co-Moore space of type $(Z_q, 2)$, where q is an integer >1. To fix a space M_q used in the sequel we put

$$M_q = S^1 \cup_q e^2 = S^1 \cup_q CS^1 .$$

Denote by

(1. 6)
$$\pi_q \text{ (or } \pi) : M_q \to S^2 \quad \text{and} \quad i_q \text{ (or } i) : S^1 \to M_q$$

the map collapsing S^1 to a point and the canonical inclusion.

The following theorem is basic in our later discussions.

Theorem 1. 1. *If* $q \not\equiv 2$ (mod 4), $1_M \in \{M_q, M_q\}$, $M = M_q$, *is of order* q. *If* $q \equiv 2$ (mod 4), *then* $q \cdot 1_M = i_q \eta \pi_q$ *for the class of Hopf map* $\eta \in \{S^2, S^1\} \cong Z_2$ *and the order of* 1_M *is* $2q$.

We sketch the proof (for details, see [3] or [9]). In the exact sequence

$$\{S^2, M_q\} \xrightarrow{\pi^*} \{M_q, M_q\} \xrightarrow{i^*} \{S^1, M_q\}$$

$i^*(1_M) = i$ is of order q, and $q \cdot 1_M$ is in the image of π^*. $i_* : \{S^2, S^1\} \to \{S^2, M_q\}$ is an epimorphism. Thus $q \cdot 1_M = 0$ or $i\eta\pi$. If q is odd then $i\eta\pi = 0$. Let q be even. $q \cdot 1_M = i\eta\pi$ if and only if the functional Sq^2 operation associated with $q \cdot 1_M$ is not zero, i. e., $Sq^2 \neq 0$ in $M_q \wedge M_q = SM_q \cup_{q \cdot 1} C(SM_q)$. By Cartan's formula, $Sq^2 = Sq^1 \wedge Sq^1 \neq 0$ if and only if $q \equiv 2$ (mod 4). This proves the theorem.

1.5. Apply (1.5) and (1.5′) for the cofibration $SX \xrightarrow{1\wedge i} X\wedge M_q \xrightarrow{1\wedge\pi} S^2X$, where $X\wedge M_q = SX\cup_f C(SX)$ and $f=1_X\wedge q=q\cdot 1_{SX}: SX\to SX$ induces the q times of the identity map. Then we have the following two exact sequences

$$(1.7) \qquad 0\to\{S^2X, W\}\otimes Z_q \xrightarrow{(1\wedge\pi)^*} \{X\wedge M_q, W\} \xrightarrow{(1\wedge i)^*} \mathrm{Tor}(\{SX, W\}, Z_q)\to 0\,,$$

$$(1.7') \qquad 0\to\{X, SX\}\otimes Z_q \xrightarrow{(1\wedge i)_*} \{Y, X\wedge M_q\} \xrightarrow{(1\wedge\pi)_*} \mathrm{Tor}(\{Y, S^2X\}, Z_q)\to 0$$

for finite dimensional Y.

By Theo. 1.1 and (1.2) we see that the order of the identity class $1_X\wedge 1_M$ of $X\wedge M_q$ is a divisor of q if $q\not\equiv 2 \pmod 4$ and is a divisor of $2q$ in general. Hence

(1.8) *$\{X\wedge M_q, W\}$ and $\{Y, X\wedge M_q\}$ are Z_{2q}-modules in general, and are Z_q-modules if $q\not\equiv 2$* (mod 4).

From this we see that the sequences (1.7) and (1.7′) split for odd prime q.

2. mod q cohomology theories

2.1. By a cohomology theory we understand, throughout the present work, a general cohomology theory defined on the category of pairs (X, A) of finite CW-complexes (or of the same homotopy type). Each cohomology theory h has its reduced cohomology theory $\tilde{h}$ defined on the category of finite CW-complexes with base points. The correspondence $h\to\tilde{h}$ is bijective, and the postulations for h have an equivalent form of postulations for $\tilde{h}$ [10]; thereby the excision axiom is replaced by the suspension isomorphism

$$\sigma: \tilde{h}^i(X) \overset{\approx}{\to} \tilde{h}^{i+1}(SX)$$

for all i.

Put

$$\tilde{h}^*(X) = \textstyle\sum_i \tilde{h}^i(X)\,.$$

Then $\tilde{h}^*$ is a functor of Z-graded abelian groups. (In case $h=K$ or KO we use $^\#$ instead of * to denote the Z-graded groups so as to avoid confusions with the periodic cohomologies.) For the sake of simplicity

$\tilde{h}^*(f)$ is denoted by

$$f^* : \tilde{h}^*(Y) \to \tilde{h}^*(X)$$

for any map $f: Y \to X$ preserving base points. From the commutativity $Sf^* \circ \sigma = \sigma \circ f^*$ (the naturality of σ) follows that f^* depends only on the stable homotopy class of f.

Let A, B be finite CW-complexes (with base points). For any element α of $\{A, B\}$, we define a homomorphism

$$\alpha^{**} : \tilde{h}^*(X \wedge B) \to \tilde{h}^*(X \wedge A)$$

by the formula

$$\alpha^{**} = (\sigma^n)^{-1}(1_X \wedge f)^* \sigma^n$$

where $f: S^n A \to S^n B$ is a map representing α. The commutativity $\sigma(1 \wedge f)^* = (1 \wedge Sf)^* \sigma$ shows that the definition does not depend on the choice of f. We have easily

(2. 1). i) *α^{**} is the identity homomorphism if α is the class of the identity map,*

ii) $$(\beta \circ \alpha)^{**} = \alpha^{**} \circ \beta^{**} ,$$

iii) $$(\alpha_1 + \alpha_2)^{**} = \alpha_1^{**} + \alpha_2^{**} ,$$

iv) α^{**} *is natural*, i. e., $(g \wedge 1_A)^* \circ \alpha^{**} = \alpha^{**} \circ (g \wedge 1_B)^*$

for any map $g: Y \to X$.

2. 2. The mod q h-cohomology theory [5], $h(\ ;Z_q)$ and $\tilde{h}(\ ;Z_q)$, is defined by

$$h^i(X, A; Z_q) = h^{i+2}(X \times M_q, X \times * \cup A \times M_q) ,$$
$$\tilde{h}^i(X; Z_q) = \tilde{h}^{i+2}(X \wedge M_q) \qquad \text{for all } i.$$

For a map $f: X \to Y$, $\tilde{h}^*(f; Z_q)$ is defined by

$$\tilde{h}^*(f; Z_q) = (f \wedge 1_M)^*, \ M = M_q ,$$

and, for the sake of simplicity, denoted sometimes by

$$f^* : \tilde{h}^*(Y; Z_q) \to \tilde{h}^*(X; Z_q)$$

The *suspension* isomorphism

$$\sigma_q : \tilde{h}^i(X; Z_q) \overset{\approx}{\to} \tilde{h}^{i+1}(SX; Z_q)$$

is defined as the composition

$$\sigma_q=(1_X\wedge T)^*\sigma:\tilde{h}^i(X\wedge M_q)\overset{\approx}{\to}\tilde{h}^{i+1}(X\wedge M_q\wedge S^1)\overset{\approx}{\to}\tilde{h}^{i+1}(X\wedge S^1\wedge M_q),$$

where $T=T(S^1, M_q)$. Since σ is natural, σ_q is also natural. With these definitions $\tilde{h}(\ ;Z_q)$ satisfies all axioms of a reduced cohomology theory. Thus $h(\ ;Z_q)$ becomes a cohomology theory.

Making use of maps (1.6) we put

$$\rho_q\ (\text{or }\rho)=(1\wedge\pi_q)^*\sigma^2:\tilde{h}^i(X)\to\tilde{h}^i(X;Z_q)$$

and

$$\delta_{q,0}\ (\text{or }\delta)=\sigma^{-1}(1\wedge i_q)^*:\tilde{h}^i(X;Z_q)\to\tilde{h}^{i+1}(X),$$

which are natural and called as the *reduction* "mod q" and the *Bockstein homomorphism* respectively. The following relations are easily seen.

(2.2) $$\sigma_q\rho_q=\rho_q\sigma,\ \delta_{q,0}\sigma_q=-\sigma\delta_{q,0}\quad\textit{and}\quad\delta_{q,0}\rho_q=0.$$

From the exact sequence of $\tilde{h}$ associated with the cofibration

$$X\wedge S^1\xrightarrow{1\wedge i}K\wedge M_q\xrightarrow{1\wedge\pi}X\wedge S^2$$

and the definitions of ρ_q and $\delta_{q,0}$ we obtain the following exact sequence

(2.3). $$\cdots\xrightarrow{q}\tilde{h}^i(X)\xrightarrow{\rho_q}\tilde{h}^i(X;Z_q)\xrightarrow{\delta_{q,0}}\tilde{h}^{i+1}(X)\xrightarrow{q}\cdots,$$

where q denotes a homomorphism to multiply every element with q, from which follows the exact universal coefficient sequence

(2.4) $$0\to\tilde{h}^i(X)\otimes Z_q\xrightarrow{\rho'}\tilde{h}^i(X;Z_q)\xrightarrow{\delta'}\operatorname{Tor}(\tilde{h}^{i+1}(X),Z_q)\to 0$$

for all i, which is natural, and the natural maps ρ' and δ' are induced respectively by ρ and δ.

Put

$$\delta_{q,r}=\rho_r\delta_{q,0}:\tilde{h}^i(X;Z_q)\to\tilde{h}^{i+1}(X;Z_r)$$

for $q, r>1$, In case $q=r$, putting $\delta_q=\delta_{q,q}$, we call it the "mod q" *Bockstein homomorphism.* The following relations follow from (2.2) and the definitions.

(2.2′) $$\sigma_r\delta_{q,r}=-\delta_{q,r}\sigma_q\quad\textit{and}\quad\delta_{r,s}\delta_{q,r}=0.$$

2.3. The following propositions follow from (2.1) and Theo. 1.1.

Proposition 2.1. *The groups $\tilde{h}^i(X;Z_q)$ are Z_q-modules if $q\not\equiv 2$* (mod 4) *and Z_{2q}-modules if $q\equiv 2$* (mod 4).

Proposition 2. 2. *If $\eta^{**}=0$ in $\tilde{h}$, then $\tilde{h}^i(X; Z_q)$ are Z_q-modules for any $q>1$.*

As an example, we consider Atiyah-Hirzebruch K-cohomology theory of complex vector bundles.

Theorem 2. 3. *Let $\alpha\in\{S^{n+r}, S^n\}$, then $\alpha^{**}=0$ in $\tilde{K}$ if $r\neq 0$. Thus $\tilde{K}^i(X; Z_q)$ are Z_q-modules for any X and $q>1$.*

Proof. Tensor products of vector bundles defines a multiplication which induces natural isomorphisms: $\tilde{K}^i(X)\otimes\tilde{K}^n(S^n)\cong\tilde{K}^{i+n}(X\wedge S^n)$. Via the naturality of these isomorphisms it is sufficient to prove the triviality of $\alpha^*:\tilde{K}^n(S^n)\to\tilde{K}^n(S^{n+r})$. By Serre [8], $\{S^{n+r}, S^n\}$ is finite for $r\neq 0$. On the other hand, $\tilde{K}^n(S^{n+r})=0$ or Z. Thus $\alpha^*=0$. q. e. d.

The above theorem does not hold for $\widetilde{KO}$-cohomology theory of real vector bundles. For, it is known [1], [9], that

$$\widetilde{KO}^{-2}(S^0; Z_2)=\widetilde{KO}^0(M_2)\cong Z_4.$$

2. 4. Take groups of coefficients Z_q, Z_r, and let a be an integer such that

$$a\cdot q\equiv 0 \pmod{r}.$$

Put $a'=aq/r$ and let $a, a': S^1\to S^1$ be the maps defined as in **1. 1.** Put $\bar{a}\,|\,S^1=a'$ and let $\bar{a}\,|\,CS^1: CS^1\to CS^1$ be the canonical extension of a. Then the map

(2. 5) $\quad \bar{a}: M_r\to M_q \quad$ *with* $\bar{a}\,|\,S^1=a'$ *and* $\pi_q\circ\bar{a}=Sa\circ\pi_r$

is well-defined and continuous. The map $\bar{a}$ induces a homomorphism $\bar{a}^{**}=(1\wedge\bar{a})^*: \tilde{h}^{i+2}(X\wedge M_q)\to\tilde{h}^{i+2}(X\wedge M_r)$ which is denoted by

(2. 6) $$a_*:\tilde{h}^i(X; Z_q)\to\tilde{h}^i(X; Z_r).$$

We see easily

(2. 7), $\dot{0}$) $0_*=0$. *If $q=r$, 1_* is the identity.*

i) $\bar{a}\circ\bar{b}=\overline{ab}$, *thus* $b_*\circ a_*=(ab)_*$.

ii) $a_*\rho_q(x)=a\cdot\rho_r(x)$ *for* $x\in\tilde{h}^i(X)$.

iii) $\delta_{r,0}a_*(x)=(aq/r)\cdot\delta_{q,0}(x)$ *for* $x\in\tilde{h}^i(X; Z_q)$.

iv) a_* *is natural*, i. e., $f^*\circ a_*=a_*\circ f^*$ *for any map* f.

We have

(2. 8). *The map $\bar{r}: M_r\to M_q$ is homotopic to zero. Thus $r_*=0:\tilde{h}^i(\ ; Z_q)\to\tilde{h}^i(\ ; Z_r)$.*

In fact, we define a homotopy $r_\theta: M_r \to M_q$ between $\bar{r}$ and a constant map as follows. Representing each point of $M_r - S^1$ (or $M_q - S^1$) by (x, t), $x \in S^1$, $0 < t \leqq 1$, with the relations $(x, 1) = *$ and $(x, 0) = r(x)$ (or $= q(x)$), we put

$$r_\theta(x) = (x, \theta) \qquad \text{for } x \in S^1 \subset M_r$$

and

$$r_\theta(y, t) = \begin{cases} (r(y), \theta), & t \leqq \theta, \\ (r(y), t), & t \geqq \theta, \end{cases} \qquad \text{for } y \in S^1,\ 0 < t \leqq 1.$$

Then r_θ is a homotopy between $r_0 = \bar{r}$ and $r_1 = \bar{0}$ (constant map).

Now let

$$C(\bar{1}) = M_{rq} \cup_{\bar{1}} CM_r$$

be a mapping cone of the map $\bar{1}: M_r \to M_{rq}$. For the map $\bar{r}: M_{rq} \to M_q$, the composition $\bar{r} \circ \bar{1}: M_r \to M_q$ is homotopic to zero by (2.7), i) and (2.8). Thus the map $\bar{r}$ is extended over a map

$$R: C(\bar{1}) \to M_q, \quad R \,\big|\, M_{rq} = \bar{r},$$

using the above homotopy. We have

(2.9). *R is a homotopy equivalence.*

In fact, R is a deformation retraction by regarding M_q as the subspace $S^1 \cup_q CS^1$ of $C(\bar{1})$ as $\bar{1} \,\big|\, S^1 = q$.

Making use of the equivalence R, we have the following exact sequence (2.10) associated with the cofibration $X \wedge M_{rq} \to X \wedge C(\bar{1}) \to X \wedge SM_r$:

$$(2.10) \qquad \cdots \to \tilde{h}^i(X; Z_q) \xrightarrow{r_*} \tilde{h}^i(X; Z_{rq}) \xrightarrow{1_*} \tilde{h}^i(X; Z_r) \xrightarrow{\delta_{r,q}} \tilde{h}^{i+1}(X; Z_q) \to \cdots.$$

2.5. Compair $\bar{a} + \bar{b}$ and $\overline{a+b}$ in $\{M_r, M_q\}$. From the exactness of the sequence $\{S^2, M_q\} \xrightarrow{\pi_r^*} \{M_r, M_q\} \xrightarrow{i_r^*} \{S^1, M_q\}$ and from the fact $i_r^*(\bar{a} + \bar{b}) = i_r^*(\overline{a+b})$ we see that $\overline{a+b} - (\bar{a} + \bar{b})$ is in $\pi_r^*\{S^2, M_q\}$ which is generated by $i_q \eta \pi_r$. If q or r is odd, then $\pi_r^*\{S^2, M_q\} = 0$. Therefore we obtain

Proposition 2.4. *If q or r is odd or if $\eta^{**} = 0$ in $\tilde{h}$, then $a_* + b_* = (a+b)_*$ as natural maps: $\tilde{h}^i(\ ; Z_q) \to \tilde{h}^i(\ ; Z_r)$; in particular, $a_*(x) = a \cdot x$, $x \in \tilde{h}^i(X; Z_q)$, when $q = r$.*

(2.7), i), (2.8) and Prop. 2.4 show the following

Proposition 2.5. *Given a homomorphism $f: Z_q \to Z_r$ and let a be an* *integer such that* $f(t) \equiv at$ (mod r). *Then, under the assumption of Prop.* 2. 4, a_* *is independent of the choice of* a, i. e., *we may write*

$$f_* = a_* : \tilde{h}^i(\ ; Z_q) \to \tilde{h}^i(\ ; Z_r).$$

Theorem 2. 6. *If q is prime to r, then we have the isomorphism*

$$(1_*, 1_*) : \tilde{h}^i(X; Z_{qr}) \overset{\approx}{\to} \tilde{h}^i(X; Z_q) \oplus \tilde{h}^1(X; Z_r).$$

Moreover, if $\eta^{**}=0$ *in* $\tilde{h}$, *the inverse of* $(1_*, 1_*)$ *is*

$$(ur)_* + (vq)_* = u \cdot r_* + v \cdot q_* : \tilde{h}^i(X; Z_q) \oplus \tilde{h}^i(X; Z_r) \overset{\approx}{\to} \tilde{h}^i(X; Z_{qr}),$$

where u und v are integers such that $ur+vq=1$.

By assumption q or r is odd. Let q be odd. For $1_* : \tilde{h}^i(X; Z_{qr}) \to \tilde{h}^i(X; Z_q)$ and $r_* : \tilde{h}^i(X; Z_q) \to \tilde{h}^i(X; Z_{qr})$, $1_* \circ r_* = r$ is an automorphism of $\tilde{h}^i(X; Z_q)$ by (2. 7), i), Props. 2. 1 and 2. 4. From the exactness of the sequence (2. 10) the first assertion of the theorem follows. The second assertion is straightforward from (2. 7), i) and Prop. 2. 4.

When q is a multiple of r, we put

$$\rho_{q,r} = 1_* : \tilde{h}^i(\ ; Z_q) \to \tilde{h}^i(\ ; Z_r). \tag{2. 11}$$

2. 6. The following theorem gives a condition for to split the exact sequence (2. 4).

Theorem 2. 7. *If* $q \not\equiv 2$ (mod 4) *or if* $\eta^{**}=0$ *in* $\tilde{h}$, *then the sequence* (2. 4) *splits*:

$$\tilde{h}^i(X; Z_q) \cong \tilde{h}^i(X) \otimes Z_q \oplus \operatorname{Tor}(\tilde{h}^{i+1}(X), Z_q).$$

Proof. Since $\operatorname{Tor}(\tilde{h}^{i+1}(X), Z_q)$ is a Z_q-module, it is isomorphic to the direct sum of cyclic groups F_k (the set of indices $\{k\}$ may be infinite [6]). Let α be a generator of F_k and let s be the order of α. It is sufficient to prove the existence of an element γ of $\tilde{h}^i(X; Z_q)$ such that $s\gamma=0$ and $\delta'\gamma=\alpha$. Put $t=q/s$. By (2. 7) we have the following commutative diagram:

$$\begin{array}{ccccccc}
\tilde{h}^i(X) & \xrightarrow{\rho_s} & \tilde{h}^i(X; Z_s) & \xrightarrow{\delta'} & \operatorname{Tor}(\tilde{h}^{i+1}(X), Z_s) & \to & 0 \\
\downarrow t & & \downarrow t_* & & \downarrow j & & \\
\tilde{h}^i(X) & \xrightarrow{\rho_q} & \tilde{h}^i(X; Z_q) & \xrightarrow{\delta'} & \operatorname{Tor}(\tilde{h}^{i+1}(X), Z_q) & \to & 0,
\end{array}$$

where j is the inclusion. Let $\beta \in \tilde{h}^i(X; Z_s)$ be chosen such as $j\delta'\beta=\alpha$, Put $\gamma=t_*\beta$. Then $\delta'\gamma=\alpha$. If $\eta^{**}=0$ in $\tilde{h}$ or if $s\not\equiv 2 \pmod 4$, then $s\beta=0$, thus $s\gamma=0$. The remaining case is that $s\equiv 2 \pmod 4$ and $q\equiv 0 \pmod 4$, then t is even. By Theo. 1.1, we have

$$\begin{aligned} s\gamma &= t_*(s\beta) = t_*(1\wedge i\eta\pi)^*\beta = t_*(1\wedge\pi)^*(1\wedge i\eta)^*\beta \\ &= t_*\rho_s\sigma^{-2}(1\wedge i\eta)^*\beta = t\cdot\rho_q\sigma^{-2}(1\wedge i\eta)^*\beta = 0\,, \end{aligned}$$

since t is even and $t\eta=0$. q. e. d.

Corollary 2.8. (2.4) *splits for* $h=K$:

$$\tilde{K}^i(X;\ Z_q)\approx\tilde{K}^i(X)\otimes Z_q\oplus \mathrm{Tor}\,(\tilde{K}^{i+1}(X),\ Z_q)$$

for any X *and* $q>1$.

By a parallel proof to that of Theo. 2.7 we obtain

Theorem 2.9. *The sequences* (1.7) *and* (1.7′) *split if* $q\not\equiv 2 \pmod 4$:

$$\begin{aligned} \{X\wedge M_q,\ W\} &\approx \{S^2X,\ W\}\otimes Z_q\oplus \mathrm{Tor}\,(\{SX,\ W\},\ Z_q)\,, \\ \{Y,\ X\wedge M_q\} &\approx \{Y,\ SX\}\otimes Z_q\oplus \mathrm{Tor}\,(\{Y,\ S^2X\},\ Z_q)\,. \end{aligned}$$

3. An axiomatic approach to multiplications in mod q cohomology theories

3.1. A cohomology theory h is said to be *multiplicative*, if it is equipped with a map

(3.1) $$\mu : h^i(X,\ A)\otimes h^j(Y,\ B)\to h^{i+j}(X\times Y,\ X\times B\cup A\times Y)$$

for all i, j, which is

(M_1) *linear*,

(M_2) *natural* (*with respect to both variables*),

(M_3) *has a bilateral unit* $1\in h^0(S^0,\ *)$, i. e., $\mu(1\otimes x)=\mu(x\otimes 1)=x$ *for any* $x\in h^i(X,\ A)$,

(M_4) *compatible with the connecting morphisms.* cf., Dold [5], p. 6.

If μ is associative, i. e., satisfies

(M_5) $$\mu(\mu\otimes 1)=\mu(1\otimes\mu)\,,$$

(where we used 1 to stand for an identity map of a group and such a kind of usage of 1 would not give rise to any confusion with the unit $1\in h^0(S^0,\ *)$), or if μ is commutative, i. e., satisfies

(M_6) $$T^*\mu(x\otimes y)=(-1)^{ij}\mu(y\otimes x)$$

for $x \in h^i(X, A)$ and $y \in h^j(Y, B)$, where $T: Y \times X \to X \times Y$ is a map swiching factors, then we say that μ is an *associative*, or *commutative*, multiplication.

To give a multiplication μ in h is equivalent to give a multiplication in $\tilde{h}$

$$\mu : \tilde{h}^i(X) \otimes \tilde{h}^j(Y) \to \tilde{h}^{i+j}(X \wedge Y) \tag{3.2}$$

for all i, j, such that they transform naturally to each other by the passages from h to $\tilde{h}$ and converse. About the multiplication (3.2) the axiom (M_4) is replaced by

(M_4') *the compatibility with the suspension isomorphism* σ:

$$\sigma\mu(x \otimes y) = (1 \wedge T)^* \mu(\sigma x \otimes y) = (-1)^i \mu(x \otimes \sigma y)$$

for $\deg x = i$, *where* $T = T(Y, S^1)$.

By the reason of this equivalence our discussions are limited only for multiplications (3.2) in $\tilde{h}$.

3.2. Let $\tilde{h}$ be a multiplicative cohomology theory with a multiplication μ. The multiplications

$$\begin{aligned} &\mu_R : \tilde{h}^i(\ ;\ Z_q) \otimes \tilde{h}^j(\ ) \to \tilde{h}^{i+j}(\ ;\ Z_q) \\ &\mu_L : \tilde{h}^i(\ ) \otimes \tilde{h}^j(\ ;\ Z_q) \to \tilde{h}^{i+j}(\ ;\ Z_q) \end{aligned} \quad \text{for all } i, j, \tag{3.3}$$

are canonically induced by requiring that the following diagrams should be commutative:

$$\begin{array}{ccccc} \tilde{h}^i(X;\ Z_q) \otimes \tilde{h}^j(Y) & & \xrightarrow{\mu_R} & & \tilde{h}^{i+j}(X \wedge Y;\ Z_q) \\ \| & & & & \| \\ \tilde{h}^{i+2}(X \wedge M_q) \otimes \tilde{h}^j(Y) & \xrightarrow{\mu} & \tilde{h}^{i+j+2}(X \wedge M_q \wedge Y) & \xrightarrow{(1 \wedge T)^*} & \tilde{h}^{i+j+2}(X \wedge Y \wedge M_q), \end{array}$$

$$\begin{array}{ccc} \tilde{h}^i(X) \otimes \tilde{h}^j(Y;\ Z_q) & \xrightarrow{\mu_L} & \tilde{h}^{i+j}(X \wedge Y;\ Z_q) \\ \| & & \| \\ \tilde{h}^i(X) \otimes \tilde{h}^{j+2}(Y \wedge M_q) & \xrightarrow{\mu} & \tilde{h}^{i+j+2}(X \wedge Y \wedge M_q). \end{array}$$

As is easily checked, μ_R and μ_L satisfy the following properties:

(H_1) *linear*;

(H_2) *natural*;

(H_3) 1 *is a right unit for* μ_R *and a left unit for* μ_L;

(H_4) *compatible with suspension isomorphisms in the sense that*

$$\sigma_q \mu_R(x\otimes y) = (1\wedge T)^* \mu_R(\sigma_q x\otimes y) = (-1)^i \mu_R(x\otimes \sigma y),$$
$$\sigma_q \mu_L(x\otimes y) = (1\wedge T)^* \mu_L(\sigma x\otimes y) = (-1)^i \mu_L(x\otimes \sigma_q y)$$

for $\deg x = i$;

(H_5) *compatible with the reduction* mod q *in the sense that*

$$\mu_R(\rho_q\otimes 1) = \rho_q \mu = \mu_L(1\otimes \rho_q);$$

(H_6) *compatible with the Bockstein homomorphisms in the sense that*

$$\delta\mu_R(x\otimes y) = \mu(\delta x\otimes y), \quad \delta\mu_L(x\otimes y) = (-1)^i \mu(x\otimes \delta y),$$
$$\delta_q \mu_R(x\otimes y) = \mu_R(\delta_q x\otimes y), \quad \delta_q \mu_L(x\otimes y) = (-1)^i \mu_L(x\otimes \delta_q y)$$

for $\deg x = i$.

If μ is associative, then the following *associativity*

$$(H_7) \qquad \begin{aligned} \mu_R(\mu_R\otimes 1) &= \mu_R(1\otimes \mu), \\ \mu_R(\mu_L\otimes 1) &= \mu_L(1\otimes \mu_R), \\ \mu_L(\mu\otimes 1) &= \mu_L(1\otimes \mu_L) \end{aligned}$$

holds; and it μ is commutative then the *commutativity*

$$(H_8) \qquad T^*\mu_L(x\otimes y) = (-1)^{ij} \mu_R(y\otimes x)$$

holds for $x\in \tilde{h}^i(X)$ and $y\in \tilde{h}^j(Y;\, Z_q)$, where $T = T(Y, X)$.

If $\eta^{**} = 0$ in $\tilde{h}$, then the following *compatibility with the homomorphisms of coefficients* holds:

$$(H_9) \qquad f_*\mu_R = \mu_R(f_*\otimes 1), \quad f_*\mu_L = \mu_L(1\otimes f_*),$$

where $f_* : \tilde{h}^i(\ ;\, Z_q) \to \tilde{h}^i(\ ;\, Z_r)$ is induced by a homomorphism $f: Z_q \to Z_r$. For general case (H_9) holds after replacing f_* by a_* (a, an integer).

3. 3. Let $\tilde{h}$ be a multiplicative cohomology theory with an associative multiplication μ. We shall discuss multiplications

$$(3.4) \qquad \mu_q : \tilde{h}^i(X;\, Z_q)\otimes \tilde{h}^j(Y;\, Z_q) \to \tilde{h}^{i+j}(X\wedge Y;\, Z_q)$$

in $\tilde{h}(\ ;\, Z_q)$ by postulating the following properties:

(Λ_0) μ_q *is a multiplication,* i. e., *satisfies* (M_1), (M_2), (M_3) *and* (M_4') *for the cohomology theory* $\tilde{h}(\ ;\, Z_q)$;

(Λ_1) *compatible with* μ_R *and* μ_L *through the reduction* mod q, i. e.,

$$\mu_R = \mu_q(1\otimes \rho_q) \quad \textit{and} \quad \mu_L = \mu_q(\rho_q\otimes 1);$$

(Λ_2) *δ_q is a derivation (in the graded sense),* i.e.,

$$\delta_q \mu_q(x\otimes y) = \mu_q(\delta_q x\otimes y)+(-1)^i \mu_q(x\otimes\delta_q y)$$

for $\deg x = i$;

(Λ_3) *"quasi-associative" in the sense that, if at least one element of $\{x, y, z\}$ is in ρ_q-images, then the associativity*

$$\mu_q(\mu_q(x\otimes y)\otimes z) = \mu_q(x\otimes\mu_q(y\otimes z))$$

holds.

(Λ_0) is a minimal requirement to call μ_q as a multiplication. (Λ_1) means a compatibility of μ_q with μ. In fact, (Λ_1) and (H_5) imply that

(Λ_1') *μ_q is compatible with μ through the reduction* mod q *in the sense that*

$$\mu_q(\rho_q\otimes\rho_q) = \rho_q\mu .$$

Denote by 1_q the bilateral unit of μ_q, then we have

Proposition 3.1. *If a multiplication μ_q satisfies (Λ_1), then*

(Λ_1'') $$1_q = \rho_q(1).$$

From (Λ_1) and (H_3) we see easily that $\rho_q(1)$ is a bilateral unit of μ_q. Then from the uniqueness of bilateral units follows (Λ_1'').

Proposition 3.2. *If a multiplication μ_q satisfies (Λ_1''), then $\tilde{h}^*(X;\ Z_q)$ is a Z_q-module for any X.*

For any $x\in\tilde{h}^*(X;\ Z_q)$,

$$\begin{aligned} q\cdot x &= \mu_q(1_q\otimes q\cdot x) = \mu_q(q\cdot\rho_q(1)\otimes x) \\ &= \mu_q(\rho_q(q\cdot 1)\otimes x) = 0 \end{aligned}$$

because $\rho_q(q\cdot 1)=0$ by (2.3). Thus Prop. 3.2 was obtained.

From Prop. 3.2 we see that, if μ_q satisfies (Λ_1''), the exact sequence of $\tilde{h}(\ ;\ Z_q)$ associated with the cofibration

$$X\wedge S^1 \xrightarrow{1\wedge i} X\wedge M_q \xrightarrow{1\wedge\pi} X\wedge S^2$$

breaks into short exact sequences

(3.5) $$0\to\tilde{h}^i(X\wedge S^2;\ Z_q)\xrightarrow{(1\wedge\pi)^*}\tilde{h}^i(X\wedge M_q;\ Z_q)\xrightarrow{(1\wedge i)^*}\tilde{h}^i(X\wedge S^1;\ Z)\to 0$$

for any X and i.

Postulation (Λ_2) and (Λ_3) are necessary to get theorems of uniqueness-type. A multiplication μ_q satisfying (Λ_0), (Λ_1), (Λ_2) and (Λ_3) is called an *admissible* multiplication.

3. 4. Assuming the existence of μ_q satisfying (Λ_1''), put

$$\kappa_2 = \pi^*\sigma_q{}^2 1_q \in \tilde{h}^2(M_q\,;\,Z_q)\,. \tag{3.6}$$

By the exactness of (3. 5) for $X=S^0$, π^* is monomorphic. Hence $\kappa_2 \neq 0$. We shall prove that there exist an element $\kappa_1 \in \tilde{h}^1(M_q\,;\,Z_q)$ such that

$$\delta_q \kappa_1 = \kappa_2 \quad \text{and} \quad i^*\kappa_1 = -\sigma_q 1_q\,. \tag{3.7}$$

Let

$$\pi' : M_q \wedge M_q \to (M_q \wedge M_q)/(S^1 \wedge S^1)$$

be a map collapsing $S^1 \wedge S^1$ to a point. The map

$$1 \wedge i : M_q \wedge S^1 \to M_q \wedge M_q \quad (\text{or } i \wedge 1 : S^1 \wedge M_q \to M_q \wedge M_q)$$

induces an injection

$$i_1 : S^3 = S^2 \wedge S^1 \to (M_q \wedge M_q)/(S^1 \wedge S^1) \quad (\text{or } i_2 : S^3 = S^1 \wedge S^2 \to (M_q \wedge M_q)/(S^1 \wedge S^1))$$

such that the commutativities

$$\pi'(1 \wedge i) = i_1(\pi \wedge 1) \quad \text{and} \quad \pi'(i \wedge 1) = i_2(1 \wedge \pi)$$

hold. Putting $i_k(S^3) = S^3_k$, $k=1$ or 2, we obtain the following cell structure

$$(M_q \wedge M_q)/(S^1 \wedge S^1) = S^3_1 \vee S^3_2 \cup_{q_1+q_2} e^4\,,$$

where $q_k : S^3 \to S^3_k$, $k=1$ or 2, is a map of degree q. Let i'_k, $k=1$ or 2, be the map i_k considered as the maps into $S^3_1 \vee S^3_2$, and

$$p_k : S^3_1 \vee S^3_2 \to S^3$$

the map collapsing S^3_l, $l \neq k$, to a point such that

$$p_k i'_k = 1_{S^3}\,.$$

Then

$$(q_1+q_2)^*(p_1^*\sigma^3 1 - p_2^*\sigma^3 1) = 0$$

since $p_k q_k = q$. Thus, by the exact sequence of $\tilde{h}$ associated with the cofibration

$$S^3_1 \wedge S^3_2 \xrightarrow{\ j\ } (M_q \wedge M_q)/(S^1 \wedge S^1) \to S^4\,,$$

there exists an element

$$\kappa\in\tilde{h}^3(M_q\wedge M_q/(S^1\wedge S^1)) \quad \text{such that} \quad j^*\kappa=p_1^*\sigma^3 1-p_2^*\sigma^3 1\,.$$

Put

$$\kappa_1=\pi'^*\kappa\in\tilde{h}^3(M_q\wedge M_q)=\tilde{h}^1(M_q\,;\,Z_q)\,,$$

which satisfies (3.7) as will be proved in the following way:

$$\begin{aligned}\delta_q\kappa_1&=(1\wedge\pi)^*\sigma(1\wedge i)^*\kappa_1=(1\wedge\pi)^*\sigma(\pi\wedge 1)^*i_1^*\kappa\\&=(\pi\wedge\pi)^*\sigma i_1'^*j^*\kappa=(\pi\wedge\pi)^*\sigma^4 1=\kappa_2\,,\\ i^*\kappa_1&=(i\wedge 1)^*\pi'^*\kappa=(1\wedge\pi)^*i_2^*\kappa\\&=(1\wedge\pi)^*i_2'^*j^*\kappa=-(1\wedge\pi)^*\sigma^3 1\\&=-\rho_q\sigma 1=\sigma_q 1_q\,.\end{aligned}$$

The choice of κ_1 is not unique. Nevertheless we fix κ_1 once for all. We remark that in the choices of κ_1 and κ_2 we did not use any special μ_q.

3.5. The next proposition gives a necessary condition for the existence of an admissible multiplication in case of $q\equiv 2$ (mod 4) which is sufficient for the existence of a multiplication μ_q satisfying (Λ_1), c.f., Cor. 5.6).

Proposition 3.3. *If* $q\equiv 2$ (mod 4) *and there exists a multiplication* μ_q *satisfing* (Λ_1''), *then*

$$\rho_q\eta^*(1)=0 \quad \textit{and} \quad (\eta\pi_q)^{**}=0 \textit{ in } \tilde{h}.$$

Proof.
$$\begin{aligned}q\cdot\kappa_1&=(q\cdot 1_M)^*\kappa_1=(i\eta\pi)^*\kappa_1 \text{ by Theo. 1.1,}\\&=(\eta\pi)^*i^*\kappa_1=-(\eta\pi)^*\sigma_q 1_q\\&=-\sigma_q\pi_q^*(\eta^* 1_q)\,.\end{aligned}$$

On the other hand $q\cdot\kappa_1=0$ by Prop. 3.2. Thus

$$\sigma_q\pi_q^*(\eta^* 1_q)=0\,.$$

Now σ_q and π_q^* are monomorphic (by (3.5)). Hence

$$\eta^* 1_q=0\,.$$

By (Λ_1''), $\eta^*1_q=\eta^*\sigma_q(1)=\rho_q\eta^*(1)$. Thus

$$\rho_q\eta^*(1)=0\,.$$

Next, $\eta\pi_q=\eta\circ S\pi_q: SM_q\to S^3\to S^2$.
For any $x\in\tilde{h}^i(X\wedge S^2)$,

$$(\eta\pi_q)^{**}x=(\eta\pi_q)^{**}\mu(x\otimes 1)=(\eta\pi_q)^{**}\mu(\sigma^{-2}x\otimes\sigma^2 1)$$
$$=\mu(\sigma^{-2}x\otimes(S\pi_q)^*\eta^*\sigma^2 1)$$
$$=\mu(\sigma^{-2}x\otimes T^*(1\wedge\pi_q)^*T_1^*(\sigma^2\eta^*1)),$$

where $T=T(M_q, S^1)$ and $T_1=T(S^1, S^2)$. Since $T_1^*=1$,

$$(\eta\pi_q)^{**}x=\mu(\sigma^{-2}x\otimes T^*(1\wedge\pi_q)^*\sigma^2\eta^*1)$$
$$=\mu(\sigma^{-2}x\otimes T^*\rho_q\eta^*(1))=0. \quad \text{q. e. d.}$$

3. 6. Next proposition can be viewed as a special kind of Künneth's theorem.

Proposition 3. 4. *If a multiplication μ_q satisfies (Λ_1''), then for any X and $x\in\tilde{h}^i(X\wedge M_q;\ Z_q)$ it can be expressed uniquely as a sum*

$$x=\mu_q(x_1\otimes\kappa_1)+\mu_q(x_2\otimes\kappa_2)$$

with $x_1\in\tilde{h}^{i-1}(X;\ Z_q)$ and $x_2\in\tilde{h}^{i-2}(X;\ Z_q)$.

Proof. Define a homomorphism

$$k:\tilde{h}^i(X\wedge S^1;\ Z_q)\to\tilde{h}^i(X\wedge M_q;\ Z_q)$$

by putting

$$k(y)=(-1)^i\mu_q(\sigma_q^{-1}y\otimes\kappa_1).$$

By an easy calculation we see that

$$(1\wedge i_q)^*k=\text{an identity map},$$

i. e., k gives a splitting of the exact sequence (3. 5). Thus, for any $x\in\tilde{h}^i(X\wedge M_q;\ Z_q)$ two elements $y\in\tilde{h}^i(X\wedge S^1;\ Z_q)$ and $y'\in\tilde{h}^i(X\wedge S^2;\ Z_q)$ are determined uniquely so as to satisfy

$$x=(1\wedge\pi)^*y'+k(y).$$

Put

$$x_1=(-1)^i\sigma_q^{-1}y \quad \text{and} \quad x_2=\sigma_q^{-2}y'.$$

Then, by (3. 6) and (3. 7), we get

$$x=\mu_q(x_1\otimes\kappa_1)+\mu_q(x_2\otimes\kappa_2).$$

The uniqueness of x_1 and x_2 follows also from the exact sequence (3. 5) and the definitions of κ_1 and κ_2.

3. 7. Let μ be an associative and commutative multiplication in $\tilde{h}$. We fix μ once for all throughout this paragraph and shall discuss relations between different admissible multiplications μ_q, μ'_q, μ''_q etc. To simplify notations we put

$$\mu_q(x\otimes y)=x\wedge y,\quad \mu'_q(x\otimes y)=x\wedge' y \quad \text{etc.}$$

For two μ_q and μ'_q we see by (Λ_1) that

(3.8) $x\wedge y=x\wedge' y$ *if either* x *or* y *is in* ρ_q*-images.*

In particular,

(3.9) $\delta_q x\wedge y=\delta_q x\wedge' y$ *and* $x\wedge\delta_q y=x\wedge'\delta_q y$ *for any* x *and* y.

Thus, by (Λ_2) we obtain

(3.9′) $\delta_q(x\wedge y)=\delta_q(x\wedge' y)$ *for any* x *and* y.

Also, by (H_8) and (Λ_1) we obtain

(3.10) $T^*(x\wedge y)=(-1)^{ij}y\wedge x$ *if either* x *or* y *is in* ρ_q*-images, where* $\deg x=i$ *and* $\deg y=j$.

By (3.6)-(3.10) we obtain

(3.11) $\kappa_i\wedge\kappa_j=\kappa_i\wedge'\kappa_j$ *if* $i=2$ *or* $j=2$,

(3.12) $T^*(\kappa_i\wedge\kappa_j)=\kappa_j\wedge\kappa_i$ *if* $i=2$ *or* $j=2$.

3.8. By Prop. 3.4 every element $x\in\tilde{h}^i(X\wedge M_q\wedge M_q;\, Z_q)$ can be expressed uniquely as

$$x=(x_1\wedge\kappa_1)\wedge\kappa_1+(x_2\wedge\kappa_2)\wedge\kappa_1+(x_3\wedge\kappa_1)\wedge\kappa_2+(x_4\wedge\kappa_2)\wedge\kappa_2$$

with $x_1\in\tilde{h}^{i-2}(X;\, Z_q)$, x_2, $x_3\in\tilde{h}^{i-3}(X;\, Z_q)$, $x_4\in\tilde{h}^{i-4}(X;\, Z_q)$.

Put

(3.13′) $$T^*(\kappa_1\wedge\kappa_1)=(a_1\wedge\kappa_1)\wedge\kappa_1+(a_2\wedge\kappa_2)\wedge\kappa_1+(a_3\wedge\kappa_1)\wedge\kappa_2+(a_4\wedge\kappa_2)\wedge\kappa_2$$

with $a_1\in\tilde{h}^0(S^0;\, Z_q)$, a_2, $a_3\in\tilde{h}^{-1}(S^0;\, Z_q)$, $a_4\in\tilde{h}^{-2}(S^0;\, Z_q)$. Apply $(i\wedge 1)^*$ on both sides of (3.13′), then, by (3.6), (3.7) and (3.10), we obtain

$$\sigma_q 1_q\wedge\kappa_1=(-\sigma_q a_1)\wedge\kappa_1+\sigma_q a_3\wedge\kappa_2\,.$$

Thus, by Prop. 3.4, we see that

$$a_1=-1_q \quad\text{and}\quad a_3=0\,.$$

Similarly, applying $(1\wedge i)^*$ on both sides of (3. 13′), we see that

$$a_2=0\,.$$

Finally, making use of (Λ_3), we get

$$T^*(\kappa_1\wedge\kappa_1)=-\kappa_1\wedge\kappa_1+a(\mu_q)\wedge(\kappa_2\wedge\kappa_2) \tag{3.13}$$

with $a(\mu_q)\in\tilde{h}^{-2}(S^0;\,Z_q)$. Apply δ_q on both sides of (3. 13), then, by (Λ_2), (3. 6), (3. 7), (3. 12) and Prop. 3. 4, we see that

$$\delta_q a(\mu_q)=0\,. \tag{3.14}$$

$a(\mu_q)$ is characteristic of μ_q.

Next, put

$$\kappa_1\wedge'\kappa_1=(b_1\wedge\kappa_1)\wedge\kappa_1+(b_2\wedge\kappa_2)\wedge\kappa_1 +(b_3\wedge\kappa_1)\wedge\kappa_2+(b_4\wedge\kappa_2)\wedge\kappa_2 \tag{3.15′}$$

with $b_1\in\tilde{h}^0(S^0;\,Z_q)$, b_2, $b_3\in\tilde{h}^{-1}(S^0;\,Z_q)$, $b_4\in\tilde{h}^{-2}(S^0;\,Z_q)$.

Apply $(i\wedge 1)^*$ on both sides of (3. 15′), then, by (3. 6), (3. 7), (3. 8) and Prop. 3. 4, we see that

$$b_1=1_q \quad\text{and}\quad b_3=0\,.$$

Similarly, applying $(1\wedge i)^*$ on both sides of (3. 15′), we see that

$$b_2=0\,.$$

Thus, making use of (Λ_3), we get

$$\kappa_1\wedge\kappa_1-\kappa_1\wedge'\kappa_1=b(\mu_q,\,\mu'_q)\wedge(\kappa_2\wedge\kappa_2) \tag{3.15}$$

with $b(\mu_q,\,\mu'_q)\in\tilde{h}^{-2}(S^0;\,Z_q)$. Apply δ_q on both sides of (3. 15), then, by (Λ_2), (3. 7), (3. 9′) and Prop. 3. 4, we see that

$$\delta_q b(\mu_q,\,\mu'_q)=0\,. \tag{3.16}$$

By (3. 8) and (3. 15) we obtain

$$b(\mu_q,\,\mu''_q)=b(\mu_q,\,\mu'_q)+b(\mu'_q,\,\mu''_q)\,. \tag{3.17}$$

Apply T^* on both sides of (3. 15) and make use of (3. 13); then, by Prop. 3. 4 we get the relation

$$a(\mu_q)-a(\mu'_q)=2b(\mu_q,\,\mu'_q)\,. \tag{3.18}$$

3. 9. Here we state the following

Lemma 3.5. *Let μ_q be an admissible multiplication in $\tilde{h}(\ ;Z_q)$. If $x\in\rho_q(\tilde{h}^i(X))$ and $a\in\tilde{h}^j(S^0;Z_q)$, then*

$$x\wedge a=(-1)^{ij}a\wedge x\,.$$

Proof. By (3.10) we have

$$(-1)^{ij}a\wedge x=T^*(x\wedge a)\,,$$

where $T=T(S^0,X)$. *Via* the identification $X\wedge S^0=S^0\wedge X=X$, we see that $T(S^0,X)=1_X$. Thus T^* is an identity map, and the lemma follows.

The next theorem shows that $b(\mu_q,\mu_q')$ measures the difference of μ_q from μ_q'.

Theorem 3.6. *Let μ_q and μ_q' be admissible multiplications in $\tilde{h}(\ ;Z_q)$. Then*

$$x\wedge y-x\wedge' y=(-1)^{i+1}b(\mu_q,\mu_q')\wedge(\delta_q x\wedge\delta_q y)$$

for any $x\in\tilde{h}^i(X;Z_q)$ and $y\in\tilde{h}^j(Y;Z_q)$.

Proof. In case $y=\kappa_1$: by (3.9)–(3.9′) we obtain

$$\delta_q(x\wedge\kappa_1)\wedge\kappa_1=\delta_q(x\wedge'\kappa)\wedge'\kappa_1\,.$$

from which, making use of (Λ_2), (Λ_3), (3.8) and (3.10), we get

$$\begin{aligned}(-1)^i(x\wedge'(\kappa_2\wedge'\kappa_1)-x\wedge(\kappa_2\wedge\kappa_1))&\\ =\delta_q x\wedge(\kappa_1\wedge\kappa_1-\kappa_1\wedge'\kappa_1)&\\ =(\delta_q x\wedge b(\mu_q,\mu_q'))\wedge(\kappa_2\wedge\kappa_2)&\\ =(b(\mu_q,\mu_q')\wedge\delta_q x)\wedge(\kappa_2\wedge\kappa_2)&\end{aligned}$$

by Lemma 3.5. Here apply $(1_X\wedge T)^*$, $T=T(M_q,M_q)$, on both sides of this equality. Making use of (Λ_3), (3.11) and (3.12), we obtain

$$\begin{aligned}&((b(\mu_q,\mu_q')\wedge\delta_q x)\wedge\kappa_2)\wedge\kappa_2\\ &\qquad=(-1)^i(x\wedge'\kappa_1-x\wedge\kappa_1)\wedge\kappa_2\,.\end{aligned}$$

Then, from the uniqueness of the expression of Prop. 3.4 follows

(*1) $$x\wedge\kappa_1-x\wedge'\kappa_1=(-1)^{i+1}b(\mu_q,\mu_q')\wedge(\delta_q x\wedge\kappa_2)\,.$$

This shows the theorem in case $y=\kappa_1$.

The theorem for $x=\kappa_1$ can be proved similarly as above by deforming the formula

$$\kappa_1\wedge\delta_q(\kappa_1\wedge y)=\kappa_1\wedge'\delta_q(\kappa_1\wedge' y)\,,$$

and we obtain

$$(*2)\qquad \kappa_1\wedge y-\kappa_1\wedge' y=b(\mu_q,\ \mu'_q)\wedge(\kappa_2\wedge\delta_q y)\,.$$

Now we shall discuss the general case. By (3.9)–(3.9′) we have

$$\delta_q(x\wedge\kappa_1)\wedge\delta_q(y\wedge\kappa_1)=\delta_q(x\wedge'\kappa_1)\wedge'\delta_q(y\wedge'\kappa_1)\,.$$

Decompose the both sides into four terms by (Λ_2), and apply $(1\wedge T\wedge 1)^*$, $T=T(Y, M_q)$, on the both sides. Remarking that we can drop many brackets by the quasi-associativity (Λ_3) we obtain

$$\begin{aligned}&(-1)^{j+1}\delta_q x\wedge\delta_q y\wedge\kappa_1\wedge\kappa_1+(-1)^j\delta_q x\wedge T^*(\kappa_1\wedge y)\wedge\kappa_2\\&\quad+(-1)^i x\wedge\delta_q y\wedge\kappa_2\wedge\kappa_1+(-1)^{i+j}x\wedge y\wedge\kappa_2\wedge\kappa_2\\&=(-1)^{j+1}\delta_q x\wedge\delta_q y\wedge(\kappa_1\wedge'\kappa_1)+(-1)^j\delta_q x\wedge T^*(\kappa_1\wedge' y)\wedge\kappa_2\\&\quad+(-1)^i(x\wedge\delta_q y\wedge\kappa_2)\wedge'\kappa_1+(-1)^{i+j}(x\wedge' y)\wedge\kappa_2\wedge\kappa_2\,.\end{aligned}$$

Thus

$$\begin{aligned}&(-1)^{j+1}\delta_q x\wedge\delta_q y\wedge(\kappa_1\wedge\kappa_1-\kappa_1\wedge'\kappa)\\&\quad+(-1)^j\delta_q x\wedge T^*(\kappa_1\wedge y-\kappa_1\wedge' y)\wedge\kappa_2\\&\quad+(-1)^i((x\wedge\delta_q y\wedge\kappa_2)\wedge\kappa_1-(x\wedge\delta_q y\wedge\kappa_2)\wedge'\kappa_1)\\&\quad+(-1)^{i+j}(x\wedge y-x\wedge' y)\wedge\kappa_2\wedge\kappa_2\\&=0\,.\end{aligned}$$

Rewrite the first three terms by making use of (*1) or (*2). Then, by using Lemma 3.5, we see that the first and second terms cancel to each other, and obtain

$$((b(\mu_q,\ \mu'_q)\wedge(\delta_q x\wedge\delta_q y)+(-1)^i(x\wedge y-x\wedge' y))\wedge\kappa_2)\wedge\kappa_2=0\,.$$

Making use of Prop. 3.4 twice, we obtain

$$b(\mu_q,\ \mu'_q)\wedge(\delta_q x\wedge\delta_q y)+(-1)^i(x\wedge y-x\wedge' y)=0\,.\quad \text{q.e.d.}$$

3.10. The following theorem shows that $a(\mu_q)$ measures the deficiency of μ_q from the commutativity.

Theorem 3.7. *Let μ_q be an admissible multiplication in $\tilde{h}(\ ;Z_q)$. Then*

$$T^*(y\wedge x)=(-1)^{ij}(x\wedge y+(-1)^i a(\mu_q)\wedge(\delta_q x\wedge\delta_q y))$$

for any $x\in\tilde{h}^i(X;Z_q)$ and $y\in\tilde{h}^j(Y;Z_q)$, where $T=T(X, Y)$.

Proof. Put $\mu'_q(x\otimes y)=(-1)^{ij}T^*(y\wedge x)$, then it is a routine matter to see that μ'_q is also an admissible multiplication. (3.13) shows that

$$\kappa_1 \wedge' \kappa_1 = \kappa_1 \wedge \kappa_1 - a(\mu_q) \wedge (\kappa_2 \wedge \kappa_2) .$$

Hence the theorem follows from Theo. 3.6.

Theorem 3.8. *Let μ_q be an admissible multiplication in $\tilde{h}(\ ; Z_q)$ and any $b \in \tilde{h}^{-2}(S^0 ; Z_q) \cap \delta_q^{-1}(0)$ given. If we put*

$$\mu_q'(x \otimes y) = x \wedge y + (-1)^i b \wedge (\delta_q x \wedge \delta_q y)$$

for $x \in \tilde{h}^i(X ; Z_q)$ and $y \in \tilde{h}^j(Y ; Z_q)$, then μ_q' is also an admissible multiplication and $b(\mu_q, \mu_q') = b$.

Proof. It is straightforward to see that μ_q' satisfies (Λ_0), (Λ_1) and (Λ_2). By a simple calculation we see that

$$(*3) \qquad \begin{aligned} &(x \wedge' y) \wedge' z - x \wedge' (y \wedge' z) \\ &= ((x \wedge y) \wedge z - x \wedge (y \wedge z)) \\ &\quad + (-1)^j (b \wedge x - x \wedge b) \wedge \delta_q y \wedge \delta_q z , \end{aligned}$$

where $j = \deg y$. If x is in ρ_q-images, then

$$b \wedge x = x \wedge b$$

by Lemma 3.5. If y or z is in ρ_q-images, then

$$\delta_q y \wedge \delta_q z = 0 .$$

Thus, if x or y, or z, is in ρ_q-images, then the second term of the left side of (*3) vanishes, and the first term also vanishes by (Λ_3) for μ_q, i.e., (Λ_3) for μ_q' was proved. q.e.d.

In the formula (*3), if b is in ρ_q-images, then

$$b \wedge x = x \wedge b$$

by a similar proof as in Lemma 3.5. Thus we obtain from (*3) that

(3.19) *if μ_q is an associative admissible multiplication and b is in ρ_q-images, then the multiplication μ_q' defined as in Theo.* 3.8 *is also associative.*

3.11. From Theos. 3.6, 3.7, 3.8, (3.18) and (3.19) we obtain the following corollaries.

Corollary 3.9. *Let μ_q and μ_q' be two admissible multiplications. The following conditions are equivalent.*

i) $\mu_q = \mu_q'$,

ii) $b(\mu_q, \mu'_q)=0$,

iii) μ_q *coinsides with* μ'_q *for the case of* $X=Y=M_q$.

Corollary 3.10. *If there exists an admissible multiplication in* $\tilde{h}(\ ;Z_q)$, *then admissible multiplications in* $\tilde{h}(\ ;Z_q)$ *are in a one-to-one correspondence with the elements of* $\tilde{h}^{-2}(S^0;Z_q)\cap\delta_q^{-1}(0)$.

Corollary 3.11. *If q is odd and admissible multiplications exist in* $\tilde{h}(\ ;Z_q)$, *then the correspondence* $\mu_q\to a(\mu_q)$ *is bijective, and there is just one commutative multiplication (which corresponds to* $a(\mu_q)=0$).

Corollary 3.12. *If q is even and admissible multiplications exist in* $\tilde{h}(\ ;Z_q)$, *then either there is no commutative one, or commutative ones are in one-to-one correspondence with the elements of* Tor $(\tilde{h}^{-2}(S^0;Z_q)\cap\delta_q^{-1}(0), Z_2)$.

Corollary 3.13. *If there exists an associative admissible multiplication in* $\tilde{h}(\ ;Z_q)$ *and* $\tilde{h}^{-2}(S^0;Z_q)=\rho_q(\tilde{h}^{-2}(S^0))$, *then every admissible multiplication in* $\tilde{h}(\ ;Z_q)$ *is associative.*

3.12. Assume that q and r are relatively prime integers, u, v are integers such that $ur+vq=1$, and $\eta^{**}=0$ in $\tilde{h}$ or qr is odd. Given a multiplication μ_{qr} in $\tilde{h}(\ ;Z_{qr})$, we define multiplication μ_q in $\tilde{h}(\ ;Z_q)$ and μ_r in $\tilde{h}(\ ;Z_r)$ respectively by the formulas

$$\mu_q(x\otimes y)=\rho_{qr,q}\mu_{qr}((ur)_*x\otimes(ur)_*y),\qquad (3.20)$$
$$\mu_r(x\otimes y)=\rho_{qr,r}\mu_{qr}((vq)_*x\otimes(vq)_*y),$$

where $(ur)_*;\tilde{h}^*(\ ;Z_q)\to\tilde{h}^*(\ ;Z_{qr})$ and $(vq)_*:\tilde{h}^*(\ ;Z_r)\to\tilde{h}^*(\ ;Z_{qr})$. If μ_{qr} satisfies (Λ''_1), then it is straightforward to see that μ_q and μ_r are multiplications satisfying (Λ''_1) by (2.7) and Prop. 2.4. Given multiplications μ_q and μ_r, we define a multiplication μ_{qr} in $\tilde{h}(\ ;Z_{qr})$ by

$$\mu_{qr}(x\otimes y)=(ur)_*\mu_q(\rho_{qr,q}x\otimes\rho_{qr,q}y)+(vq)_*\mu_r(\rho_{qr,r}x\otimes\rho_{qr,r}y).\qquad (3.21)$$

Also in this case, if μ_r and μ_q satisfies (Λ''_1), then μ_{qr} becomes a multiplication satisfying (Λ''_1).

Theorem 3.14. *Under the assumptions that q and r are relatively prime, and* $\eta^{**}=0$ *or qr is odd, the correspondences* $\mu_{qr}\to(\mu_q,\mu_r)$ *and* $(\mu_q,\mu_r)\to\mu_{qr}$, *defined by* (3.20) *and* (3.21) *respectively, are bijections of multiplications satisfying* (Λ'_1) *which are the inverses of each other.* μ_{qr} *satisfies* (Λ_1), (Λ_2) *or* (Λ_3) *if and only if* μ_q *and* μ_r *do so.*

Proof. The first assertion follows from a simple calculation to check that the two correspondences are the inverses of each other (making use of Props. 2.4, 2.5 and (2.7)). The assertions for (Λ_1) and (Λ_3) are also easily checked.

To prove the assertion for (Λ_2), first we remark that, by (2.7), ii) and iii), when $as \equiv 0 \pmod t$ and $as^2/t \equiv 0 \pmod t$ the following diagram is commutative:

$$\begin{array}{ccc} \tilde{h}^i(X;\, Z_s) & \xrightarrow{a_*} & \tilde{h}^i(X;\, Z_t) \\ \downarrow{\scriptstyle \delta_s} & & \downarrow{\scriptstyle \delta_t} \\ \tilde{h}^{i+1}(X;\, Z_s) & \xrightarrow{(as/t)_*} & \tilde{h}^{i+1}(X;\, Z_t). \end{array}$$

In the following calculations the above commutativity is used.

Assume that μ_{qr} satisfies (Λ_2), then

$$\begin{aligned} \delta_r \mu_r &= \delta_r \rho_{qr,r} \mu_{qr}((vq)_* \otimes (vq)_*) \\ &= q \cdot \delta_{qr} \mu_{qr}((vq)_* \otimes (vq)_*) \\ &= q \cdot \rho_{qr,r} \mu_{qr}(\delta_{qr}(vq)_* \otimes (vq)_* \pm (vq)_* \otimes \delta_{qr}(vq)_*) \\ &= \rho_{qr,r} \mu_{qr}(\delta_{qr}(vq^2)_* \otimes (vq)_* \pm (vq)_* \otimes \delta_{qr}(vq^2)_*) \\ &= \rho_{qr,r} \mu_{qr}((vq)_* \delta_r \otimes (vq)_* \pm (vq)_* \otimes (vq)_* \delta_r) \\ &= \mu_r(\delta_r \otimes 1) \pm \mu_r(1 \otimes \delta_r). \end{aligned}$$

That is, μ_r satisfies (Λ_2). Similarly μ_q satisfies (Λ_2).

Next, assume that μ_q and μ_r satisfies (Λ_2), then

$$\begin{aligned} \delta_{qr} \mu_{qr} &= \delta_{qr}(u^2r^2)_* \mu_q(1_* \otimes 1_*) + \delta_{qr}(v^2q^2)_* \mu_r(1_* \otimes 1_*) \\ &= (u^2r)_* \delta_q \mu_q(1_* \otimes 1_*) + (v^2q)_* \delta_r \mu_r(1_* \otimes 1_*) \\ &= (u^2r)_* \mu_q(\delta_q 1_* \otimes 1_* \pm 1_* \otimes \delta_q 1_*) \\ &\quad + (v^2q)_* \mu_r(\delta_r 1_* \otimes 1_* \pm 1_* \otimes \delta_r 1_*) \\ &= (u^2r)_* \mu_q(r \cdot 1_* \delta_{qr} \otimes 1_*) + (v^2q)_* \mu_r(q \cdot 1_* \delta_{qr} \otimes 1_*) \\ &\quad \pm ((u^2r)^* \mu_q(1_* \otimes r \cdot 1_* \delta_{qr}) + (v^2q)_* \mu_r(1_* \otimes q \cdot 1_* \delta_{qr})) \\ &= \mu_{qr}(\delta_{qr} \otimes 1) \pm \mu_{qr}(1 \otimes \delta_{qr}), \end{aligned}$$

i.e., μ_{qr} satisfies (Λ_2). q.e.d.

4. Stable homotopy of some elementary complexes. I.

4.1. The results in the following table are well known.

	$i<0$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4, 5$
$\{S^{n+i}, S^n\}$	0	Z	Z_2	Z_2	Z_{24}	0
generators		1	η	$\eta^2=\eta\eta$	ν	

From (1.7) and (1.7′) we have the exact sequences

$$(4.1') \qquad \begin{aligned} 0 \to \{S^{n+i-1}, S^n\} \otimes Z_q \xrightarrow{\pi^*} \{S^{n+i-3}M_q, S^n\} \\ \xrightarrow{i^*} \mathrm{Tor}\,(\{S^{n+i-2}, S^n\}, Z_q) \to 0\,, \\ 0 \to \{S^{n+i}, S^{n+1}\} \otimes Z_q \xrightarrow{i_*} \{S^{n+i}, S^n M_q\} \\ \xrightarrow{\pi_*} \mathrm{Tor}\,(\{S^{n+i}, S^{n+2}\}, Z_q) \to 0\,. \end{aligned}$$

(4.1) *When q is even, there exist elements $\bar{\eta} \in \{S^n M_q, S^n\}$ and $\tilde{\eta} \in \{S^{n+3}, S^n M_q\}$ such that $i^*\bar{\eta} = \bar{\eta} i = \eta$ and $\pi_* \tilde{\eta} = \pi \tilde{\eta} = \eta$.*

(4.1) is clear from the exactness of (4.1′) for $i=3$. We see also that the groups $\{S^n M_q, S^n\}$ and $\{S^{n+3}, S^n M_q\}$ are of order 4 when q is even.

(4.2′) *If $q \equiv 2$ (mod 4), then $2\bar{\eta} = \eta^2 \pi$ and $2\tilde{\eta} = i\eta^2$.*

Because: making use of Theo. 1.1, we see that

$$2\bar{\eta} = 2\bar{\eta} + (q-2)\bar{\eta} = \bar{\eta}(q \cdot 1) = \bar{\eta} i \eta \pi = \eta^2 \pi\,,$$

and

$$2\tilde{\eta} = 2\tilde{\eta} + (q-2)\tilde{\eta} = (q \cdot 1)\tilde{\eta} = i\eta\pi\tilde{\eta} = i\eta^2\,.$$

From (4.2′) and Theo. 2.9 applied to (4.1′) we obtain

(4.2). *The groups $\{S^{n+i-3}M_q, S^n\}$ and $\{S^{n+i}, S^n M_q\}$ are both isomorphic to the corresponding groups in the following table:*

	$i \leq 0$	$i=1$	$i=2$	$i=3$	$i=4$
q : odd	0	Z_q	0	0	$Z_{(q,24)}$
$q \equiv 0$ (mod 4)	0	Z_q	Z_2	$Z_2 + Z_2$	$Z_2 + Z_{(q,24)}$
$q \equiv 2$ (mod 4)	0	Z_q	Z_2	Z_4	$Z_2 + Z_{(q,24)}$
generators of $\{S^{n+i-3}M_q, S^n\}$		π	$\eta\pi$	$\bar{\eta}$, $\eta^2\pi$	$\eta\bar{\eta}$, $\nu\pi$
generators of $\{S^{n+i}, S^n M_q\}$		i	$i\eta$	$\tilde{\eta}$, $i\eta^2$	$\tilde{\eta}\eta$, $i\nu$

4.2. For even q, we use the following notations

$$(4.3) \qquad \eta_1 = i\bar{\eta}, \quad \eta_2 = \tilde{\eta}\pi \in \{S^{n+1}M_q, S^n M_q\}\,,$$

(4.4) *When $q \equiv 0$ (mod 4), there exists an element $\eta_3 \in \{S^{n+2}M_q, S^n M_q\}$ such that $\pi\eta_3 i = \eta$ and $2\eta_3 = 0$. Thus we may choose $\bar{\eta}$ and $\tilde{\eta}$ such as $\bar{\eta} = \pi\eta_3$ and $\tilde{\eta} = \eta_3 i$, then*

$$\eta_1=(i\pi)\eta_3 \quad and \quad \eta_2=\eta_3(i\pi).$$

By Theo. 2.9, there exists an element η_3 such that $\eta_3 i=i^*\eta_3=\tilde{\eta}$ and $2\eta_3=0$. Then $\pi\eta_3 i=\pi\tilde{\eta}=\eta$. Changing $\tilde{\eta}$ by $\pi\eta_3$ if necessary, we see that (4.4) holds.

From (1.7) and (1.7′) we have the following exact sequences:

$$0\to\{S^{n+i+2},\ S^n M_q\}\otimes Z_q\xrightarrow{\pi^*}\{S^{n+i}M_q,\ S^n M_q\}\xrightarrow{i^*}\mathrm{Tor}\,(\{S^{n+i+1},\ S^n M_q\},\ Z_q)\to 0,$$

$$0\to\{S^{n+i}M_q,\ S^{n+1}\}\otimes Z_q\xrightarrow{i_*}\{S^{n+i}M_q,\ S^n M_q\}\xrightarrow{\pi_*}\mathrm{Tor}\,(\{S^{n+i}M_q,\ S^{n+2}\},\ Z_q)\to 0.$$

We obtain

Theorem 4.1. *The group $\{S^{n+i}M_q,\ S^n M_q\}$ is isomorphic to the corresponding group in the following table:*

	$i<-1$	$i=-1$	$i=0$	$i=1$	$i=2$
q : odd	0	Z_q	Z_q	0	$Z_{(q,24)}$
generators		$i\pi$	1		$i\nu\pi$
$q\equiv 0 \pmod 4$	0	Z_q	Z_q+Z_2	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_{(q,24)}$
generators		$i\pi$	$1,\ i\eta\pi$	$\eta_1,\ \eta_2,\ i\eta^2\pi$	$\eta_1^2,\ \eta_2^2,\ \eta_3,\ i\nu\pi$
$q\equiv 2 \pmod 4$	0	Z_q	Z_{2q}	Z_2+Z_2	$Z_2+Z_2+Z_{(q,24)}$
generators		$i\pi$	1	$\eta_1,\ \eta_2$	$\eta_1^2,\ \eta_2^2,\ i\nu\pi$

Proof. In case $q\not\equiv 2 \pmod 4$, the above two sequences split by Theo. 2.9, and the results follow easily from (4.2). In case $q\equiv 2$ (mod 4): for $i=1, 2$, combining the above two sequences we see that the sequences split, then we have the results; for $i\leqq -1$, the proof is obvious; for $i=0$, the results follows from Theo. 1.1. q.e.d.

Corollary 4.2. (i) *In case $q\equiv 0$* (mod 4): *$\{S^{n+i}M_q,\ S^n M_q\}$ are multiplicatively generated for $i\leqq 2$ by $i\pi$, η_3 and $i\nu\pi$,* i.e., *putting $\delta=i\pi$ we have the relations:*

$$i\eta\pi=\delta\eta_3\delta,\quad \eta_1=\delta\eta_3,\quad \eta_2=\eta_3\delta,$$
$$i\eta^2\pi=\delta\eta_3\delta\eta_3\delta,\quad \eta_1^2=\delta\eta_3\delta\eta_3,\quad \eta_2^2=\eta_3\delta\eta_3\delta,$$
$$\eta_2\eta_1=0,\quad \eta_1\eta_2=0 \ \ or \ \ 12(i\nu\pi),$$

and
$$\delta\delta=\delta(i\nu\pi)=(i\nu\pi)\delta=0.$$

(ii) *In case $q\equiv 2$* (mod 4): *$\{S^{n+i}M_q,\ S^n M_q\}$ are multiplicatively* *generated for $i\leqq 2$ by $i\pi$ $(=\delta)$, η_1, η_2 and $i\nu\pi$ with the relations*:

$$\delta\delta=\delta\eta_1=\eta_2\delta=\delta(i\nu\pi)=(i\nu\pi)\delta=0\,,$$
$$\eta_1\delta=\delta\eta_2=i\eta\pi=q\cdot 1,\quad \eta_1\eta_2=\eta_2\eta_1=0\,,$$

and

$$\eta_1{}^2\delta=\delta\eta_2{}^2=i\eta^2\pi=0\,.$$

Proof. The proof is easy except the relations on $\eta_1\eta_2$. By (4.3) $\eta_1\eta_2=i\bar{\eta}\tilde{\eta}\pi$. By (4.2)-(4.2′) $2\bar{\eta}=(q/2)\eta^2\pi$. Then,

$$2\bar{\eta}\tilde{\eta}=(q/2)\eta^2\pi\tilde{\eta}=(q/2)\eta^3=6q\nu\,.$$

This implies that

$$\bar{\eta}\tilde{\eta}\equiv 3q\nu \mod 12\nu\,.$$

Thus

$$\eta_1\eta_2\equiv 0 \mod 12(i\nu\pi)\,,$$

where $12(i\nu\pi)=0$ if $q\equiv 2 \pmod 4$. q.e.d.

4.3. We shall see that $M_q\wedge M_q$ is homotopy equivalent (in stable range) to the following mapping cone

(4.5) $$\bar{N}_q=SM_q\cup_{\bar{g}}C(SM_q)\,,$$

where

$$\bar{g}=\begin{cases}(Si)\eta(S\pi): SM_q\to S^3\to S^2\to S^3\subset SM_q & \text{if } q\equiv 2 \pmod 4,\\ 0 \text{ (constant map)} & \text{if } q\not\equiv 2 \pmod 4.\end{cases}$$

We denote also by N_q a subcomplex of $\bar{N}_q$ obtained by removing the 3-cell SM_q-S^2, i.e.,

(4.5′) $$N_q=S^2\cup_g C(SM_q)\,,$$

where

$$g=\begin{cases}\eta(S\pi): SM_q\to S^3\to S^2 & \text{if } q\equiv 2 \pmod 4,\\ 0 & \text{if } q\not\equiv 2 \pmod 4.\end{cases}$$

Obviously $\bar{N}_q=N_q\cup SM_q$ and $N_q\cap SM_q=S^2$.

The cell structures of $\bar{N}_q$ and N_q can be interpreted as follows:

(4.6) (i) *if* $q\not\equiv 2 \pmod 4$,

$$\bar{N}_q=SM_q\vee S^2M_q \quad \textit{and} \quad N_q=S^2\vee S^2M_q\,;$$

(ii) *if* $q\equiv 2 \pmod 4$,

$$\bar{N}_q=(SM_q\vee S^3)\cup e^4 \quad \textit{and} \quad N_q=(S^2\vee S^3)\cup e^4\,,$$

where e^4 *is attached to* $S^2 \vee S^3$ *by a map representing the sum of* $\eta \in \{S^3, S^2\}$ *and* $S^2 q = q \cdot 1_3 \in \{S^3, S^3\}$.

We use the following notations:

(4.7) $j : N_q \subset \bar{N}_q$, *the inclusion*;
$\bar{i}_0 : SM_q \subset \bar{N}_q$, $i_0 : S^2 \subset N_q$, *the inclusions*;
$\bar{i}_1 : S^3 \subset \bar{N}_q$, $i_1 : S^3 \subset N_q$, *the inclusions*;
$p : \bar{N}_q \to S^3$, *the map collapsing* N_q:
$\bar{\pi}_0 : \bar{N}_q \to S^2 M_q$, $\pi_0 : N_q \to S^2 M_q$, *the map collapsing* SM_q *or* S^2.

Hereafter, these mappings will be fixed as to satisfy the following relations:

(4.7′) $ji_0 = \bar{i}_0(Si)$, $\bar{i}_1 = ji_1$, $p\bar{i}_0 = S\pi$, $\pi_0 = \bar{\pi}_0 j$ *and* $\bar{\pi}_0 i_1 = \pi_0 i_1 = S^2 i$.

Lemma 4.3. *Teere exists an element* $\bar{\alpha}$ *of* $\{\bar{N}_q, M_q \wedge M_q\}$ *satisfying the following three conditions*:
(4.8), (i). $\bar{\alpha}$ *is a homotopy equivalence*, i.e., *there is a* (*uniquely determined*) *inverse* $\bar{\beta} \in \{M_q \wedge M_q, \bar{N}_q\}$ *of* $\bar{\alpha}$ *such that* $\bar{\alpha}\bar{\beta} = 1$ *and* $\bar{\beta}\bar{\alpha} = 1$.

(ii). $\bar{\alpha}\bar{i}_0 = 1_M \wedge i$, *thus* $\bar{\beta}(1_M \wedge i) = \bar{i}_0$.

(iii). $(1_M \wedge \pi)\bar{\alpha} = \bar{\pi}_0$, *thus* $\bar{\pi}_0\bar{\beta} = 1_M \wedge \pi$.

Proof. In general, a homotopy between f and $g : X \to Y$ gives a homotopy equivalence $h : Y \cup_f CX \to Y \cup_g CX$ such that $h \big| Y = 1_Y$ and, by callapsing Y, h induces a mapping $\bar{h} : SX \to SX$ homotopic to 1_{SX}. By Theo. 1.1, $\bar{g}$ is homotopic to $q \cdot 1_{SM}$ in stable range. On the other hand $M_q \wedge N_q$ is a mapping cone of $q \cdot 1_{SM}$. Thus the lemma follows. q. e. d.

Put

(4.9) $\alpha_0 = \bar{\alpha}\bar{i}_1 \in \{S^3, M_q \wedge M_q\}$ *and* $\beta_0 = p\bar{\beta} \in \{M_q \wedge M_q, S^3\}$.

It follows from (ii), (iii) of (4.8)

(4.9′). $$(1_M \wedge \pi)\alpha_0 = S^2 i \quad \textit{and} \quad \beta_0(1_M \wedge i) = S\pi .$$

Remark that

(4.9″) *if* α_0' *and* β_0' *satisfies* (4.9′) *then*

$$\alpha_0 - \alpha_0' = 0 \textit{ or } = (i \wedge i)\eta \quad \textit{and} \quad \beta_0 - \beta_0' = 0 \textit{ or } = \eta(\pi \wedge \pi),$$

where $(i \wedge i)\eta = \eta(\pi \wedge \pi) = 0$ *if* q *is odd.*

For, $\alpha_0-\alpha'_0\in(1_M\wedge i)_*\{S^3, SM_q\}$ and $\beta_0-\beta'_0\in(1_M\wedge\pi)^*\{S^2M_q, S^3\}$ ((*c f.* (4.2)).

Lemma 4.4. (i). *Let* $\bar{\alpha}\in\{\bar{N}_q, M_q\wedge M_q\}$ *be an element satisfying* (4.8). *Any element* $\bar{\alpha}'\in\{\bar{N}_q, M_q\wedge M_q\}$ *satisfies* (4.8) *if and only if*

$$\bar{\alpha}'=\bar{\alpha}+(1_M\wedge i)\gamma\bar{\pi}_0+x(i\wedge i)\eta p \qquad \textit{for some } \gamma\in\{S^2M_q, SM_q\},$$

where $x=0$ *if* $q\not\equiv 2 \pmod 4$ *and* $x=0$ *or* 1 *if* $q\equiv 2 \pmod 4$.

(ii). *For any* $\alpha_0\in\{S^3, M_q\wedge M_q\}$ *and* $\beta_0\in\{M_q\wedge M_q, S^3\}$ *satisfying* (4.9′) *there exists* $\bar{\alpha}\in\{\bar{N}_q, M_q\wedge M_q\}$ *which satisfies* (4.8) *and* (4.9). *Such an element* $\bar{\alpha}$ *is unique if q is odd,* $\bar{\alpha}$ *or* $\bar{\alpha}+(i\wedge i)\eta^2(S^2\pi)\bar{\pi}_0$ *if* $q\equiv 0 \pmod 4$, $\bar{\alpha}$ *or* $\bar{\alpha}+(i\wedge i)\eta p$ *if* $q\equiv 2 \pmod 4$.

Proof. (i). Assume that $\bar{\alpha}$ and $\bar{\alpha}'$ satisfy (4.8). There exists $\gamma'\in\{\bar{N}_q, SM_q\}$ such that $(1_M\wedge i)\gamma'=\bar{\alpha}'-\bar{\alpha}$ since $(1_M\wedge\pi)(\bar{\alpha}'-\bar{\alpha})=0$. Then $(1_M\wedge i)\gamma'\bar{i}_0=(\bar{\alpha}'-\bar{\alpha})\bar{i}_0=0$. The kernel of $(1_M\wedge i)_*$: $\{SM_q, SM_q\}\rightarrow\{SM_q, M_q\wedge M_q\}$ is $q\{SM_q, SM_q\}$ which vanishes if $q\not\equiv 2 \pmod 4$ and is generated by $(Si)\eta(S\pi)$ if $q\equiv 2 \pmod 4$. We have $(Si)\eta p\bar{i}_0=(Si)\eta(S\pi)$ by (4.7′). Thus $(\gamma'-x(Si)\eta p)\bar{i}_0=0$ for some $x(\in Z_2$ if $q\equiv 2$, $=0$ if $q\not\equiv 2)$. Then there exists $\gamma\in\{S^2M_q, SM_q\}$ such that $\gamma\bar{\pi}_0=\gamma'-x(Si)\eta p$, and

$$\bar{\alpha}'=\bar{\alpha}+(1_M\wedge i)\gamma'=\bar{\alpha}+(1_M\wedge i)\gamma\bar{\pi}_0+x(i\wedge i)\eta p\,.$$

Conversely, if $\bar{\alpha}$ satisfies (ii), (iii) of (4.8) then so does $\bar{\alpha}'$. $\bar{\alpha}$ and $\bar{\alpha}'$ induce the same homomorphisms of ordinary cohomology groups. Thus $\bar{\alpha}'$ is a homotopy equivalence if so is $\bar{\alpha}$.

(ii). If q is odd α_0 and β_0 are unique by (4.9″). Also $\bar{\alpha}$ is unique since $\{S^2M_q, SM_q\}=0$ if q is odd. Thus (ii) is obvious for odd q.

Let q be even and choose an element $\bar{\alpha}'$ satisfying (4.8). By (i) and (4.1), any $\bar{\alpha}''$ satisfying (4.8) can be written in the form

$$(*) \qquad \bar{\alpha}''=\bar{\alpha}'+(1_M\wedge i)(x\eta_1+y\eta_2)\bar{\pi}_0+z\delta$$

with $x, y, z\in Z_2$, where $\delta=(i\wedge i)\eta p$ if $q\equiv 2 \pmod 4$, and $\delta=(1_M\wedge i)(Si)\eta^2(S^2\pi)\bar{\pi}_0=(i\wedge i)\eta^2(S^2\pi)\bar{\pi}_0$ if $q\equiv 0 \pmod 4$.

By a caluculation making use of (4.1), (4.3) and (4.7′) we see that

$$\alpha''_0=\alpha'_0+x(i\wedge i)\eta\,,$$

where $\alpha'_0=\bar{\alpha}'\bar{i}_1$ and $\alpha''_0=\bar{\alpha}''_0\bar{i}_1$. Putting $\beta'_0=p\bar{\beta}'$ and $\beta''_0=p\bar{\beta}''$ ($\bar{\beta}'$ and $\bar{\beta}''$ are the inverses of $\bar{\alpha}'$ and $\bar{\alpha}''$ respectively), by a similar calculation we see that

$$(\beta'_0+y\eta(\pi\wedge\pi))\bar{\alpha}''=p\,.$$

On the other hand

$$\beta_0''\bar{\alpha}'' = p\,.$$

Since $\bar{\alpha}''$ is a homotopy equivalence we obtain

$$\beta_0'' = \beta_0' + y\eta(\pi\wedge\pi)\,.$$

Now, for the given α_0 and β_0 satisfying (4.9′), by (4.9″) we can put

$$\alpha_0 - \alpha_0' = x'(i\wedge i)\eta \qquad \text{and} \qquad \beta_0 - \beta_0' = y'\eta(\pi\wedge\pi)$$

with $x', y' \in Z_2$. Since, for arbitrarily chosen x, y and z, the element $\bar{\alpha}''$ satisfying (4.8) and $(*)$ exists uniquely by (i), if we put $x = x'$ and $y = y'$, then the determined $\bar{\alpha}''$ is the required element. Thereby z can not be determined by given α_0 and β_0 but may have two possible values, which corresponds to the conclusions of the lemma (ii). q. e. d.

4. 5. Consider the groups $\{M_q\wedge M_q,\ S^2M_q\}$ and $\{SM_q,\ M_q\wedge M_q\}$. From (1.17) and (1.17′) we have the following exact sequences:

$$0 \to \{S^2M_q,\ S^2M_q\}\otimes Z_q \xrightarrow{(1\wedge\pi)^*} \{M_q\wedge M_q,\ S^2M_q\} \xrightarrow{(1\wedge i)^*} \{SM_q,\ S^2M_q\}\,,$$

$$0 \to \{SM_q,\ SM_q\}\otimes Z_q \xrightarrow{(1\wedge i)_*} \{SM_q,\ M_q\wedge M_q\} \xrightarrow{(1\wedge\pi)_*} \{SM_q,\ S^2M_q\}\,.$$

By (4.2) and Theo. 4.1, $(S\pi)^* : \{S^3, S^2M_q\} \to \{SM_q,\ S^2M_q\}$ is an isomorphism. The formula $\beta_0(1_M\wedge i) = S\pi$ of (4.9′) implies that $\beta_0^*(S\pi)^{*-1}$ is a right inverse of $(1\wedge i)^*$. Thus the first sequence splits. Similarly the second sequence splits since $\alpha_{0*}(S^2i)_*^{-1}$ is a right inverse of $(1\wedge\pi)_*$. Then it follows from Theorem 4.1

(4.10).

	$q\not\equiv 0 \pmod 4$	$q\equiv 0 \pmod 4$
$\{M_q\wedge M_q,\ S^2M_q\}\cong\{SM_q,\ M_q\wedge M_q\}\cong$	Z_q+Z_q	$Z_q+Z_q+Z_2$
generators of $\{M_q\wedge M_q,\ S^2M_q\}$	$1_M\wedge\pi,\ (S^2i)\beta_0$	$1_M\wedge\pi,\ (S^2i)\beta_0,\ (S^2i)\eta(\pi\wedge\pi)$
generators of $\{SM_q,\ M_q\wedge M_q\}$	$1_M\wedge i,\ \alpha_0(S\pi)$	$1_M\wedge i,\ \alpha_0(S\pi),\ (i\wedge i)\eta(S\pi)$

Lemma 4. 5. *For each* $q>1$, *there exist elements* $\alpha_0 = \alpha_{0,q} \in \{S^3,\ M_q\wedge M_q\}$ *and* $\beta_0 = \beta_{0,q} \in \{M_q\wedge M_q,\ S^3\}$ *which satisfy* (4.9′) *and the following relations*:

(4.11)

$$\text{(i)}\quad (1_M\wedge\pi)T = 1_M\wedge\pi + (S^2i)\beta_0\,,$$

$$\text{(ii)}\quad T(1_M\wedge i) + 1_M\wedge i = \alpha_0(S\pi)\,,$$

Proof. Let G be a subgroup of $\{M_q\wedge M_q,\ S^2M_q\}$ generated by

$(S^2 i)\eta(\pi\wedge\pi)$ if $q\equiv 0 \pmod 4$, or consisting only of zero if $q\not\equiv 0 \pmod 4$. Then, using an element β_0 satisfying (4. 9′), we can put

$$(\sharp 1) \qquad (1\wedge\pi)T\equiv a(1\wedge\pi)+b(S^2 i)\beta_0 \mod G,$$

for some $a, b\in Z_q$ by (4. 10).

Let us use the ordinary mod q reduced cohomology $\tilde{H}^*(\ ; Z_q)$ and the generator $g_i\in\tilde{H}^i(M_q; Z_q)$, $i=1, 2$, such that

$$i^* g_1 = -\sigma 1_q \quad \text{and} \quad \pi^*(\sigma^2 1_q) = g_2,$$

i. e., $g_1=\kappa_i$ in the sense of **3. 4.** Then, by (3. 7) and Prop. 3. 3 $\delta_q g_1=g_2$ and the four elements $g_i\wedge g_j=\mu_q(g_i\otimes g_j)$ form a base of $\tilde{H}^*(M_q\wedge M_q; Z_q)$, where μ_q is the reduced cross product.

From ($\sharp$1) we obtain the identity

$$T^*(1\wedge\pi)^* = a(1\wedge\pi)^*+b\cdot\beta_0^*(S^2 i)^*$$

of cohomology maps. Applying this to $\sigma^2 g_1=g_1\wedge\sigma^2 1_q$, we have

$$g_2\wedge g_1 = a(g_1\wedge g_2)-b\cdot\beta_0^*(\sigma^3 1_q).$$

Since the class $\beta_0^*(\sigma^3 1_q)$ is integral,

$$\beta_0^*(\sigma^3 1_q) = x(g_2\wedge g_1-g_1\wedge g_2)$$

for some $x\in Z_q$. It follows from (4. 9′) that

$$\begin{aligned}\sigma g_2 &= (S\pi)^*(\sigma^2 1_q) = (1\wedge i)^*\beta_0^*(\sigma^3 1_q)\\ &= -x(g_2\wedge\sigma 1_q) = -x\cdot\sigma g_2.\end{aligned}$$

Thus, $x=-1$ and

$$g_2\wedge g_1 = (a-b)(g_1\wedge g_2)+b(g_2\wedge g_1).$$

That is, $a=b=1$, and

$$(\sharp 2) \qquad (1\wedge\pi)T\equiv(1\wedge\pi)+(S^2 i)\beta_0 \mod G.$$

This shows that, in case $q\not\equiv 0 \pmod 4$, arbitrarily chosen β_0 satisfies (i).

In case $q\equiv 0 \pmod 4$, put

$$(1\wedge\pi)T-(1\wedge\pi)-(S^2 i)\beta_0 = y(S^2 i)\eta(\pi\wedge\pi), \qquad y\in Z_2.$$

If $y=0$, then β_0 satisfies (i). If $y\neq 0$, put $\beta_0'=\beta_0+\eta(\pi\wedge\pi)$, then β_0' satisfies (4. 9′) and (i) as is easily checked.

Thus the existence of β_0 satisfying (4. 9′) and (i) was proved.

The proof of the existence of α_0 satisfying (4. 9′) and (ii) is completely parallel to the above, and is left to the readers. q. e. d.

From the above proof and (4. 9″) we see that

(4. 12) *the elements* α_0 *and* β_0 *satisfying* (4. 9′) *and the relations* (i) *and* (ii) *of* (4. 11) *is unique if* $q \not\equiv 2$ (mod 4); *if* $q \equiv 2$ (mod 4), *any elements* α_0 *and* β_0 *satisfying* (4. 9′) *satisfy* (i) *and* (ii) *of* (4. 11).

4. 6. Next we compute the groups $\{M_q, M_t\}$ $(q, t>1)$, By (1. 7) we get the following exact sequence

$$0 \to \{S^2, M_t\} \otimes Z_q \xrightarrow{\pi_q^*} \{M_q, M_t\} \xrightarrow{i_q^*} \mathrm{Tor}(\{S^1, M_t\}, Z_q) \to 0 .$$

Let $d=(q, t)$ be the greatest common divisoa of g and t. Tor $(\{S^1, M_t\}, Z_q)$ is isomorphic to Z_d and generated by $(t/d)\cdot i_t$. By (2. 5).

$$i_q^*(\overline{q/n}) = (\overline{q/d})i_q = (t/d)\cdot i_t .$$

From (4. 2) it follows that $\{M_q, M_t\}$ is generated by $\overline{q/d}$ and $i_t\eta\pi_q$, where $d(q/d)=0$ or $i_t\eta\pi_q$, and $i_t\eta\pi_q \neq 0$ if and only if q and t are even. We have

(4. 13′) $d(\overline{q/d})=i_t\eta\pi \neq 0$, i. e., $\overline{q/d}$ *is of order* $2d$, *if and only if* $q\equiv t\equiv 2$ (mod 4).

To see (4. 13′), we may assume that q and t are even. q/d or t/d is odd since they are relatively prime. Assume that q/d is odd. Then, using Theo. 1. 1,

$$d\cdot(\overline{q/d}) = (q/d)d\cdot(\overline{q/d}) = q\cdot(\overline{q/d}) = 0 \qquad \text{if} \quad q \not\equiv 2 \pmod 4$$

and, if $q\equiv 2$ (mod 4)

$$d\cdot(\overline{q/d}) = q\cdot(\overline{q/d}) = (\overline{q/d})i_q\eta\pi_q = (t/d)\cdot i_t\eta\pi_q$$

by (2. 5), which prove (4. 13′) in case q/d is odd. In case t/d is odd, (4. 13′) can be proved similarly.

From (4. 13′) and the above exact sequence we obtain

(4. 13).

	q or t : odd	$q\equiv t\equiv 2$ (mod 4)	others
$\{M_q, M_t\}\cong$	Z_d	Z_{2d}	Z_d+Z_2
generators	$\overline{q/d}$	$\overline{q/d}$	$\overline{q/d}$, $i_t\eta\pi_q$

where $d=(q, t)$.

Let $q>1$, $r\geqq 1$, and consider the elements $(i_q\wedge i_q)\eta(S^2\pi_{qr})\in\{SM_{qr}, M_q\wedge M_q\}$ and $(S^2 i_{qr})\eta(\pi_q\wedge\pi_q)\in\{M_q\wedge M_q, S^2M_{qr}\}$. By (4.13), $(Si_q)\eta(S^2\pi_{qr})\neq 0$ if and only if q and qr are even. The kernel of the homomorphism

$$(1\wedge i)_*: \{SM_{qr}, SM_q\}\to\{SM_{qr}, M_q\wedge M_q\}$$

is $q\{SM_{qr}, SM_q\}$ which is generated by $(Si_q)\eta(S\pi_{qr})$ if $q\equiv qr\equiv 2 \pmod 4$ and vanishes otherwise. Thus we conclude that

(4.14) $(i_q\wedge i_q)\eta(S^2\pi_{qr})\neq 0$ *if and only if* $q\equiv 0 \pmod 4$ *or* $q\equiv 2$, $qr\equiv 0 \pmod 4$.

Similarly we see that

(4.14′) $(S^2 i_{qr})\eta(\pi_q\wedge\pi_q)\neq 0$ *if and only if* $q\equiv 0 \pmod 4$ *or* $q\equiv 2$, $qr\equiv 0 \pmod 4$.

The following table (4.15) and the relation (4.15′) are verified from (1.7), (1.7′), (4.2) and Theo. 1.1 (*cf.*, Theo. 4.1).

(4.15)

	q : odd	$q\equiv 0$ (mod 4)	$q\equiv 2$ (mod 4)
$\{S^3, M_q\wedge M_q\}\cong\{M_q\wedge M_q, S^3\}\cong$	Z_q	Z_q+Z_2	Z_{2q}
generators of $\{S^3, M_q\wedge M_q\}$	α_0	α_0, $(i_q\wedge i_q)\eta$	α_0
generators of $\{M_q\wedge M_q, S^3\}$	β_0	β_0, $\eta(\pi_q\wedge\pi_q)$	β_0

where α_0 *and* β_0 *are arbitrarily chosen elements satisfying* (4.9′).

(4.15′) $q\cdot\alpha_0=(i_q\wedge i_q)\eta\neq 0$ *and* $q\cdot\beta_0=\eta(\pi_q\wedge\pi_q)\neq 0$ *if* $q\equiv 2 \pmod 4$.

4.7. Lemma 4.6. (i) *There exist sequences* $\{\alpha_{0,q}\}$, $\{\beta_{0,q}\}$, $q>1$, *of elements* $\alpha_{0,q}\in\{S^3, M_q\wedge M_q\}$ *and* $\beta_{0,q}\in\{M_q\wedge M_q, S^3\}$ *which satisfy* (4.9′), (4.11) *and*

(4.16) $(\bar r\wedge\bar r)\alpha_{0,qr}=r\cdot\alpha_{0,q}$, $\beta_{0,qr}(\bar 1\wedge\bar 1)=r\cdot\beta_{0,q}$ *for the maps* $\bar r: M_{qr}\to M_q$, $\bar 1: M_q\to M_{qr}$ *of* (2.5), $r\geqq 1$.

(ii) *There are just two sequences* $\{\alpha_{0,q}\}$ *and* $\{\alpha'_{0,q}\}$ *satisfying* (4.9′) *and* (4.16); $\alpha_{0,q}=\alpha'_{0,q}$ *if* $q\not\equiv 2 \pmod 4$ *and* $\alpha'_{0,q}=\alpha_{0,q}+(i_q\wedge i_q)\eta$ *if* $q\equiv 2 \pmod 4$.

(iii) *There are just two sequences* $\{\beta_{0,q}\}$ *and* $\{\beta'_{0,q}\}$ *satisfying* (4.9′) *and* (4.16); $\beta_{0,q}=\beta'_{0,q}$ *if* $q\not\equiv 2 \pmod 4$ *and* $\beta'_{0,q}=\beta_{0,q}+\eta(\pi_q\wedge\pi_q)$ *if* $q\equiv 2 \pmod 4$.

(iv) *If two sequences* $\{\alpha_{0,q}\}$ *and* $\{\beta_{0,q}\}$ *satisfy* (4.9′) *and* (4.16), *then they satisfy* (4.11).

Proof. (i) For each $q>1$ choose a pair of elements $\{\alpha''_{0,q}\}$ and $\{\beta''_{0,q}\}$ satisfying (4.9′) and (4.11).

For each (q, r), $q>1$, $r\geqq 1$ the element

$$(\bar{r}\wedge\bar{r})\alpha''_{0,qr}-r\cdot\alpha''_{0,q} \qquad (\bar{r}: M_{qr}\to M_q)$$

is in the kernel of

$$(1\wedge\pi_q)_*: \{S^3, M_q\wedge M_q\}\to\{S^3, S^2M_q\}$$

since

$$(1\wedge\pi_q)r\cdot\alpha''_{0,q}=r\cdot S^2i_q \qquad \text{by (4.9′)}$$

$$\begin{aligned}(1\wedge\pi_q)(\bar{r}\wedge\bar{r})\alpha''_{0,qr}&=(\bar{r}\wedge r\cdot\pi_{qr})\alpha''_{0,qr} && \text{by (2.5)}\\ &=r\cdot(\bar{r}\wedge 1)S^2i_{qr} && \text{by (4.9′)}\\ &=r\cdot S^2i_q && \text{by (2.5)}.\end{aligned}$$

By (1.7′) and (4.2), the kernel of $(1\wedge\pi_q)_*$ is generated by $(1\wedge i_q)(Si_q)\eta=(i_q\wedge i_q)\eta$ which vanishes if and only if q is odd. Hence

$$(\natural 1) \qquad (\bar{r}\wedge\bar{r})\alpha''_{0,qr}=r\cdot\alpha''_{0,q}+x_{q,r}\cdot(i_q\wedge i_q)\eta,$$

where $x_{q,r}\in Z_2$ if q is even, and $x_{q,r}=0$ if q is odd.

Compose $(\bar{r}\wedge\bar{r})$ to the equation

$$(T_{qr}+1)(1\wedge i_{qr})=T_{qr}(1\wedge i_{qr})+1\wedge i_{qr}=\alpha''_{0,qr}(S\pi_{qr})$$

of (4.11) from the left, where $T_{qr}=T(M_{qr}, M_{qr})$. Then, making use of (1.2), (2.5), (4.11) and ($\natural$1), we have that

$$\begin{aligned}\text{the left side}&=(T_q+1)(\bar{r}\wedge\bar{r})(1\wedge i_{qr})=(T_q+1)(1\wedge\bar{r}i_{qr})(\bar{r}\wedge 1)\\ &=(T_q+1)(1\wedge i_q)S\bar{r}=\alpha''_{0,q}(S\pi_r)S\bar{r}=r\cdot\alpha''_{0,q}(S\pi_{qr}),\\ \text{the right side}&=(\bar{r}\wedge\bar{r})\alpha''_{0,qr}(S\pi_{qr})\\ &=r\cdot\alpha''_{0,q}(S\pi_{qr})+x_{q,r}(i_q\wedge i_q)\eta(S\pi_{qr}),\end{aligned}$$

i.e., $\qquad x_{q,r}(i_q\wedge i_q)\eta(S\pi_{qr})=0$.

Hence, by (4.14), we have

$$(\natural 2) \qquad x_{q,r}=0 \text{ if } q\equiv 0 \pmod 4 \text{ or if } q\equiv 2,\ qr\equiv 0 \pmod 4.$$

From ($\natural$1) and ($\natural$2) we have

$$(\natural 3) \qquad (\bar{r}\wedge\bar{r})\alpha''_{0,qr}=r\cdot\alpha''_{0,q} \text{ if } q\not\equiv 2 \pmod 4 \text{ or } qr\not\equiv 2 \pmod 4.$$

Now we put

(♮4) $$\alpha_{0,q}=\begin{cases}\alpha''_{0,q} & \text{if } q\not\equiv 2 \pmod 4\\ \alpha''_{0,q}+x_{q/2,2}\cdot(i_q\wedge i_q) & \text{if } q\equiv 2 \pmod 4)\,.\end{cases}$$

By (4 12) the sequence $\{\alpha_{0,q}\}$ satisfies (4.9′) and (4.11). We shall check (4.16) for this sequence $\{\alpha_{0,q}\}$. In case $q\not\equiv 2$ and $qr\not\equiv 2 \pmod 4$, (4.16) is obvious by (♮3) and (♮4). In case $q\not\equiv 2$ and $qr\equiv 2 \pmod 4$, q is odd, hence $(i_q\wedge i_q)\eta=0$ and (4.16) is easily seen. In case $q\equiv 2$ and $qr\not\equiv 2 \pmod 4$, r is even and we obtain (4.16) as follows:

$$\begin{aligned}(\bar{r}\wedge\bar{r})\alpha_{0,qr}&=(\bar{r}\wedge\bar{r})\alpha''_{0,qr}=r\cdot\alpha''_{0,q}\\&=r\cdot\alpha_{0,q}+r\cdot x_{q/2,2}\cdot(i_q\wedge i_q)\eta=r\cdot\alpha_{0,q}\,.\end{aligned}$$

Finally consider the case $q\equiv qr\equiv 2 \pmod 4$. Since $\{\alpha_{0,q}\}$ satisfies (4.9′) and (4.11), we can put, by (♮1),

$$(\bar{r}\wedge\bar{r})\alpha_{0,qr}=r\cdot\alpha_{0,q}+y_{q,r}(i_q\wedge i_q)\eta$$

for some $y_{q,r}\in Z_2$. Compose $(\overline{q/2}\wedge\overline{q/2})$, $\overline{q/2}:M_2\to M_q$, to this equation from the left. Then, by (♮4), (2.7) and (2.5),

$$\begin{aligned}\text{the left side}&=(\overline{qr/2}\wedge\overline{qr/2})(\alpha''_{0,q}+x_{qr/2,2}(i_{qr}\wedge i_{qr})\eta)\\&=(qr/2)\cdot\alpha''_{0,2}+2\cdot x_{qr/2,2}(i_2\wedge i_2)\eta\\&=(qr/2)\cdot\alpha''_{0,2},\\ \text{the right side}&=r\cdot(\overline{q/2}\wedge\overline{q/2})(\alpha''_{0,q}+x_{q/2,2}(i_q\wedge i_q)\eta)+y_{q,r}(i_2\wedge i_2)\eta\\&=r\cdot((q/2)\cdot\alpha''_{0,2}+2x_{q/2,2}(i_2\wedge i_2)\eta)+y_{q,r}(i_2\wedge i_2)\eta\\&=(qr/2)\cdot\alpha''_{0,2}+y_{q,r}(i_2\wedge i_2)\eta\,,\end{aligned}$$

i.e., $$y_{q,r}(i_2\wedge i_2)\eta=0\,.$$

Since $(i_2\wedge i_2)\eta\neq 0$ by (4.15′) we obtain

$$y_{q,r}=0$$

which proves (4.16) for the considered case.

Thus we have proved the existence of the sequence $\{\alpha_{0,q}\}$ satisfying (4.9′), (4.11) and (4.16).

The proof of the existence of a sequence $\{\beta_{0,q}\}$ satisfying (4.9′), (4.11) and (4.16) is completely dual to the above one, and the details are left to the readers.

(ii) Assume that $\{\alpha_{0,q}\}$ and $\{\alpha'_{0,q}\}$ satisfy (4.9′) and (4.16). By (4.9″) we can put

$$\alpha'_{0,q}=\alpha_{0,q}+z_q\cdot(i_q\wedge i_q)\eta\,,\qquad z_q\in Z_2\,,$$

where $z_q=0$ if q is odd. Let q be even. By (4.16) and (2.5)

$$\begin{aligned}(q/2)\cdot\alpha'_{0,2} &= (\overline{q/2}\wedge\overline{q/2})\alpha'_{0,q} \\ &= (\overline{q/2}\wedge\overline{q/2})(\alpha_{0,q}+z_q\cdot(i_q\wedge i_q)\eta) \\ &= (q/2)\cdot\alpha_{0,2}+z_q\cdot(i_2\wedge i_2)\eta \\ &= (q/2)\cdot\alpha'_{0,2}+((q/2)z_2+z_q)(i_2\wedge i_2)\eta\,.\end{aligned}$$

Since $(i_2\wedge i_2)\eta\neq 0$ by (4.15′), we have

$$(q/2)z_2\equiv z_q \pmod 4\,,$$

i. e.,

$$\begin{aligned}z_q &= 0 \quad \text{if} \quad q\equiv 0 \pmod 4\,, \\ &= z_2 \quad \text{if} \quad q\equiv 2 \pmod 4\,.\end{aligned}$$

Thus there are at most two sequences satisfying (4.9′) and (4.16).

Next let $\{\alpha_{0,q}\}$ be a sequence of (i). Put $\alpha''_{0,q}=\alpha_{0,q}$ for $q>2$ and $\alpha''_{0,2}=\alpha_{0,2}+(i_2\wedge i_2)\eta$, then $\{\alpha''_{0,q}\}$ satisfies (4.9′) and (4.11) by (4.12). Repeating the proof of (i) to this sequence $\{\alpha''_{0,q}\}$ we get a sequence $\{\alpha'_{0,q}\}$ satisfying (4.9′) and (4.16). We see in (♮4) that

$$\alpha'_{0,2}=\alpha''_{0,2}=\alpha_{0,2}+(i_2\wedge i_2)\eta\neq\alpha_{0,2}$$

since $x_{1,2}=0$. Thus $\{\alpha_{0,q}\}$ and $\{\alpha'_{0,q}\}$ are two different sequences satisfying (4.9′) and (4.16), and we have proved (ii).

(iii) The proof of (iii) is a dual of that of (ii).

(iv) By (ii), there are just two sequences $\{\alpha_{0,q}\}$ and $\{\alpha'_{0,q}\}$ satisfying (4.9′) and (4.16). But, as is seen in the last half of the proof of (ii), both sequences are constructed by a method employed in the proof of (i), hence they satisfy (4.11). Thus (iv) was proved. q. e. d.

4. 8. Take a pair of sequences $\{\alpha_{0,q}\}$ and $\{\beta_{0,q}\}$ of Lemma 4.6, (i). In virtue of Lemma 4,4, (ii), we see that

(4.17) *there exists a sequence* $\{\bar\alpha_q\}$, $q>1$, *of elements* $\bar\alpha=\bar\alpha_q\in\{\bar N_q, M_q\wedge M_q\}$, *of which each element satisfies* (4.8) *and, putting* $\bar\alpha_q\bar i_1=\alpha_{0,q}=\alpha_0$ *and* $p\bar\beta_q=\beta_{0,q}=\beta_0$, *the sequences* $\{\alpha_{0,q}\}$ *and* $\{\beta_{0,q}\}$ *satisfy* (4.9′), (4.11) *and* (4.16).

In the following, we fix a sequence $\{\bar\alpha_q\}$ of (4.17), and use as $\bar\alpha$ only the elements of this sequence.

We put

(4.18) $$\alpha=\bar\alpha j\in\{N_q,\ M_q\wedge M_q\}\,.$$

This element α will play an important role in the following paragraphs. By (4.8) and (4.7′) we have

(4.18′) $$\alpha i_1=\alpha_0,\ \alpha i_0=i\wedge i \quad \textit{and} \quad (1_M\wedge\pi)\alpha=\pi_0\,.$$

Proposition 4.7. (i) $(1_M\wedge\pi)T\alpha=\pi_0$.

(ii) $$T(1_M\wedge i)+(1_M\wedge i)=\alpha i_1(S\pi).$$

Proof. (i) $$(1_M\wedge\pi)T\alpha=(1_M\wedge\pi)\alpha+(S^2 i)\beta_0\alpha$$
$$=\pi_0+(S^2 i)pj=\pi_0$$

by (4.9), (4.11), (4.18), (4.18′) and (4.8).

(ii) follows from (4.11), (ii) and (4.18′). q.e.d.

5. Existence of admissible multiplications

5.1. Let μ be an associative multiplication in a reduced cohomology theory $\tilde{h}$. In this paragraph we define a multiplication μ_q in $\tilde{h}(\ ;Z_q)$ for each $q>1$, and prove that μ_q is admissible. Thereby, we need some assumptions on μ and $\tilde{h}$ in case $q\equiv 2 \pmod 4$.

Using the notations of **4.3**, the cofibration

$$S^2\xrightarrow{i_0}N_q\xrightarrow{\pi_0}S^2M_q$$

yields, for any object (finite CW-complex with a base point) W of $\tilde{h}$, a cofibration

$$W\wedge S^2\xrightarrow{1\wedge i_0}W\wedge N_q\xrightarrow{1\wedge\pi_0}W\wedge S^2M_q.$$

In the exact sequence of $\tilde{h}$ associated with this cofibration,

$$(1_W\wedge S^n g)^*:\tilde{h}^k(W\wedge S^{n+2})\to\tilde{h}^k(W\wedge M_q\wedge S^{n+1})$$

becomes a trivial map if $q\not\equiv 2 \pmod 4$, or if $q\equiv 2 \pmod 4$ and $(\eta\pi)^{**}=0$, since the attaching map $g=0$ in the former case and $=\eta(S\pi)$ in the latter case. Thus,

(5.1) *the $\tilde{h}$-cohomology sequence associated with the above cofibration breaks into the following short exact sequences*

$$0\to\tilde{h}^k(W\wedge S^2M_q)\xrightarrow{(1\wedge\pi_0)^*}\tilde{h}^k(W\wedge N_q)\xrightarrow{(1\wedge i_0)^*}\tilde{h}^k(W\wedge S^2)\to 0$$

if $q\not\equiv 2 \pmod 4$, *or if* $q\equiv 2 \pmod 4$ *and* $(\eta\pi)^{**}=0$ *in* $\tilde{h}$.

When $q\not\equiv 2 \pmod 4$, $N_q=S^2\vee S^2M_q$ by (4.6), (i). Let

$$i':S^2M_q\to N_q \quad\text{and}\quad \pi':N_q\to S^2$$

be the inclusion and the map collapsing S^2M_q respectively. Obviously we have

(5.2). $$\pi' i_0 = 1,\ \pi_0 i' = 1,\ \pi' i' = 0 \quad and \quad \pi' i_1 = 0\,.$$

Lemma 5.1. *If* (i) $q \not\equiv 2$ (mod 4), *or if* (ii) $q \equiv 2$ (mod 4) *and* $\eta^{**}=0$ *in* $\tilde{h}$, *then there exists an element* γ_0 *of* $\tilde{h}^2(N_q)$ *satisfying*

(5.3) $$i_0^* \gamma_0 = \sigma^2 1 \quad and \quad i_1^* \gamma_0 = 0\,.$$

In case (i), $\gamma_0 = \pi'^*(\sigma^2 1)$ *satisfies these relations.*

If (iii) $q \equiv 2$ (mod 4) *and* $(\eta\pi)^{**}=0$ *in* $\tilde{h}$, *then there exists* $\gamma_0 \in \tilde{h}^2(N_q)$ *satisfying*

(5.3′) $$i_0^* \gamma_0 = \sigma^2 1\,.$$

Proof. Case (i) follows from (5.2) by putting $\gamma_0 = \pi'^*(\sigma^2 1)$. In cases (ii) and (iii), $(\eta\pi)^{**}=0$. Then, by (5.1) for $W=S^0$, there exists $\gamma_0' \in \tilde{h}^2(N_q)$ such teat $i_0^* \gamma_0' = \sigma^2 1$. Thus the case (iii) is proved by putting $\gamma_0 = \gamma_0'$. In the remaining case (ii); (4.6), (ii) implies that $i_1(S^2 q)$ is homotopic to $i_0 \eta$. Thus $(S^2 q)^* i_1^* \gamma_0' = 0$. From the exactness of the sequence

$$\tilde{h}^2(S^2 M_q) \xrightarrow{(S^2 i)^*} \tilde{h}^2(S^3) \xrightarrow{(S^2 q)^*} \tilde{h}^2(S^3)$$

follows that $(S^2 i)^* x = i_1^* \gamma_0'$ for some $x \in \tilde{h}^2(S^2 M_q)$. Put

$$\gamma_0 = \gamma_0' - \pi_0^* x\,.$$

Then we have

$$i_0^* \gamma_0 = i_0^* \gamma_0' - i_0^* \pi_0^* x = \sigma^2 1$$

and

$$\begin{aligned} i_1^* \gamma_0 &= i_1^* \gamma_0' - i_1^* \pi_0^* x = (S^2 i)^* x - (\pi_0 i_1)^* x \\ &= 0 \end{aligned}$$

by (4.7′). q.e.d.

5.2. Making use of γ_0 of Lemma 5.1, hence at least under the assumption of $(\eta\pi)^{**}=0$ if $q \equiv 2$ (mod 4), we define a homomorphism

$$\gamma = \gamma_W : \tilde{h}^k(W \wedge N_q) \to \tilde{h}^k(W \wedge S^2 M_q)$$

by the formula

(5.4) $$\gamma_W(x) = (1_W \wedge \pi_0)^{*-1}(x - \mu(\sigma^{-2}(1_W \wedge i_0)^* x \otimes \gamma_0))$$

for $x \in \tilde{h}^k(W \wedge N_q)$. Since

$$\begin{aligned} (1_W \wedge i_0)^* \mu(\sigma^{-2}(1 \wedge i_0)^* x \otimes \gamma_0) &= \mu(\sigma^{-2}(1 \wedge i_0)^* x \otimes \sigma^2 1) \\ &= (1 \wedge i_0)^* x\,, \end{aligned}$$

$x - \mu(\sigma^{-2}(1 \wedge i_0)^* x \otimes \gamma_0)$ is in the kernel of $(1_W \wedge i_0)^*$. By (5.1), $(1_W \wedge \pi_0)^*$ is monomorphic. Thus the map γ is a well-defined homomorphism.

As is easily checked by (5.2), we have

(5.4'). $\gamma_W = (1_W \wedge i')^*$ *if* $q \not\equiv 2 \pmod 4$ *and* $\gamma_0 = \pi'^*(\sigma^2 1)$.

Lemma 5.2. (i) *γ_W is a left inverse of $(1_W \wedge \pi_0)^*$, i.e., $\gamma_W(1_W \wedge \pi_0)^*$ = an identity map; hence the sequence of* (5.1) *splits*:

$$\tilde{h}^k(W \wedge N_q) \simeq \tilde{h}^k(W \wedge S^2) \oplus \tilde{h}^k(W \wedge S^2 M_q).$$

(ii) *γ is natural in the sense that*

$$(f \wedge S^2 1_M)^* \gamma_W = \gamma_{W'}(f \wedge 1_N)^*,$$

where $f: W' \to W$, $M = M_q$ and $N = N_q$.

(iii) *γ is compatible with the suspension in the sense that*

$$(1_W \wedge T'')^* \sigma \gamma_W = \gamma_{SW}(1_W \wedge T')^* \sigma,$$

where $T' = T(S^1, N_q)$ and $T'' = T(S^1, S^2 M_q)$.

(iv) *The relation*

$$\mu(y \otimes \gamma_W(x)) = \gamma_{Y \wedge W} \mu(y \otimes x)$$

holds, where $x \in \tilde{h}^k(W \wedge N_q)$ and $y \in \tilde{h}^j(Y)$.

Proof. (i) If $x = (1 \wedge \pi_0)^* y$, then $(1 \wedge i_0)^* x = 0$ and (i) follows from (5.4).

(ii) Since $(1_{W'} \wedge \pi_0)^*$ is monomorphic, it it sufficient to prove the equality

$$(1_{W'} \wedge \pi_0)^* (f \wedge S^2 1_M)^* \gamma_W(x) = (1_{W'} \wedge \pi_0)^* \gamma_{W'}(f \wedge 1_N)^*(x).$$

Now, the left ride $= (f \wedge \pi_0)^* \gamma_W(x) = (f \wedge 1_N)^* (1_W \wedge \pi_0)^* \gamma_W(x)$

$$\begin{aligned} &= (f \wedge 1_N)^* x - \mu(\sigma^{-2} S^2 f^* (1 \wedge i_0)^* x \otimes \gamma_0) \\ &= (f \wedge 1_N)^* x - \mu(\sigma^{-2} (1_{W'} \wedge i_0)^* (f \wedge 1_N)^* x \otimes \gamma_0) \\ &= \text{the right side.} \end{aligned}$$

(iii) Since $T_0 = T(S^1, S^2)$ is a map of degree 1, we have

$$(1_{SW} \wedge i_0)^* (1_W \wedge T')^* \sigma = (1_W \wedge T_0)^* \sigma (1_W \wedge i_0)^* = \sigma (1_W \wedge i_0)^*.$$

Making use of this identity, we have

$$\begin{aligned} &(1_{SW} \wedge \pi_0)^* \gamma_{SW} (1_W \wedge T')^* \sigma(x) \\ &\quad = (1_W \wedge T')^* \sigma x - \mu(\sigma^{-2} (1_{SW} \wedge i_0)^* (1_W \wedge T')^* \sigma x \otimes \gamma_0) \\ &\quad = (1_W \wedge T') \sigma x - (1_W \wedge T')^* \sigma \mu(\sigma^{-2} (1_W \wedge i_0)^* x \otimes \gamma_0) \\ &\quad = (1_W \wedge T')^* \sigma (1_W \wedge \pi_0)^* \gamma_W(x) \\ &\quad = (1_{SW} \wedge \pi_0)^* (1_W \wedge T'')^* \sigma \gamma_W(x), \end{aligned}$$

from which follows (iii) since $(1_{SW}\wedge\pi_0)^*$ is monomorphic.

$$
\begin{aligned}
\text{(iv)}\quad & (1_{Y\wedge W}\wedge\pi_0)^*\gamma_{Y\wedge W}\mu(y\otimes x)\\
&=\mu(y\otimes x)-\mu(\sigma^{-2}(1_{Y\wedge W}\wedge i_0)^*\mu(y\otimes x)\otimes\gamma_0)\\
&=\mu(y\otimes x)-\mu(\mu(y\otimes\sigma^{-2}(1_W\wedge i_0)^*x)\otimes\gamma_0)\\
&=\mu(y\otimes(1_W\wedge\pi_0)^*\gamma_W(x))\\
&=(1_{Y\wedge W}\wedge\pi_0)^*\mu(y\otimes\gamma_W(x))\,,
\end{aligned}
$$

from which follows (iv). q. e. d.

Lemma 5.3. *If α_0 satisfies* (5.3), *then the relation*

$$(1_W\wedge S^2 i)^*\gamma_W=(1_W\wedge i_1)^*$$

holds for the inclusions $i:S^1\subset M_q$ and $i_1:S^3\subset N_q$.

Proof. Since $S^2 i=\pi_0 i_1$ by (4.7′) and $i_1^*\gamma_0=0$ by (5.3), we see that

$$
\begin{aligned}
(1_W\wedge S^2 i)^*\gamma_W(x)&=(1_W\wedge i_1)^*(1_W\wedge\pi_0)^*\gamma_W(x)\\
&=(1_W\wedge i_1)^*x-\mu(\sigma^{-2}(1_W\wedge i_0)^*x\otimes i_1^*\gamma_0)\\
&=(1_W\wedge i)^*x\,.\quad \text{q. e. d.}
\end{aligned}
$$

To prove the quasi-associativity of the multiplication μ_q to be defined in 5.3, we need a special kind of commutativity, i. e.,

(5.5) *if $\gamma_0=\pi'^*(\sigma^2 1)$ in case $q\not\equiv 2$* (mod 4), *or if μ is commutative, then there holds a commutativity*

(5.5′) $$\mu(z\otimes\gamma_0)=T'^*\mu(\gamma_0\otimes z)$$

for any $z\in\tilde{h}^i(Z)$, where $T'=T(Z,N_q)$.

The proof is clear.

Lemma 5.4. *If γ_0 satisfies* (5.5′), *then there holds a relation*

$$(1_W\wedge T'')^*\mu(\gamma_W(x)\otimes z)=\gamma_{W\wedge Z}(1_W\wedge T')^*\mu(x\otimes z)$$

for any $x\in\tilde{h}^k(W\wedge N_q)$ and $z\in\tilde{h}^i(Z)$, where $T'=T(Z,N_q)$ and $T''=T(Z,S^2M_q)$.

Proof.
$$
\begin{aligned}
&(1_{W\wedge Z}\wedge\pi_0)^*(1_W\wedge T'')^*\mu(\gamma_W(x)\otimes z)\\
&=(1_W\wedge T')^*(1_W\wedge\pi_0\wedge 1_Z)^*\mu(\gamma_W(x)\otimes z)\\
&=(1_W\wedge T')^*\mu(x\otimes z)-(1_W\wedge T')^*\mu(\mu(\sigma^{-2}(1_W\wedge i_0)^*x\otimes\gamma_0)\otimes z)\\
&=(1_W\wedge T')^*\mu(x\otimes z)-\mu(\sigma^{-2}(1_W\wedge i_0)^*x\otimes T'^*\mu(\gamma_0\otimes z))\,,
\end{aligned}
$$

and $$(1_{W\wedge Z}\wedge\pi_0)^*\gamma_{W\wedge Z}(1_W\wedge T')^*\mu(x\otimes z)$$
$$=(1_W\wedge T')^*\mu(x\otimes z)-\mu(\mu(\sigma^{-2}(1_W\wedge i_0)^*x\otimes z)\otimes\gamma_0)$$
$$=(1_W\wedge T')^*\mu(x\otimes z)-\mu(\sigma^{-2}(1_W\wedge i_0)^*x\otimes\mu(z\otimes\gamma_0)).$$

Thus (5. 5′) concludes the lemma.

5. 3. Making use of the homomorphism γ defined by (5. 4) and the element $\alpha\in\{N_q,\ M_q\wedge M_q\}$ of (4. 18), we define a map

$$\mu_q:\tilde{h}^i(X;\ Z_q)\otimes\tilde{h}^j(Y;\ Z_q)\to\tilde{h}^{i+j}(X\wedge Y;\ Z_q) \tag{5. 6}$$

as the composition

$$\mu_q=\sigma^{-2}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T\wedge 1_M)^*\mu: \tag{5. 6′}$$
$$\tilde{h}^i(X;\ Z_q)\otimes\tilde{h}^j(Y;\ Z_q)=\tilde{h}^{i+2}(X\wedge M)\otimes\tilde{h}^{j+2}(Y\wedge M_q)$$
$$\to\tilde{h}^{i+j+4}(X\wedge M_q\wedge Y\wedge M_q)\to\tilde{h}^{i+j+4}(X\wedge Y\wedge M_q\wedge M_q)$$
$$\to\tilde{h}^{i+j+4}(X\wedge Y\wedge N_q)\to\tilde{h}^{i+j+4}(X\wedge Y\wedge M_q\wedge S^2)$$
$$\to\tilde{h}^{i+j+2}(X\wedge Y\wedge M_q)=\tilde{h}^{i+j}(X\wedge Y;\ Z_q),$$

where $T=T(Y,\ M_q)$.

μ_q is defined only if $q\not\equiv 2$ (mod 4) or if $q\equiv 2$ (mod 4) and $(\eta\pi)^{**}=0$. The definition of μ_q depends on the choices of γ_0 and α which are but fixed during the subsequent proofs of properties of an admissible multiplication.

Note that

$$\mu_q=\sigma^{-2}(\alpha i')^{**}(1_X\wedge T\wedge 1_M)^*\quad \textit{if}\quad q\not\equiv 2\ (\text{mod } 4)\ \textit{and}\ \gamma_0=\pi'^*(\sigma^2 1). \tag{5. 6″}$$

5. 4. Theorem 5. 5. *The map μ_q of* (5. 6) *is a multiplication satisfying* (Λ_1).

Proof. The linearity and the naturality of μ_q is obvious.

To prove (Λ_1): putting $T'=T(Y,\ M_q)$, $T_1=T(S^2,\ Y\wedge M_q)$, $T_2=T(S^2,\ M_q)$ and $T=T(M_q,\ M_q)$, by definitions of ρ_q and μ_q we have

$$\mu_q(\rho_q\otimes 1)=\sigma^{-2}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T'\wedge 1_M)^*\mu((1_X\wedge\pi)^*\sigma^2\otimes 1_{Y\wedge M})$$
$$=\sigma^{-2}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T'\wedge 1_M)^*(1_X\wedge\pi\wedge 1_{Y\wedge M})^*(1_X\wedge T_1)^*\sigma^2\mu$$
$$=\sigma^{-2}\gamma_{X\wedge Y}\alpha^{**}(1_{X\wedge Y}\wedge 1_M\wedge\pi)^*(1_{X\wedge Y}\wedge T_2)^*\sigma^2\mu$$
$$=\sigma^{-2}\gamma_{X\wedge Y}(T_2(1_M\wedge\pi)\alpha)^{**}\sigma^2\mu$$
$$=\sigma^{-2}\gamma_{X\wedge Y}((1_{X\wedge Y}\wedge\pi)T\alpha)^{**}\sigma^2\mu$$
$$=\sigma^{-2}\gamma_{X\wedge Y}(1_{X\wedge Y}\wedge\pi_0)^*\sigma^2\mu \qquad \text{by Prop. 4. 7, (i),}$$
$$=\sigma^{-2}\sigma^2\mu=\mu=\mu_L \qquad \text{by Lemma 5. 2, (i).}$$

Similarly we see that

$$\mu_q(1\otimes\rho_q)=(1_X\wedge T')^*\mu=\mu_R\,,$$

i. e., (Λ_1) was proved.

From (Λ_1) and (H_3) follows that $\rho_q(1)$ is a bilateral unit of μ_q, i. e., the existence of 1_q is obtained.

To prove the compatibility of μ_q with σ_q: putting $T=T(Y, M_q)$, $T_1=T(Y, S^1)$, $T_2=T(S^1, M_q)$, $T_3=T(S^1, Y\wedge M_q)$, $T_4=T(S^1, Y\wedge N_q)$, $T_5=T(S^1, N_q)$ and $T_6=T(S^1, S^2M_q)$, by definitions of μ_q and σ_q we have

$$\begin{aligned}&(1_X\wedge T_1)^*\mu_q(\sigma_q\otimes 1)\\ &\quad=(1_X\wedge T_1\wedge 1_M)^*\sigma^{-2}\gamma_{SX\wedge Y}\alpha^{**}(1_{SX}\wedge T\wedge 1_M)^*\mu((1_X\wedge T_2)^*\sigma\otimes 1^*_{Y\wedge M})\\ &\quad=(1_X\wedge T_1\wedge 1_M)^*\sigma^{-2}\gamma\alpha^{**}(1_{SX}\wedge T\wedge 1_M)^*(1_X\wedge T_2\wedge 1_{Y\wedge M})^*(1_{X\wedge M}\wedge T_3)^*\sigma\mu\\ &\quad=(1_X\wedge T_1\wedge 1_M)^*\sigma^{-2}\gamma(1_X\wedge T_4)^*(1_{X\wedge Y}\wedge S\alpha)^*(1_X\wedge T_2\wedge 1_{SM})^*\sigma\mu\\ &\quad=\sigma^{-2}(1_X\wedge T_1\wedge S^2 1_M)^*\gamma(1_X\wedge T_4)^*\sigma(1_{X\wedge Y}\wedge\alpha)^*(1_X\wedge T\wedge 1_M)^*\mu\,.\end{aligned}$$

Here

$$\begin{aligned}&(1_X\wedge T_1\wedge S^2 1_M)^*\gamma_{SX\wedge Y}(1_X\wedge T_4)^*\sigma\\ &\quad=\gamma_{X\wedge SY}(1_X\wedge T_1\wedge 1_N)^*(1_X\wedge T_4)^*\sigma \qquad\text{by Lemma 5.2, (ii),}\\ &\quad=\gamma_{X\wedge SY}(1_{X\wedge Y}\wedge T_5)^*\sigma\\ &\quad=(1_{X\wedge Y}\wedge T_6)^*\sigma\gamma_{X\wedge Y} \qquad\text{by Lemma 5.2, (iii).}\end{aligned}$$

Thus

$$\begin{aligned}&(1_X\wedge T_1)^*\mu_q(\sigma_q\otimes 1)\\ &\quad=(1_{X\wedge Y}\wedge T_2)^*\sigma^{-1}\gamma_{X\wedge Y}\wedge(1_{X\wedge Y}\wedge\alpha)^*(1_X\wedge T\wedge 1_M)^*\mu\\ &\quad=(1_{X\wedge Y}\wedge T_2)^*\sigma\sigma^{-2}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T\wedge 1_M)^*\mu\\ &\quad=\sigma_q\mu_q\,.\end{aligned}$$

Similarly we see that

$$(-1)^i\mu(1\otimes\sigma_q)=\sigma_q\mu_q\,. \quad \text{q. e. d.}$$

The above theorem, combined with Prop. 3.3, shows

Corollary 5.6. *When* $q\equiv 2 \pmod 4$, *the condition that* $(\eta\pi)^{**}=0$ *in* $\tilde{h}$ *is necessary and sufficient for the existence of a multiplication* μ_q *satisfying* (Λ_1).

5.5. Theorem 5.7. *If* γ_0 *satisfies* (5.3), *then the multiplication* μ_q *of* (5.6) *satisfies* (Λ_2).

Proof. By Theo. 5.5 we can use (Λ_1) for μ_q. Putting $T=T(M_q, M_q)$, $T'=T(Y, M_q)$ and $T''=T(Y\wedge M_q, S^1)$, we have (on

$\tilde{h}^i(X: Z_q)\otimes\tilde{h}^j(Y; Z))$

$$\begin{aligned}
&\mu_q(\delta_q\otimes 1)+(-1)^i\mu_q(1\otimes\delta_q)\\
&\quad=\mu_q(\rho\delta\otimes 1)+(-1)^i\mu_q(1\otimes\rho\delta)\\
&\quad=\mu_L(\delta\otimes 1)+(-1)^i\mu_R(1\otimes\delta)\\
&\quad=\mu(\sigma^{-1}(1_X\wedge i)^*\otimes 1^*_{Y\wedge M})+(-1)^i(1_X\wedge T')^*\mu(1^*_{X\wedge M}\otimes\sigma^{-1}(1_Y\wedge i)^*)\\
&\quad=(\sigma^{-1}(1_X\wedge T'')^*(1_X\wedge i\wedge 1_{Y\wedge M})^*+\sigma^{-1}(1_X\wedge ST')^*(1_{X\wedge M\wedge Y}\wedge i)^*)\mu\\
&\quad=\sigma^{-1}((1_{X\wedge Y\wedge M}\wedge i)^*(1_{X\wedge Y}\wedge T)^*+(1_{X\wedge Y\wedge M}\wedge i)^*(1_X\wedge T'\wedge 1_M)^*\mu\\
&\quad=\sigma^{-1}(T(1_M\wedge i)+(1_M\wedge i))^{**}(1_X\wedge T'\wedge 1_M)^*\mu\\
&\quad=\sigma^{-1}\pi^{**}i_1^{**}\alpha^{**}(1_X\wedge T'\wedge 1_M)^*\mu \qquad \text{by Prop. 4.7, (ii),}\\
&\quad=\sigma^{-1}\pi^{**}i^{**}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T'\wedge 1_M)^*\mu \qquad \text{by Lemma 5.3,}\\
&\quad=\rho\delta\mu_q=\delta_q\mu_q. \quad \text{q. e. d.}
\end{aligned}$$

Theorem 5.8. *If* γ_0 *satisfies* (5.5′), *then the multiplication* μ_q *of* (5.6) *satisfies* (Λ_3).

Proof. Since μ_q satisfies (Λ_1) it is sufficient to prove the following three relations:

$$\mu_q(\mu_L\otimes 1)=\mu_L(1\otimes\mu_q), \tag{5.7}$$

$$\mu_q(\mu_R\otimes 1)=\mu_q(1\otimes\mu_L), \tag{5.7′}$$

$$\mu_R(\mu_q\otimes 1)=\mu_q(1\otimes\mu_R). \tag{5.7″}$$

To prove (5.7): discussing on $\tilde{h}^*(X; Z_q)\otimes\tilde{h}^*(Y; Z_q)\otimes\tilde{h}^*(Z; Z_q)$ and putting $T_1=T(Z, M_q)$, we have

$$\begin{aligned}
\mu_q(\mu_L\otimes 1)&=\sigma^{-2}\gamma_{X\wedge Y\wedge Z}\alpha^{**}(1_{X\wedge Y}\wedge T_1\wedge 1_M)^*\mu(\mu\otimes 1)\\
&=\sigma^{-2}\gamma_{X\wedge Y\wedge Z}\mu(1\otimes\alpha^{**}(1_Y\wedge T_1\wedge 1_M)^*\mu)\\
&=\sigma^{-2}\mu(1\otimes\gamma_{Y\wedge Y}\alpha^{**}(1_Y\wedge T_1\wedge 1_M)^*\mu) \qquad \text{by Lemma 5.2, (iv),}\\
&=\mu(1\otimes\sigma^{-2}\gamma_{Y\wedge Z}\alpha^{**}(1_Y\wedge T_1\wedge 1_M)^*\mu)\\
&=\mu_L(1\otimes\mu),
\end{aligned}$$

i. e., (5.7) is proved.

In a similar way we can easily see (5.7′), and using Lemma 5.4 instead of Lemma 5.2, (iv), we can see (5.7″). The details are left to the readers. q. e. d.

5.6. As a corollary of Theos. 5.5, 5.7, 5.8, Lemma 5.1 and (5.5), we obtain

Theorem 5.9. (Existence theorem). *In case* $q\not\equiv 2$ (mod 4) *admissible* *multiplications* μ_q *exist always*; *In case* $q\equiv 2$ (mod 4), *if we assume that* $\eta^{**}=0$ *in* $\tilde{h}$ *and* μ *is commutative, then admissible ones* μ_q *exist.*

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