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ON A GENERALIZATION OF THE RING THEORY

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1. Introduction. A ring of endomorphisms of a module plays a very important role in many parts of mathematics; the property of a ring itself is also clarified when we consider it as a ring of endomorphisms of a module. As a generalization of this idea, we can consider a set of homomorphisms of a module to another module which is closed under the addition and subtraction defined naturally but has no more a structure of a ring since we can not define the product. However, suppose that we have an additive group M consisting of homomorphisms of a module A to a module B and that we have also an additive group N consisting of homomorphisms of B to A. In this case we can define the product of three elements f_1, g and f_2 where f_1 and f_2 are elements of M and g is an element of N. If this product f_1gf_2 is also an element of M for every f_1, g and f_2, we say that M is closed under the multiplication using N between. Similarly we can define that N is closed under the multiplication using M between. Take f_1, f_2 and f_3 in M and g_1 and g_2 in N in the above case. Then we have

$$(f_1g_1f_2)g_2f_3 = f_1g_1(f_2g_2f_3) = f_1(g_1f_2g_2)f_3\,.$$

When we define this situation abstractly, we can get a new algebraic system.

DEFINITION. Let M be an additive group whose elements are denoted by $a, b, c, \cdots$, and Γ another additive group whose elements are $\gamma, \beta, \alpha, \cdots$. Suppose that $a\gamma b$ is defined to be an element of M and that $\gamma a\beta$ is defined to be an element of Γ for every a, b, γ and β. If the products satisfy the following three conditions:

1)
$$\begin{aligned}(a_1+a_2)\gamma b &= a_1\gamma b+a_2\gamma b\,,\\ a(\gamma_1+\gamma_2)b &= a\gamma_1 b+a\gamma_2 b\,,\\ a\gamma(b_1+b_2) &= a\gamma b_1+a\gamma b_2\,,\end{aligned}$$

2) $$(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c\,,$$

3) if $a\gamma b=0$ for any a and b in M, then $\gamma=0$,

then M is called *a Γ-ring*

The purpose of this note is to determine the structure of Γ-rings under the following conditions which are called semi-simple and simple according to the usual ring theory.

DEFINITION. Let M be a Γ-ring as above. If for any non-zero element a of M there exists such an element γ (depending on a) in Γ that $a\gamma a\neq 0$, we say that M is *semi-simple*. If for any non-zero elements a and b of M there exists γ (depending on a and b) in Γ such that $a\gamma b\neq 0$, we say that M is *simple*.

The main result obtained in this note is that a simple Γ-ring which satisfies the chain condition for left and right ideals (defined in § 3) is the set $D_{n,m}$ of all rectangular matrices of type $n\times m$ over some division ring D and Γ is $D_{m,n}$ of type $m\times n$. The product $a\gamma b$ is the same as the usual matrix product of elements a, γ and b of $D_{n,m}$, $D_{m,n}$ and $D_{n,m}$. This is a generalization of the theorem of Wedderburn on simple rings. Subsequently, a semi-simple Γ-ring satisfying the chain condition for left and right ideals will be shown to be a direct sum of simple Γ_i-rings, where $\Gamma=\Gamma_1+\Gamma_2+\cdots+\Gamma_n$ (direct).

2. Examples. Suppose we have a right R-module M with an operator ring R. Take a submodule Γ of $\mathrm{Hom}_R(M, R)$. Then M is a Γ-ring as follows: If a and b are elements of M and if γ is an element of Γ, then we define

$$a\gamma b = a\cdot\gamma(b)\,,$$

where $\gamma(b)$ is an image of b by γ and is an element of R. It is easy to verify that

$$(a\gamma b)\beta c = (a\cdot\gamma(b))\cdot\beta(c) = a(\gamma(b)\beta(c)) = a\cdot\gamma(b\cdot\beta(c)) = a\gamma(b\beta c)\,.$$

We also define that

$$\gamma b\beta = \beta\cdot\gamma(b)_l \quad (\beta \text{ operating first})\,,$$

where $\gamma(b)_l$ means the left multiplication of $\gamma(b)$. Then

$$(a\gamma b)\beta c = a(\gamma(b)\beta(c)) = a(\gamma b\beta)c\,.$$

The conditions 1) and 3) hold naturally and M is a Γ-ring. But it will be shown in § 3 that every Γ-ring is given in this way.

To illustrate further this new algebraic system, we introduce the definition and examples of cubic rings.

DEFINITION. We call that M is *a cubic ring* when we can define the product of three elements of M which is an additive group such that it satisfies

$$\begin{aligned}(a_1+a_2)bc &= a_1bc+a_2bc\,,\\ a(b_1+b_2)c &= ab_1c+ab_2c\,,\\ ab(c_1+c_2) &= abc_1+abc_2\,,\end{aligned} \tag{4}$$

$$ab(cde) = (abc)de\,, \tag{5}$$

6) if $abc=0$ for all a and c, then $b=0$.

If we take the product in a cubic ring M as the product of two elements of M using one element of $\Gamma=M$ between, then conditions 1) and 3) for a Γ-ring are satisfied. Also the first part of 2) is satisfied. Hence, in order that M is a Γ-ring, we must be able to define the product $\Gamma\times M\times\Gamma$ such that the latter part of 2) holds. In the following examples, we can find it easily.

EXAMPLE 1. Let $V_n(F)$ be a vector space of dim n over a field F. If a, b and c are vectors in it, we define $abc=(a\cdot b)c$, where $(a\cdot b)$ is the inner product of a and b. It is easy to see that $V_n(F)$ is a cubic ring. Now we define $(bcd)'=b(c\cdot d)$. Then $ab(cde)=(a\cdot b)(c\cdot d)e=a(bcd)'e$, i.e., $V_n(F)$ is a Γ-ring with $\Gamma=V_n(F)$.

EXAMPLE 2. Let $D_{n,m}$ be the set of all rectangular matrices of type $n\times m$ over a division ring D. If a, b and c are elements in it, we define $abc=ab^tc$, where b^t is the transpose of a matrix b and the above product is well-defined. Then $D_{n,m}$ is clearly a cubic ring. Now we define $(bcd)'=dc^tb$. Then $ab(cde)=ab^tcd^te=a(bcd)'e$, i.e., $D_{n,m}$ is a Γ-ring with $\Gamma=D_{n,m}$.

EXAMPLE 3. Let I be the set of all purely imaginary complex numbers. Then it is a cubic ring with the usual multiplication. Also it is a Γ-ring with $\Gamma=I$. However, even with the same I, we can define another cubic ring. For example, if a, b and c are elements in I, we define the product of a, b and c as $a\bar{b}c$ where $\bar{b}$ is the conjugate of b, i.e., $-b$. This product also satisfies 4), 5) and 6) of the definition of cubic rings. In this case, we put $(bcd)'=-bcd$.

3. The operator rings and ideals. Let M be a Γ-ring. Consider the additive group generated by pairs (γ, a), where $\gamma\in\Gamma$ and $a\in M$ with defining relations $(\gamma_1+\gamma_2, a)=(\gamma_1, a)+(\gamma_2, a)$ and $(\gamma, a_1+a_2)=(\gamma, a_1)+(\gamma, a_2)$. We define the multiplication of the elements of this additive group such that

$$(\gamma, a)(\beta, b) = (\gamma, a\beta b).$$

Using the condition 2), we can verify that

$$((\gamma, a)(\beta, b))(\alpha, c) = (\gamma, a)((\beta, b)(\alpha, c)).$$

Thus we get a ring which we denote by F. Now we can see that F is a right operator ring of M by the following definition:

$$a(\gamma, b) = a\gamma b,$$

for, we have

$$(a(\gamma, b))(\beta, c) = (a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma, b\beta c) = a((\gamma, b)(\beta, c)).$$

The set of all elements of F that annihilate M forms an ideal which we denote by A, and we denote F/A by R and call it *the right operator ring of M.* We use γa for an element of R which is gained from (γ, a). Thus $a\gamma b=a(\gamma b)$. Then, take an element γ of Γ. It induces an R-homomorphisms of M to R such that $\gamma(a)=\gamma a$. The condition 3) implies that Γ induces the zero homomorphism if and only if $\gamma=0$. Thus Γ is considered to be a subset of the total set of R-homomorphisms of M to R; $\Gamma\subset\mathrm{Hom}_R(M, R)$.

Similary we can define *the left operator ring L of M.* We start with (a, γ) and define the product such that $(a, \gamma)(b, \beta)=(a\gamma b, \beta)$. Also we define the left operation such that $(a, \gamma)b=a\gamma b$, and so on. $a\gamma$ is an element of L given from (a, γ) and $a\gamma b=(a\gamma)b$. And we can say that $\Gamma\subset\mathrm{Hom}_L(M, L)$.

DEFINITION. R-submodules of M are called *right ideals* of M, and L-submodules of M are *left ideals.*

A right ideal $\mathfrak{r}$ is nothing but a submodule of M such that $\mathfrak{r}\Gamma M\subset\mathfrak{r}$. A left ideal $\mathfrak{l}$ is a submodule of M such that $M\Gamma\mathfrak{l}\subset\mathfrak{l}$.

4. Peirce decomposition in semi-simple Γ-rings. Assume that M is semi-simple, and let $\mathfrak{r}$ be a minimal right ideal. Then by semi-simplicity there exists an element ε in Γ such that $a\varepsilon a\neq 0$ for a non-zero element a in $\mathfrak{r}$. Then $0\neq a\varepsilon\mathfrak{r}\subset\mathfrak{r}$ and hence $\mathfrak{r}=a\varepsilon\mathfrak{r}$, for $\mathfrak{r}$ is minimal. Therefore $a=a\varepsilon e$ with some element e of $\mathfrak{r}$. Then $e=e\varepsilon e$, since from $a=a\varepsilon e=(a\varepsilon e)\varepsilon e$

we have $a\mathcal{E}(e-e\mathcal{E}e)=0$ which means $e-e\mathcal{E}e=0$, for a set $\{c \mid a\mathcal{E}c=0, c\in\mathfrak{r}\}$ is a right ideal contained in a minimal ideal $\mathfrak{r}$ and is $\{0\}$. Since $e\in\mathfrak{r}$, $eR\subset\mathfrak{r}$, i.e., $eR=\mathfrak{r}$. $\mathcal{E}M$ being a right ideal of R, $e\mathcal{E}M$ is a right ideal of M contained in $\mathfrak{r}$, and hence $e\mathcal{E}M=\mathfrak{r}$. Thus we get

Lemma 1. *If M is semi-simple and $\mathfrak{r}$ is a minimal right ideal, then $\mathfrak{r}=eR=e\mathcal{E}M$ with $e\in\mathfrak{r}$ and $\mathcal{E}\in\Gamma$, where $e\mathcal{E}e=e$.*

Now we use the idea of Peirce decomposition of the ring theory. Suppose that we have a right ideal $\mathfrak{r}=e\mathcal{E}M$ such that $e\mathcal{E}e=e$. Then

$$M = e\mathcal{E}M+M_1 \qquad \text{(direct)},$$

where $M_1=\{b \mid e\mathcal{E}b=0\}$, since any element a of M is written

$$a = e\mathcal{E}a+(a-e\mathcal{E}a),$$

and $e\mathcal{E}(a-e\mathcal{E}a)=0$. M_1 is clearly a right ideal of M. Now we can get a decomposition theorem.

Theorem 1. *If M is semi-simple and satisfies the minimum condition for right ideals, then*

$$M = e_1R+e_2R+\cdots+e_nR \qquad (\textit{direct}),$$

where e_iR are minimal right ideals and $e_iR=e_i\mathcal{E}_iM$, and $e_i\mathcal{E}_ie_i=e_i$ and $e_i\mathcal{E}_ie_j=0$ if $i\neq j$.

Proof. Suppose that we have

$$M = e_1\mathcal{E}_1M+\cdots+e_{k-1}\mathcal{E}_{k-1}M+M_{k-1} \qquad \text{(direct)}$$

such that $e_i\mathcal{E}_iM$ are minimal right ideals and

$$e_i\mathcal{E}_ie_j = \begin{cases} e_i & \text{if } i=j, \\ 0 & \text{if } i\neq j, \end{cases}$$

and that $e_i\mathcal{E}_ia=0$ if $a\in M_{k-1}$ for $i=1, 2, \cdots, k-1$. This is true for $k=2$ as above. Apply the above discussion on M_{k-1}, and we get

$$M_{k-1} = e_k\mathcal{E}'_kM+M_k \qquad \text{(direct)}$$

as in the above. Here $e_i\mathcal{E}_ie_k=0$ if $i<k$, but we can not say that $e_k\mathcal{E}'_ke_i=0$. So, we change $\mathcal{E}'_k$ suitably. Put

$$\mathcal{E}_k = \mathcal{E}'_k-\mathcal{E}'_k(e_1\mathcal{E}_1+\cdots+e_{k-1}\mathcal{E}_{k-1}).$$

Then we can see that $e_k\mathcal{E}_ke_k=e_k$ and $e_k\mathcal{E}_ke_i=0$. Thus we have a decomposition for k. Since M satisfies the minimum condition for right ideals, we can get the decomposition in Theorem 1.

Similarly we can get

Theorem 1′. *If M is semi-simple and satisfies the minimum condition for left ideals, then*

$$M = Ld_1 + Ld_2 + \cdots + Ld_m \qquad (direct),$$

where Ld_i are minimal left ideals and $Ld_i = M\delta_i d_i$, and $d_i\delta_i d_i = d_i$ and $d_j\delta_i d_i = 0$ if $i \neq j$.

5. Simple Γ-rings. Assume M is simple and satisfies the minimum condition for right and left ideals in this section. First we want to show that e_iR and e_jR are isomorphic as R-modules. M being simple, we can find an element γ in Γ such that $e_i\gamma e_j \neq 0$. Then $e_i\gamma \mathfrak{r}_j = \mathfrak{r}_i$ where $\mathfrak{r}_i$ and $\mathfrak{r}_j$ are e_iR and e_jR. By a correspondence:

$$(\mathfrak{r}_j \ni)x \longrightarrow e_i\gamma x(\in \mathfrak{r}_i)$$

we have a one-one mapping of $\mathfrak{r}_j$ onto $\mathfrak{r}_i$. If $x \neq 0$, $e_i\gamma x \neq 0$, because $\{c \mid e_i\gamma c = 0, c \in \mathfrak{r}_j\}$ is a right ideal contained in $\mathfrak{r}_j$ and is $\{0\}$ as $\mathfrak{r}_j$ is minimal. This mapping is "onto" because $\mathfrak{r}_i$ is minimal. Since $x(\beta c) = x\beta c$ corresponds to $e_i\gamma(x\beta c) = (e_i\gamma x)(\beta c)$, this mapping is an R-homomorphism, i.e., an R-isomorphism. Similarly $Ld_i \cong Ld_j$ (L-isomorphic). Next, we want to show that all L-endomorphisms of M are given by the right multiplication of R. Let ϕ be an L-endomorphism of M and put $\phi(d_i) = u_i$. Since $d_i = d_i\delta_i d_i$, $u_i = d_i\delta_i u_i$. Therefore, $u_i = d_i(\sum_j \delta_j d_j \delta_j u_j)$ where $\sum_j \delta_j d_j \delta_j u_j$ is an element of R. On the other hand, by the definition of the right operator ring, R is considered to be the set of all L-endomorphisms of M. Then the ring theory shows us that the latter ring is a matrix ring D_m over a division ring D, where D_m is $D_{m,m}$. Matrix units $E_{r,s}$ of D_m map d_r to d_s and d_t to 0 if $t \neq r$.

Now we can determine M with respect to R which is identified with D_m as above. Since minimal right ideals of D_m are $E_{r,r}D_m$, e_iD_m ($=e_iR$ in Theorem 2) $= e_iE_{r,r}D_m$ with some r. Then put $e_iE_{r,s} = e_{i,s}$. We get $e_{i,s}$ ($i = 1, 2, \cdots, n$; $s = 1, 2, \cdots, m$) such that

$$e_{i,s}E_{r,t} = \begin{cases} e_{i,t} & s = r, \\ 0 & s \neq r. \end{cases}$$

Thus we can say that $M = \sum_{i,s} e_{i,s}D$, i.e., $e_{i,s}$ are matrix units of $D_{n,m}$ and M is (isomorphic to) $D_{n,m}$ as a right D_m-module.

Next we must determine Γ. An element γ of Γ is considered to induce a mapping from M to R as in § 3, and Γ is considered to be a subset of the set of all R-homomorphisms of $M=D_{n,m}$ to $R=D_m$. On the other hand, D_m-homomorphisms of $D_{n,m}$ to D_m are induced by the left multiplications of elements of $D_{m,n}$. In fact, suppose ϕ is a D_m-homomorphism of $D_{n,m}$ to D_m such that

$$\phi(e_{i,s}) = \sum_{p,q} E_{p,q} T_{p,q}(i, s)$$

with $T_{p,q}(i, s)$ in D. Multiply $E_{s,s}$, and we can see $T_{p,q}(i, s)=0$ if $q \neq s$. Multiply $E_{s,t}$, and we can see $T_{p,s}(i, s)=T_{p,t}(i, t)$. Putting $T_{p,s}(i, s)=T_p(i)$, we have

$$\phi(e_{i,s}) = \sum_{p} E_{p,s} T_p(i) = (\sum_{p,j} e'_{p,j} T_p(j)) e_{i,s},$$

where $e'_{p,i}$ are matrix units of $D_{m,n}$ such that

$$e'_{p,j} e_{i,s} = \begin{cases} E_{p,s} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Hence ϕ is induced by the left multiplication of an element $A=\sum e'_{p,j} T_p(j)$ of $D_{m,n}$. Identifying γ which induces ϕ and A which corresponds to ϕ, we can say that $\Gamma \subset D_{m,n}$. What we want to show is that $\Gamma=D_{m,n}$. But Γ is a two sided D_m-D_n module and must be identical with $D_{m,n}$. Summarizing all the dicussions, we get the main theorem.

Theorem 2. *If M is a simple Γ-ring satisfying the minimum condition for left and right ideals, then M is $D_{n,m}$ and Γ is $D_{m,n}$. The product $a\gamma b$ is the usual matrix product of three elements a, γ and b of $D_{n,m}$, $D_{m,n}$ and $D_{n,m}$.*

6. Semi-simple Γ-rings. Let M be a semi-simple Γ-ring which satisfies the minimum condition for left and right ideals in this section. Arranging suitably, we can see that M is expressed as follows:

$$M = Ld_1^{(1)} + \cdots + Ld_{m(1)}^{(1)} + Ld_1^{(2)} + \cdots + Ld_{m(2)}^{(2)} + \cdots + Ld_1^{(q)} + \cdots + Ld_{m(q)}^{(q)},$$

where $d_i^{(j)}$ are some d_k of Theorem 1′. Moreover we take the order such that in the above $Ld_i^{(j)} \cong Ld_k^{(j)}$ (L-isomorphic) and $Ld_i^{(j)} \not\cong Ld_k^{(j')}$ if $j \neq j'$. Then, R is, as the right multiplication ring of the L-module M, equal to a direct ring sum $\sum_j D_{m(j)}^{(j)}$, where $D_{m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $m(j) \times m(j)$. Furthermore $D_{m(j)}^{(j)}$ operate on $Ld_i^{(j)}$ as usual and are zero on $Ld_i^{(j')}$ if $j \neq j'$. On the other hand, we have in Theorem 1

$$M = e_1R+e_2R+\cdots+e_nR .$$

e_iR being minimal, $e_iR=e_iD^{(j)}_{m(j)}$ with some j. Rearranging the order suitably, we have

$$M = e_1^{(1)}R+\cdots+e_{n(1)}^{(1)}R+e_1^{(2)}R+\cdots+e_{n(2)}^{(2)}R+\cdots+e_1^{(q)}R+\cdots+e_{n(q)}^{(q)}R ,$$

where $e_i^{(j)}R=e_i^{(j)}D^{(j)}_{m(i)}$ and $e_i^{(j)}$ are some e_k. Hence $n=\sum_j n(j)$. With the same discussion as in §5, we can say that

$$\begin{aligned} e_1^{(1)}R+\cdots+e_{n(1)}^{(1)}R &= D^{(1)}_{n(1),m(1)} , \\ e_1^{(2)}R+\cdots+e_{n(2)}^{(2)}R &= D^{(2)}_{n(2),m(2)} , \\ &\cdots\cdots \\ e_1^{(q)}R+\cdots+e_{n(1(}^{(q)}R &= D^{(q)}_{n(q),m(q)} , \end{aligned}$$

i.e.,

$$M = D^{(1)}_{n(1),m(1)}+\cdots+D^{(q)}_{n(q),m(q)} ,$$

where $D^{(j)}_{n(j),m(j)}$ are matrix rings over division rings $D^{(j)}$ of type $n(j)\times m(j)$. Naturally $D^{(j)}_{m(j)}$ operate on $D^{(j)}_{n(j),m(j)}$ as usual and are zero on $D^{(j')}_{n(j'),m(j')}$ if $j\neq j'$. Γ is then a set of R-homomorphisms of M to R and is contained in $\sum_j D^{(j)}_{m(j),n(j)}$. Here the product of elements of $D^{(j)}_{m(j),n(j)}$ and of $D^{(j')}_{n(j'),m(j')}$ is performed as usual if $j=j'$ and is 0 if $j\neq j'$. On the other hand, the condition of semi-simplicity means that for any non-zero element a of $D^{(j)}_{n(j),m(j)}$ there exists γ in Γ such that $a\gamma a\neq 0$. Now we want to show that each $D^{(j)}_{n(j),m(j)}$ is a simple Γ_j-ring. Let V and V' be left $D^{(j)}$-modules of dim $n(j)$ and of dim $m(j)$. $D^{(j)}_{n(j),m(j)}$ and $D^{(j)}_{m(j),n(j)}$ are considered to be the sets of all $D^{(j)}$-homomorphisms of V to V' and of V' to V. When we notice that $D^{(j')}_{m(j'),n(j')}$ induce zero mapping on V' if $j\neq j'$, we can say that elements of Γ induce mappings of V' to V. In this case we can show that $X\Gamma=V$ for any subspace X of dim 1 of V'. For, suppose that $X\Gamma\subsetneqq V$. Then we can find an element a in $D^{(j)}_{n(j),m(j)}$ such that $Va=X$ and $(X\Gamma)a=0$. Then $a\gamma a=0$ for every γ in Γ, which is a contradiction. Now this fact implies the existance of γ such that $a\gamma b\neq 0$ for any non-zero a and b, for we can take a subspace of Va as X and take γ such that $(X\gamma)b\neq 0$. Thus we can conclude that $\Gamma=\sum_j D^{(j)}_{m(j),n(j)}$. Now put $\Gamma_j=D^{(j)}_{m(j),n(j)}$.

Theorem 3. *If M is a semi-simple Γ-ring satisfying the minimum condition for left and right ideals, then M is a direct sum of simple Γ_i-rings where $\Gamma=\Gamma_1+\cdots+\Gamma_q$ (direct):*

$$M = M_1 + M_2 + \cdots + M_q \qquad (direct),$$

where M_i *are simple* Γ_i*-rings and* $M_i \Gamma M_j = 0$ *if* $i \neq j$, *and* $M_i \Gamma_j M_i = 0$ *if* $i \neq j$.

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