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BRUNNIAN SYSTEMS OF 2-SPHERES IN 4-SPACE

Dedicated to Professor H. Terasaka on his 60th birthday

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Introduction

A link with multiplicity n in a 3-space E^3 is called a *Brunnian link*[1], if they are unsplittable but every proper sublink is completely splittable. In this paper, we shall construct an example of the systems of 2-spheres in 4-space E^4, of the similar type called *Brunnian systems of 2-spheres in 4-space.* The definition of the *splittability* of the system of 2-spheres in 4-space is similar to the definition about the links in 3-space given in [1]. And the construction of the present example follows the method given by R. H. Fox in [3; Chap. 3, p. 132∼139], and the proof is due to a theorem in [4].

The problem to construct Brunnian systems of 2-spheres in E^4, found in [3; Question 38, p. 175], was first suggested to the author by T. Yajima.

1. Preliminary lemmas. In this paper we consider everything from the semi-linear point of view. A system of n disjoint, flat[2] 2-spheres in 4-space E^4 (in 4-sphere S^4) shall be called a *link in* E^4 (*in* S^4) with multiplicity n. A sublink of a link is a subsystem of 2-spheres. For convenience, we denote a link L with the system of 2-spheres $K_0, K_1, \cdots, K_{n-1}$ by $L=K_0 \cup K_1 \cup \cdots \cup K_{n-1}$. A link $L=K_0 \cup K_1 \cup \cdots \cup K_{n-1}$ is called *splittable* in E^4 if and only if there is a *polyhedral*[3] 3-sphere S in E^4 such that $L \cap S=\phi$, $L \cap \text{int.} S \neq \phi$, and that $L \cap \text{ext.} S \neq \phi$. A link L is called *completely splittable* in E^4 if and only if there is a system of n disjoint, polyhedral 3-spheres $S_0, S_1, \cdots, S_{n-1}$ such that $L \cap S_i=\phi$, and that

1) See [1], [2].

2) A 2-sphere K in E^4 is called flat if $K \cap \partial V$ is an unknotted simple closed curve in the 3-sphere ∂V for each sufficiently small neighborhood V in E^4 of each point of K.

3) int. S denotes the closure of the bounded component of E^4-S, and ext. S, denotes the closure of the unbounded component. Here, "polyhedral" means not only a subcomplex of E^4 but both int. S and ext. S, attached with the infinity point, are combinatorial 4-cells.

$K_i \subset \text{int.}\, S_i \subset \text{ext.}\, S_j$ for $i, j=0, 1, \cdots, n-1$, $i \neq j$. If a link L is splittable (completely splittable) in E^4 (in S^4), a link $h(L)$ is also splittable (completely splittable) in E^4 (in S^4), where h is a semi-linear homeomorphism of E^4 (of S^4) onto itself. Since a 4-sphere S^4 is constructed by attaching the infinity point to a 4-space E^4, we can define the splittability of the link in S^4 similarly to that of the link in E^4.

Lemma 1. *A link $L=K_0 \cup K_1 \cup \cdots \cup K_{n-1}$ in E^4 is completely splittable in E^4, if L is splittable and each proper sublink is completely splittable in E^4.*

Proof. Since L is splittable, there is a polyhedral 3-sphere S such that $\text{int.}\, S \supset K_{m+1} \cup K_{m+2} \cup \cdots \cup K_{n-1}$ and $\text{ext.}\, S \supset K_0 \cup K_1 \cup \cdots \cup K_m$, and as S is polyhedral in E^4, there is a semi-linear isotopy h of E^4 such that h is identical on E^4-U and that $\text{int.}\, S$ is carried into a sufficiently small 4-simplex Δ in the interior of $\text{int.}\, S$ by h, where U denotes a small neighborhood of $\text{int.}\, S$. Then the system of the flat 2-spheres $h(K_0) \cup h(K_1) \cup \cdots \cup h(K_m)$ is a proper sublink of $h(L)$, and it is completely splittable in E^4. Suppose that S'_0, $S'_1, \cdots, S'_m$ is one of the system of polyhedral 3-spheres which completely splits the proper sublink and does not contain Δ in $\text{int.}\, S'_i$ for $i=0, 1, \cdots, m$. Let S_i be $h(S'_i)$ for $i=0, 1, \cdots, m$, then the system of polyhedral 3-spheres S_0, $S_1, \cdots, S_m$ completely splits the link $L'=K_0 \cup K_1 \cup \cdots \cup K_m$ in E^4 and each 3-sphere S_i does not meet the original 3-sphere S for $i=0, 1, \cdots, m$. Again, there is a semi-linear isotopy h' of E^4 such that h' is identical on E^4-V and that $\text{int.}\, S_i$ is carried into a sufficiently small 4-simplexes Δ_i in the interiors of $\text{int.}\, S_i$ by h', where V denotes the union of the small neighborhoods of $\text{int.}\, S_i$ for $i=0, 1, \cdots, m$. Then there is a system of polyhedral 3-spheres $S'_{m+1}, S'_{m+2}, \cdots, S'_{n-1}$ which completely splits the link $h'(K_{m+1} \cup K_{m+2} \cup \cdots \cup K_{n-1})$ in E^4, and we may suppose that $\text{int.}\, S_j$ does not contain Δ_i for all $i=0, 1, \cdots, m$ and $j=m+1, m+2, \cdots, n-1$. Let S_j be $h'^{-1}(S'_j)$ for $j=m+1, m+2, \cdots, n-1$, then the system of polyhedral 3-spheres S_0, $S_1, \cdots, S_{n-1}$ is one of the system of polyhedral 3-spheres which splits the link L completely in E^4.

Lemma 1 holds for the link L in S^4, and its proof is almost quite analogous.

Lemma 2. *If a link $L=K_0 \cup K_1 \cup \cdots \cup K_{n-1}$ in E^4 (in S^4) is completely splittable and each 2-sphere K_i is unknotted*[4)] *in E^4 (in S^4) for $i=0, 1, \cdots, n-1$, then $\pi_2(E^4-L)=0$ ($\pi_2(S^4-L)=0$).*

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4) K_i bounds a combinatorial 3-cell in E^4 (in S^4).

Lemma 3. If $K_i \cap E_0^3 = k_i$ are non-empty and connected for $i=0, 1, \cdots, n-1$, where $L = K_0 \cup K_1 \cup \cdots \cup K_{n-1}$ and $k_0 \cup k_1 \cup \cdots \cup k_{n-1}$ are links in E^4 and in E_0^3 respectively, then a basis of $H_1(E_0^3 - L \cap E_0^3)$ can be identical with a basis of $H_1(E^4 - L)$. Here, we denote by E_t^3 the 3-space defined by $x_4 = t$ in the 4-space where x_i is the i-th coordinate of E^4.

We make use of two lemmas without proof.

2. Example. We construct $n+1$ disjoint flat 2-spheres $K_0, K_1, \cdots, K_{n-1}$ and Σ in E^4 as described in Fig. 1, where k_i^t denotes the intersection of K_i and E_t^3 for $i=0, 1, \cdots, n-1$, and σ^t denotes the intersection of Σ and E_t^3. The saddle-point-transformations are performed on K_0 at the level-spaces $E_{\frac{1}{2}}^3$ and $E_{-\frac{1}{2}}^3$ as in Fig. 2. Consider a link $L = K_0 \cup K_1 \cup \cdots \cup K_{n-1}$, then it is easily seen that each proper sublink L' of L is completely splittable in E^4. Now, we shall show that L is not splittable in E^4. By Lemma 1, it is sufficient to prove that L is not completely splittable in E^4. Clearly each 2-sphere K_i is unknotted in E^4, and by Lemma 2, we have only to prove that $\pi_2(E^4 - L) \neq 0$.

Lemma 4. *Σ links homotopically with L in E^4*

The proof follows the method by J. J. Andrews and M. L. Curtis in their paper [4].

Proof. Suppose that Σ does not link with L homotopically, then the homeomorphism ξ of the unit 2-sphere S^2 onto Σ can be extended to the continuous mapping φ of the unit 3-ball B into $E^4 - L$, where we suppose that the equator s of S^2 is mapped onto $\sigma^0 = \Sigma \cap E_0^3$ by φ. Let s' be the 1-cycle generating $H_1(s)$. $\varphi^{-1}(\varphi(B) \cap E_0^3)$ contains a 2-complex whose boundary 1-cycle is s'. Denote this relative 2-complex with boundary s' by Q'. Let $G = \pi_1(E_0^3 - L \cap E_0^3)$ and let $\varphi_\# : \pi_1(Q) \to G$ and let $\varphi_* : H_1(Q') \to H_1(E_0^3 - L \cap E_0^3)$ be the homomorphisms induced by φ, where Q denotes a finite polyhedron carrying Q' in the 3-ball B.

First, we consider the group of the link in Fig. 1.

$G = \pi_1(E_0^3 - L \cap E_0^3)$ is as follows:

generators: $x_i,\ y_i,\ u_i,\ w_i \qquad i = 0, 1, \cdots, n-1.$

relations :

$$x_{i-1} w_i x_{i-1}^{-1} u_i^{-1} = 1$$
$$y_{i-1} w_i y_{i-1}^{-1} v_i^{-1} = 1$$
$$x_{i-1} w_i u_{i-1}^{-1} w_i^{-1} = 1$$
$$y_{i-1} w_i v_{i-1}^{-1} w_i^{-1} = 1$$
$$u_i x_{i-1} u_i^{-1} v_i y_{i-1}^{-1} v_i^{-1} = 1 \quad \mathrm{mod.}\ n$$

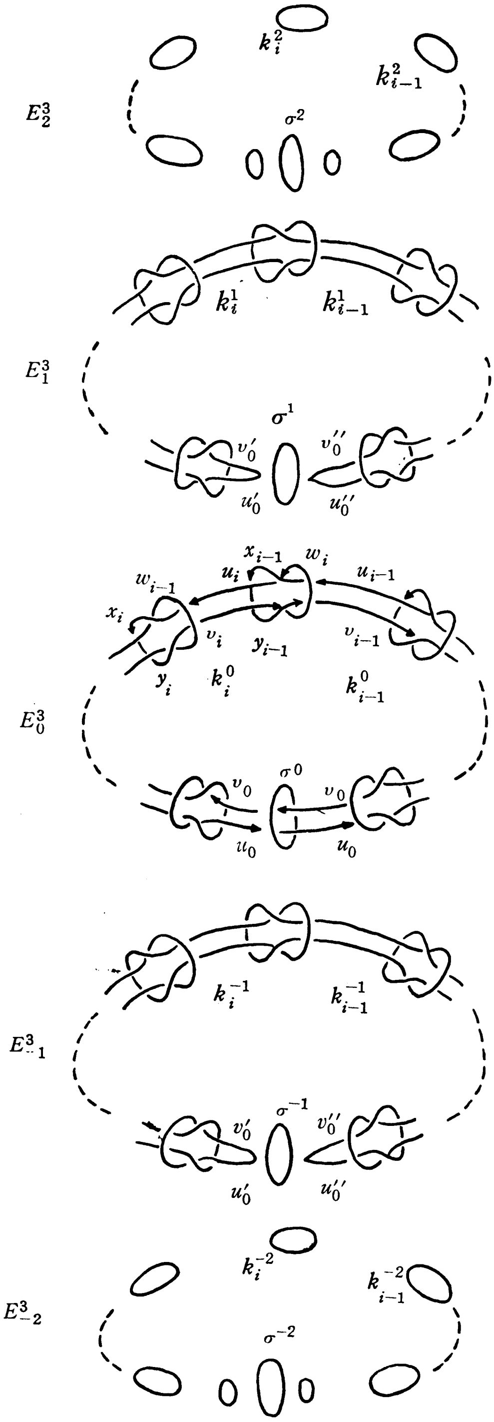
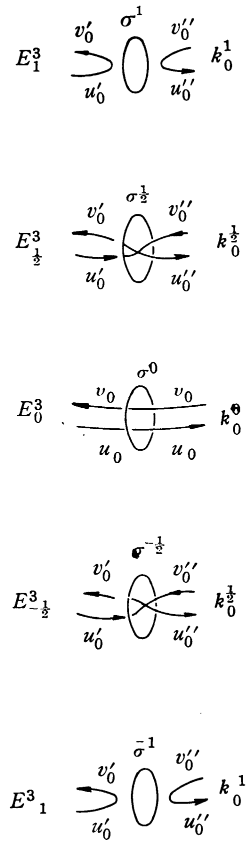


Fig. 1.



Σ

Fig. 2.

There is a representation of G onto $\mathfrak{S}_3$ such that u_i, $y_i \to (12)$ v_i, $x_i \to (13)$. By making use of this representation, we have that the class $\varphi_\#(\{s\}) = \{\sigma^0\}$ does not belong to $G^{(2)}$[5)]. Since it is clear that $\varphi_\#(\{s\}) = \{\sigma^0\}$ belongs to $G^{(1)}$, we can apply the theorem by Andrews and Curtis in their paper [4].

Theorem (Andrews and Curtis). If $\varphi_\#(\{s\})$ belongs to $G^{(1)}$ and not to $G^{(2)}$, then there exists a 1-cycle z in Q' such that $\varphi_*(z) \neq 0$.

The link L in Fig. 1 satisfies the conditions of Lemma 3, therefore the basis of $H_1(E_0^3 - L \cap E_0^3)$ is also the basis of $H_1(E^4 - L)$. Then the image of the 1-cycle z given in the above theorem is not only non-trivial in $H_1(E_0^3 - L \cap E_0^3)$ but also in $H_1(E^4 - L)$, but the 1-complex z' carrying the 1-cycle z in Q must be contractible in $\varphi(B)$ contained in $E^4 - L$. This is a contradiction, and we have:

Theorem. *The system of* 2*-spheres described in Fig.* 1 *is of the Brunnian type.*

3. Another example. In [1], it is shown that there exist links in E^3 with n components such that every proper sublink with at most $k-1$ components is completely splittable but no sublink with components more than $k-1$ is splittable. In this section, we consider the same type of links of 2-spheres in 4-space. Here, the link described in Fig. 3 is one of the links of the desired type; that is, every proper sublink with at most two components is completely splittable, but no sublink with components more than two is splittable. The slice link, at the level $x_4 = 0$ in Fig. 3, is the link shown in p. 22 Fig. 3 of [1]. Each proper sublink

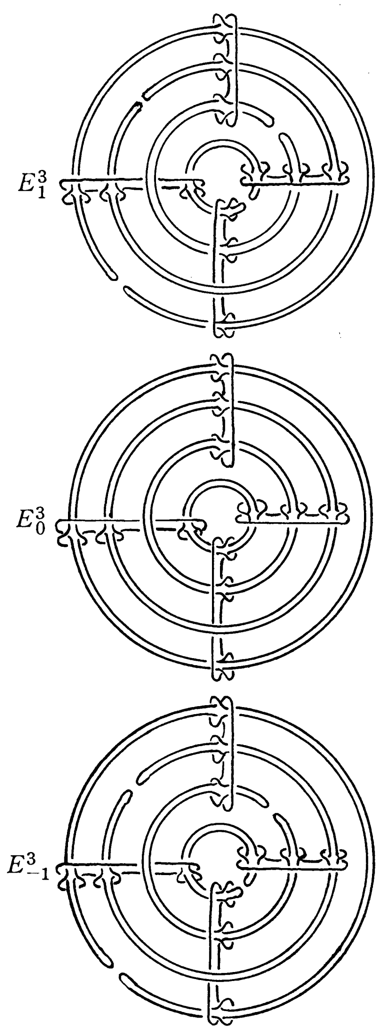


Fig. 3.

5) $G^{(1)}$ denotes $[G, G]$ and $G^{(2)}$ denotes $[G^{(1)}, G^{(1)}]$.

with three components is almost similar to the link ($n=3$) in the section 2. Therefore each proper sublink with two components is completely splittable and the sublink with three components is not splittable. And the link is not splittable because of the cyclic situation of components.

The present example will imply that there is a link of 2-spheres with n components such that each proper sublink with at most $k-1$ components is completely splittable but no sublink with components more than $k-1$ is splittable for integers n, and $n>k>1$.

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