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SOME CRITERIA FOR HEREDITARITY OF CROSSED PRODUCTS

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Let $\mathfrak{O}$ be the integral closure of a discrete rank one valuation ring R with maximal ideal $\mathfrak{p}$ in a finite Galois extension L of the quotient field of R. Auslander, Goldman and Rim have proved in [1] and [2] that a crossed product Λ over $\mathfrak{O}$ with trivial factor sets is a maximal order in K_n if and only if a prime ideal $\mathfrak{P}$ in $\mathfrak{O}$ over $\mathfrak{p}$ is unramified and Λ is a hereditary if and only if $\mathfrak{P}$ is tamely ramified. Recently Williamson has generalized those results in [11] to a crossed product Λ with any factor sets in $U(\mathfrak{O})$, where $U(\mathfrak{O})$ means the set of units in $\mathfrak{O}$, namely if $\mathfrak{P}$ is tamely remified, then Λ is hereditary and the rank[1] of Λ is determined.

In this paper, we shall modify the Williamson's method by making use of a property of crossed product over a ring.

Let G, S and H be the Golois group of L, decomposition group of $\mathfrak{P}$ and inertia group of $\mathfrak{P}$, respectively. We denote a crossed propuct Λ with factor sets $\{a_{\sigma,\tau}\}$ in $U(\mathfrak{O})$ by $(a_{\sigma,\tau}, G, \mathfrak{O})$. Then we shall prove in Theorem 1 that Λ is a hereditary order if and only if so is $(a_{\sigma,\tau}, H, \mathfrak{O}_{\mathfrak{P}_H})$ where $\mathfrak{P}_H=\mathfrak{P}\cap\mathfrak{O}_H$, and $\mathfrak{O}_H$ is the integral closure of R in the inertia field $\mathfrak{L}_H$. Using this fact and the structure of hereditary orders [7], [8] we obtain the above results in [1], [2] and [11].

Furthermore, we shall show that Λ is hereditary if and only if $\mathfrak{P}$ is tamely ramified under an assumptions that $R/\mathfrak{p}$ is a perfect field.

Finally, we give a complete description of hereditary orders in a generalized quaternions over rationals in Theorem 3.

1. Reduction theorem

In this paper we always assume that R is a discrete rank one valuation ring with maximal ideal $\mathfrak{p}$ and p in the characteristic of $R/\mathfrak{p}$. Let L be a finite Golois extension of the quotient field of R with Galois

1) The rank means the number of maximal two-sided ideals in Λ.

group G, and $\mathfrak{O}$ the integral closure of R in L. For a prime ideal $\mathfrak{P}$ in $\mathfrak{O}$ over $\mathfrak{p}$ we denote the decomposition group and the inertia group of $\mathfrak{P}$ by S and H and their fields and the integral closure by L_S, L_H and $\mathfrak{O}_S$, $\mathfrak{O}_H$ and so on.

We note that $\mathfrak{O}$ is a semi-local Dedekind domain and hence, $\mathfrak{O}$ is a principal ideal domain. Let $\{\mathfrak{P}_i\}_{i=1}^g$ be the set of prime ideals in $\mathfrak{O}$ and S_i and H_i be decomposition group and inertia group of $\mathfrak{P}_i$. Let $\mathfrak{p}\mathfrak{O}=\Pi\mathfrak{P}_i^e=P^e$, where $P=\Pi\mathfrak{P}_i$. Since $(\mathfrak{P}_i, \mathfrak{P}_j)=\mathfrak{O}$ for $i\neq j$, $\mathfrak{O}/P^n=\mathfrak{O}/\mathfrak{P}_1^n\oplus\cdots\oplus\mathfrak{O}/\mathfrak{P}_g^n$. We note that $(\mathfrak{O}/\mathfrak{P}_i^n)^\sigma=\mathfrak{O}/(\mathfrak{P}_i^\sigma)^n$ for $\sigma\in G$. Then $\mathfrak{O}_{H_i}/\mathfrak{P}_{H_i}$ is the separable closure of $R/\mathfrak{p}$ in $\mathfrak{O}/\mathfrak{P}_i$ and $\mathfrak{O}_{H_i}/\mathfrak{P}_{H_i}$ is a Galois extension of $R/\mathfrak{p}$ with Galois group S_i/H_i, (see [10], p. 290).

Let Λ be a crossed product over $\mathfrak{O}$ with factor sets $\{a_{\sigma,\tau}\}$ in $U(\mathfrak{O})$: $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$. Since $P^\sigma=P$ for all $\sigma\in G$, $P^n\Lambda=\Lambda P^n$ is a two-sided ideal in Λ. Let $\bar{\Lambda}(n)=\Lambda/P^n\Lambda=(\bar{a}_{\sigma,\tau}, G, \mathfrak{O}/P^n)=\Sigma\oplus(\bar{a}_{\sigma,\tau}, G, \mathfrak{O}/\mathfrak{P}_i^n)$ as a module. We put $\bar{\Lambda}(S_i, n)=(\bar{a}_{\sigma,\tau}, S_i, \mathfrak{O}/\mathfrak{P}_i^n)$. Since $\bar{u}_\sigma^{-1}(\bar{u}_\tau\mathfrak{O}/\mathfrak{P}_i^n)\bar{u}_\sigma=\bar{u}_{\sigma^{-1}\tau\sigma}(\mathfrak{O}/\mathfrak{P}_i^\sigma)^n$, $\bar{u}_\sigma^{-1}\Lambda(S_i, n)\bar{u}_\sigma=\bar{\Lambda}(S_i^\sigma, n)$, where $S_i^\sigma=\sigma^{-1}S_i\sigma$. Thus we have

$$\bar{\Lambda}(S_i, n)\bar{u}_\sigma=\bar{u}_\sigma\Lambda(S_i^\sigma, n)\,. \tag{1}$$

Let $G=\sigma_{i1}S_i+\sigma_{i2}S_i+\cdots+\sigma_{ig}S_i=S_i\sigma_{i1}+\cdots+S_i\sigma_{ig}$, $\sigma_{i1}S_i=S_i$, since G is a finite group. Then

$$\begin{aligned}\bar{\Lambda}(n) = {} & \bar{\Lambda}(S, n)+\bar{u}_{\sigma_{11}}\bar{\Lambda}(S, n)+\cdots+\bar{u}_{\sigma 1g}\bar{\Lambda}(S, n)\\ & +\bar{\Lambda}(S_2, n)+\bar{a}_{\sigma 22}\Lambda(S_2, n)\cdots+\bar{a}_{\sigma 2g}\Lambda(S_2, n)\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ & +\bar{\Lambda}(S_g, n)+\bar{a}_{\sigma g2}\Lambda(S_g, n)\cdots+\bar{a}_{\sigma g,g}\bar{\Lambda}(S_g, n)\,,\end{aligned} \tag{2}$$

where $S=S_1$.

Let p_{ij} be projections of $\bar{\Lambda}(n)$ to $\bar{u}_{\sigma_{ij}}\bar{\Lambda}(S_i, n)$. For a two-sided ideal $\mathfrak{A}$ in $\bar{\Lambda}(n)$ we have $\mathfrak{A}\supseteq\Sigma p_{ij}(\mathfrak{A})$. Since $\bar{u}_{\sigma_{ij}}$ is unit, $p_{ij}(\mathfrak{A})=u_{\sigma 1i}P_{i1}(\mathfrak{A})$ for all j. Let $\bar{e}$ be the unit element in $\bar{\Lambda}(S, n)$. Then $\bar{\Lambda}(S_i, n)\bar{e}=0$ for $i\neq 1$ and $\bar{e}\bar{u}_{\sigma_{1j}}\Lambda(S, n)\subseteq\bar{u}_{\sigma_{1j}}\bar{\Lambda}(S^{\sigma_{1j}}, n)\bar{\Lambda}(S, n)=0$ for $j\neq 1$. Hence, $\bar{e}\mathfrak{A}\bar{e}=p_{11}(\mathfrak{A})$. Furthermore, since $S_i=S^{\sigma_{1j}}$, $p_{i1}(\mathfrak{A})=\bar{u}_{\sigma_i}^{-1}p_{11}(\mathfrak{A})\bar{u}_{\sigma_{1i}}=p_{11}(\mathfrak{A})^{\sigma_{1i}}$. Therefore,

$$\mathfrak{A}=\sum_{i,j}u_{\sigma_{ij}}\mathfrak{A}_0^{\sigma_{1i}} \tag{3}$$

for a two-sided ideal of $\mathfrak{A}_0$ in $\bar{\Lambda}(S, n)$. Conversely, the above ideal is a two-sided ideal in $\bar{\Lambda}(n)$ for a two-sided ideal $\mathfrak{A}_0$ in $\bar{\Lambda}(S, n)$.

Thus, we have

Lemma 1. *Let $\bar{\Lambda}(n)$ and $\bar{\Lambda}(S, n)$ be as above. Then we have a one-to-one correspondence betueen two-sided ideals of $\bar{\Lambda}(n)$ and $\bar{\Lambda}(S, n)$ as above.*

We note that the above correspondence preserves product of ideals.

Next we shall consider $\Lambda_S=(a_{\sigma,\tau}, S, \mathfrak{O})$ ($\subseteq\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$), where S is the decomposition group of $\mathfrak{P}$. Since $\mathfrak{O}_S$ is contained in the center of Λ_S, we may regard Λ_S as an order over $\mathfrak{O}_S$. Let $\mathfrak{P}_S$ be the prime ideal in $\mathfrak{O}_S$ over $\mathfrak{p}$. Then $\mathfrak{O}_{\mathfrak{P}_S}/\mathfrak{P}^n_{\mathfrak{P}_S}=\mathfrak{O}/\mathfrak{P}^n$. If we set $\Gamma=(a_{\sigma,\tau}, S, \mathfrak{O}_{\mathfrak{P}_S})$ $=(\Lambda_S)_{\mathfrak{P}_S}$, $\bar{\Gamma}(n)=\Gamma/\mathfrak{P}^n\Gamma\approx\bar{\Lambda}(S, n)$. In Γ we may regard $K=L_S$ and $\mathfrak{O}=\mathfrak{O}_{\mathfrak{P}_S}$. Let H be the inertia group of a unique prime ideal $\mathfrak{P}$ in $\mathfrak{O}$. Then H is a normal subgroup of S, (see [10], p. 290) and we have $S=H+\sigma_2H+\cdots+\sigma_fH$. Let $\Gamma_H=(a_{\sigma,\tau}, H, \mathfrak{O})$, then $\Gamma\mathfrak{P}^n\cap\Gamma_H=\Gamma_H\mathfrak{P}^n$. Hence $\bar{\Gamma}=\bar{\Gamma}(n)=\Gamma/\mathfrak{P}^n\Gamma\supseteq\bar{\Gamma}_H(n)=\bar{\Gamma}_H$. Furthermore,

$$\bar{\Gamma} = \bar{\Gamma}_H+\bar{u}_{\sigma_2}\bar{\Gamma}_H+ \cdots + \cdots +\bar{u}_{\sigma_f}\bar{\Gamma}_H\,.$$

By a similar argument as above, we have $\bar{u}_\sigma^{-1}\bar{\Gamma}_H\bar{u}_\sigma=\bar{\Gamma}_H$. We denote this automorphism by f_σ. Then the restriction of f_σ on $\mathfrak{O}/\mathfrak{P}^n$ conincides with σ. Let $\mathfrak{R}_H$ be the radical of $\bar{\Gamma}_H$. Then $\mathfrak{R}_H\supseteq\mathfrak{P}\bar{\Gamma}_H$. We put $\mathfrak{R}=\mathfrak{R}_H+u_{\sigma_2}\mathfrak{R}_H+\cdots+u_{\sigma_f}\mathfrak{R}_H$, then $\mathfrak{R}$ is a two-sided ideal of $\bar{\Gamma}$ and $\mathfrak{R}^m=\mathfrak{R}^m_H+\cdots+u_{\sigma_f}\mathfrak{R}^m_H\subseteq\mathfrak{P}^n\Gamma$ for some m. $\bar{\Gamma}/\mathfrak{R}=\bar{\Gamma}_H/\mathfrak{R}_H+\tilde{u}_{\sigma_2}\bar{\Gamma}_H/\mathfrak{R}_H+\cdots+\tilde{u}_{\sigma_f}\bar{\Gamma}_H/\mathfrak{R}_H$ and $\bar{\Gamma}_H/\mathfrak{R}_H\supseteq\mathfrak{O}/\mathfrak{P}$. Now we consider a crossed product of $\bar{\Gamma}_H/\mathfrak{R}_H$ with automorphisms $\{f_\sigma\}$ and factor sets $\{\tilde{a}_{\sigma,\tau}\}$. We define a two-sided $\bar{\Gamma}_H/\mathfrak{R}_H$-module $\bar{\Gamma}_H/\mathfrak{R}_H$ as follows: for $\tilde{x}, \tilde{y}\in\bar{\Gamma}_H/\mathfrak{R}_H$ $\tilde{x}*\tilde{y}=\widetilde{x^{f_\sigma}y}$ and $\tilde{y}*\tilde{x}=\widetilde{yx}$, and denote it by $(\sigma, \bar{\Gamma}_H/\mathfrak{R}_H)$. Since $\bar{\Gamma}_H/\mathfrak{R}_H$ is semi-simple, $(\sigma, \bar{\Gamma}_H/\mathfrak{R}_H)$ is completely reducible. Furthermore, $\{\sigma\}$ is the complete set of automorphisms of $\mathfrak{O}/\mathfrak{P}$ (see [10], p. 290). Hence $\{f_\sigma\}$ is a complete outer-Galois, namely for any two-sided $\bar{\Gamma}_H/\mathfrak{R}_H$-module $A\supseteq B$ in $(\sigma, \bar{\Gamma}_H/\mathfrak{R}_H)$ A/B is not isomorphic to some of those forms in $(1, \bar{\Gamma}_H/\mathfrak{R}_H)$ if $\sigma\neq1$. Therefore, for any two-sided ideal $\mathfrak{A}$ in $\bar{\Gamma}/\mathfrak{R}$ we have by [3], Theorem 48.2

$$\mathfrak{A} = \Sigma\tilde{u}_{\sigma_i}\mathfrak{A}_0\,, \tag{3}$$

where $\mathfrak{A}_0$ is a twe-sided ideal in $\bar{\Gamma}_H/\mathfrak{R}_H$ and $\mathfrak{A}_0^{f_\sigma}=\mathfrak{A}_0$ for all f_σ, and it is a one-to-one correspondence. Hence, $\bar{\Gamma}/\mathfrak{R}$ is semi-simple, and $\mathfrak{R}$ is the radical of $\bar{\Gamma}$. From the definition of f_σ we have

$$(\tilde{u}_\tau\tilde{\lambda})^{f_\sigma} = \tilde{u}_{\sigma^{-1}\tau\sigma}\tilde{\lambda}^\sigma a_{\sigma,\tau}/a_{\sigma,\sigma^{-1}\tau\sigma} \tag{4}$$

for $\sigma\in S$, $\tau\in H$, $\tilde{\lambda}\in\mathfrak{O}/\mathfrak{P}$, and $\tilde{u}_\tau\in\bar{\Gamma}_H/\mathfrak{R}_H$.
Furthermore, let $\bar{\Gamma}_H/\mathfrak{R}_H=\mathfrak{A}_1\oplus\cdots\oplus\mathfrak{A}_k$, where the $\mathfrak{A}_i$'s are simple components of $\bar{\Gamma}_H/\mathfrak{R}_H$. If we classify those ideals $\mathfrak{A}, \mathfrak{B}$ by a relation

$$\mathfrak{A}\sim\mathfrak{B} \text{ if and only if } \mathfrak{A}^{f_\sigma}=\mathfrak{B} \text{ for some } f_\sigma, \tag{5}$$

then the number of maximal two-sided ideals in $\bar{\Gamma}/\mathfrak{R}$ is equal to this class number.

Thus, we have

Lemma 2. *Let L be a Galois extension of the field K with Galois group G such that $S=G$, $\Gamma=(a_{\sigma,\tau}, S, \mathfrak{O})$, and $\Gamma_H=(a_{\sigma,\tau}, H, \mathfrak{O})$. If we denote the radicals of Γ and Γ_H by $\mathfrak{N}$, $\mathfrak{N}_H$, then, $\mathfrak{N}^t \equiv \Sigma \tilde{u}_\sigma \mathfrak{N}_H^t (mod\, \mathfrak{P}^n\Gamma)$ for some $t<n$, and there exists a one-to-one correspondence between two-sided ideals in $\Gamma/\mathfrak{N}$ and $\Gamma_H/\mathfrak{N}_H$ which is given by* (3) *and* (4).

Lemma 3. *Let Ω be an order over R in a central simple K-algebra Σ and $\mathfrak{N}$ the radical of Ω. Then Ω is hereditary if and only if $\mathfrak{N}^t=\alpha\Omega=\Omega\alpha$ for some $t>0$ and $\alpha\in\Sigma$.*

Proof. If $\mathfrak{N}^t=\alpha\Omega$, then the left (right) order of $\mathfrak{N}=\Omega$, and $\mathfrak{N}\mathfrak{N}^{t-1}\alpha^{-1}=\Omega$. Hence $\mathfrak{N}$ is inversible in Ω, which implies that Ω is hereditary by [7], Lemma 3.6. The converse is clear by [7], Theorem 6.1.

Theorem 1. *Let R be a discrete rank one valuation ring and K its quotient field, and L a Galois extension of K with group G. Let S and H be decomposition group and inertia group of a prime ideal $\mathfrak{P}$ in the integral closure $\mathfrak{O}$ of R in L. Let $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$, $\Lambda_S=(a_{\sigma,\tau}, S, \mathfrak{O}_{\mathfrak{P}_S})$, and $\Lambda_H=(a_{\sigma,\tau}, H, \mathfrak{O}_{\mathfrak{P}_H})$. Then the following statement is equivalent*

1) *Λ is hereditary,*
2) *Λ_S is hereditary,*
3) *Λ_H is hereditary.*

In this case the rank of Λ is equal to that of Λ_S and is equal or less than that of Λ_H.

Proof. 1)→2). Let $\mathfrak{N}$, $\mathfrak{N}_S$ be the radicals of Λ and Λ_S and P be the product of the prime ideals as in the beginning. Then $\mathfrak{N}^t=P\Lambda$. For $n>t$ we have $\mathfrak{N}_S^t\equiv\mathfrak{P}\Lambda_S \pmod{\mathfrak{P}^n\Lambda_S}$ by Lemma 1 and remark after that. Hence $\mathfrak{N}_S^t=\mathfrak{P}\Lambda_S$ since $\mathfrak{N}_S^t\supseteq\mathfrak{P}^n\Lambda_S$. Therefore, Λ_S is hereditary by Lemma 3. The remaining parts are proved similarly by using Lemmas 1, 2, and 3, and a remark before Lemma 2.

If $(|H|, p)=1$, then $\Lambda/\mathfrak{P}\Lambda$ is separable by [11], Theorem 1, (see Lemma 4 below) and hence Λ is herediatry, where $|H|$ means the order of group H. Therefore, we have

Corollary 1. ([11]). *If $\mathfrak{P}$ is tamely ramefied, i.e. $(|H|, p)=1$, then $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$ is hereditary of the same rank as that of $\Lambda_S=(a_{\sigma,\tau}, S, \mathfrak{O}_{\mathfrak{p}_S})$ and its rank is equal to the class number of ideals defined by* (5).

Corollary 2. ([1, 2]). *If $\{a_{\sigma,\tau}\}=\{1\}$, then Λ is hereditary if and only if a prime ideal $\mathfrak{P}$ in $\mathfrak{O}$ over $\mathfrak{p}$ is tamely ramified. In this case the rank of Λ is equal to the ramification index of $\mathfrak{P}$.*

Proof. $\{a_{\sigma,\tau}\}=\{1\}$, then $\Sigma=(a_{\sigma,\tau}, G, L)=K_n$. We assume that Λ is hereditary, then Λ_H is also hereditary by Theorem 1. $\Lambda_H L=(L_H)_h$, where $h=|H|$, $(\mathfrak{O}_H)_h$ is a maximal order in $\Lambda_H L$. Furtheremore, the composition length of left ideals of $(\mathfrak{O}_H)_h$ modulo the radical $(\mathfrak{P}_H)_h$ is equal to h, which is invariant for hereditary orders in $\Lambda_H L$ by [8], Corollary to Lemma 2.5. On the other hand $[\Lambda_H/\mathfrak{P}\Lambda_H: \mathfrak{O}/\mathfrak{P}]=h$. Hence, $\mathfrak{P}\Lambda_H$ is the radical and $\Lambda_H/\mathfrak{P}\Lambda_H$ is semi-simple which is a group ring of H over $\mathfrak{O}/\mathfrak{P}$. Therefore, $(|H|, p)=1$. In this case $\mathfrak{A}=(\sum_{\sigma\in H}u_\sigma)\cdot\mathfrak{O}/\mathfrak{P}$ is a two-sided ideal in $\Lambda_H/\mathfrak{P}\Lambda_H$ which is invariant under automorphisms f_σ of (4). $\mathfrak{A}$ is a minimal two-sided ideal in $\Lambda_H/\mathfrak{P}\Lambda_H$ which is invariant under f_σ. Hence, $\Lambda_S/\mathfrak{M}\approx\sum_{\{\sigma H\}}u_{\sigma H}\mathfrak{A}$ for some maximal ideal $\mathfrak{M}$ in Λ_S. Furtheremore, since Λ_S is principal[2], $\Lambda_S/\mathfrak{M}\approx\Lambda_S/\mathfrak{M}'$ for any maximal ideal $\mathfrak{M}'$ in Λ_S by [8], Theorem 4.1. Therefore, there exists h two-sided ideals in $\Lambda_H/\mathfrak{P}\Lambda_H$ which is invariant under f_σ, since $[\mathfrak{A}:\mathfrak{O}/\mathfrak{P}]=1$.

By the same argument as in the proof of Theorem 1 we have

Proposition 1. *We assume that $R/\mathfrak{p}$ is a perfect field, and we use the same notations as in Theorem* 1. *Let V be the second ramification group*[2] *and $\Lambda_V=(a_{\sigma,\tau}, V, \mathfrak{O}_{\mathfrak{P}_V})$. Then Λ is hereditary if and only if so is Λ_V.*

Proof. By virtue of Theorem 1 we may assume $G=H$. Let $G=V+\sigma V+\cdots+\rho V$. Then $\Lambda=\Lambda_V+u_\sigma\Lambda_V+\cdots+u_\rho\Lambda_V$. Since V is a normal subgroup of G by [10], p. 295, an inner-automorphism by u_σ in Λ reduces an automorphism f_σ in Λ_V. Let $\mathfrak{N}_V$ be the radical of Λ_V and $\mathfrak{N}=\mathfrak{N}_V+u_\sigma\mathfrak{N}_V+\cdots+u_\rho\mathfrak{N}_V$. We shall show that $\mathfrak{N}$ is the radical of Λ. By assumption that $R/\mathfrak{p}$ is perfect, $\bar{\Lambda}_V=\Lambda_V/\mathfrak{N}_V$ is separable. Therefore, there exist x_i, y_i in $\bar{\Lambda}_V$ such that $\sum_i x_i y_i=1$ and $\sum_i \lambda x_i\otimes y_i^*=\sum_i x_i\otimes(y_i\lambda)^*$, where $y\to y^*$ gives an anti-isomorphism of Λ to Λ^*. Furthermore, we note that $|G/V|=t$ is relative prime to p by [10], p. 296. Let $\theta=1/t(\sum_{\tau,i}\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\tau x_i\otimes(\bar{u}_{\tau^{-1}}y_i^{f_{\tau^{-1}}})^*)=1/t(\sum\bar{a}^{-1}_{\tau,\tau^{-1}}\sum_i\bar{u}_\tau x_i\otimes(y_i^{f_{\tau^{-1}}})^*\bar{u}^*_{\tau^{-1}})$. Then $1/t(\sum_\tau\bar{a}^{-1}_{\tau,\tau^{-1}}\sum\bar{u}_\tau x_i\bar{u}_{\tau^{-1}}y_i^{f_{\tau^{-1}}})=1$. We show that $\{(\eta\otimes 1^*)-(1\otimes\eta^*)\}\theta=0$ for any $\eta\in\bar{\Lambda}$. Let γ be in $\bar{\Lambda}_V$. $(\gamma\otimes 1^*)\theta=1/t(\sum\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\tau\gamma^{f_\tau}x_i\otimes(\bar{u}_{\tau^{-1}}y_i^{f_{\tau^{-1}}})^*)$ and $(1\otimes\gamma^*)\theta=1/t(\sum_{i,\tau}\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\tau x_i\otimes(\bar{u}_{\tau^{-1}}y_i^{f_{\tau^{-1}}}\gamma)^*)=1/t(\sum\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\tau x_i\otimes(y_i^{f_{\tau^{-1}}}\gamma)^*\bar{u}^*_{\tau^{-1}})$. We can naturally define $\{f_\sigma\}$ on $\bar{\Lambda}_V\otimes\bar{\Lambda}^*_V$ by setting $(\gamma\otimes\gamma'^*)^{f_\sigma}=(\gamma\otimes\gamma'^{f_\sigma *})$. Since $\sum\gamma^{f_\tau}x_i\otimes y_i^*=\sum x_i\otimes(y_i\gamma^{f_\tau})^*$, we obtain $\sum\gamma^{f_\tau}x_i\otimes(y_i^{f_{\tau^{-1}}})^*=\sum x_i\otimes(y_i^{f_{\tau^{-1}}}\gamma)^*$. Therefore, $\{(\gamma\otimes 1^*)-(1\otimes\gamma^*)\}\theta=0$. $(\bar{u}_\sigma\otimes 1)\theta=1/t(\sum\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\sigma\bar{u}_\tau x_i\otimes u_{\tau^{-1}}y_i^{f_{\tau^{-1}}})^*$ $=1/t(\sum\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{a}_{\sigma,\tau}\bar{u}_{\sigma\tau}x_i\otimes(\bar{u}_{\tau^{-1}}y_i^{f_{\tau^{-1}}})^*)$. $(1\otimes\bar{u}^*_\sigma)6=1/t(\sum\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\tau x_i\otimes$

2) See the definition in [10].

$(\bar{u}_{\tau^{-1}}y_i^{f_{\tau^{-1}}}\bar{u}_\sigma)^*)=1/t(\sum \bar{a}^{-1}_{\tau,\tau^{-1}}\bar{u}_\tau x_i\otimes(\bar{a}_{\tau^{-1},\sigma}(y_i^{f_{\tau^{-1}\sigma}})^*u^*_{\tau^{-1}\sigma})$. However, we obtain $\bar{a}^{-1}_{\tau,\tau^{-1}}\bar{a}_{\sigma,\tau}=\bar{a}^{-1}_{\sigma\tau,(\sigma\tau)^{-1}}\bar{a}_{\tau^{-1},\sigma}$ by the relation of $\bar{a}_{\sigma,\tau}$. Hence $\{(\bar{u}_\sigma\otimes 1)^*-(1\otimes\bar{u}^*_\sigma)\}\theta=0$. Therefore, $\{(\bar{u}_\sigma\gamma\otimes 1^*)-(1\otimes(\bar{u}_\sigma\gamma)^*)\}\theta=(\bar{u}_\sigma\otimes 1^*)(\gamma\otimes 1-1\otimes\gamma^*)\theta+(1\otimes\gamma^*)(\bar{u}_\sigma\otimes 1-1\otimes\bar{u}^*_\sigma)\theta=0$. Thus we have proved that $\mathfrak{N}$ is the radical of Λ. We can prove the proposition similarly to Theorem 1 by Lemma 3.

2. Tamely ramification

In this section we always assume that $R/\mathfrak{p}$ is a perfect field.

Theorem 2. *Let L be a Galois extension of K with Golois group G, and $\Lambda=(a_{\sigma\tau}, G, \mathfrak{O})$ a crossed product with a factor set $\{a_{\sigma,\tau}\}$ in $U(\mathfrak{O})$. We assume $R/\mathfrak{p}$ is a perfect field. Then Λ is hereditary if and only if every prime ideal $\mathfrak{P}$ in $\mathfrak{O}$ over $\mathfrak{p}$ is tamely ramified, where $U(\mathfrak{O})$ is the set of unit elements in $\mathfrak{O}$.*

Proof. If $\mathfrak{P}$ is tamely ramified, then Λ is hereditary by Corollary 1. We assume that Λ is hereditary. Then by virtue of Proposition 1 we may assume that G is equal to the second ramification group V. Since the elements of G operate trivially on $\mathfrak{O}/\mathfrak{P}$, $\bar{\Lambda}=\Lambda/\mathfrak{P}\Lambda=\bar{\mathfrak{O}}+\bar{u}_\sigma\bar{\mathfrak{O}}+\cdots+\bar{u}_\tau\bar{\mathfrak{O}}$ is a generalized group ring. Furthermore, from a relation on a factor set we have $a^{|G|}_{\sigma,\tau}=A'_\sigma A'_\tau/A'_{\sigma\tau}$, where $A'=\prod_{\rho\in G}\bar{a}_{\rho,\sigma}$. Since $R/\mathfrak{p}=\mathfrak{O}/\mathfrak{P}$ is perfect and G is a p-group by [10], p. 296, we have $\bar{a}_{\sigma,\tau}=A_\sigma A_\tau/A_{\sigma\tau}$, $A_\sigma\in\bar{\mathfrak{O}}$. Therefore, $\bar{\Lambda}$ is a group ring of G over $\bar{\mathfrak{O}}$. As well known (see [5], p. 435), the radical $\bar{\mathfrak{N}}$ of $\bar{\Lambda}$ is equal to $\sum(1-\bar{u}_\sigma)\bar{\mathfrak{O}}$ and $\bar{\Lambda}/\bar{\mathfrak{N}}=\bar{\mathfrak{O}}$. Hence Λ is a unique maximal order by [2], Theorem 3.11. Let σ be an element in G. $(u_\sigma)^i=u_{\sigma^i}C_{\sigma^i}$; $C_{\sigma^i}\in U(\mathfrak{O})$. Hence, if we replace a basis $\{u_\rho\}$ by $\{u'_\rho\}$; $u'_{\sigma^i}=(u_\sigma)^i$, and $u'_\tau=u_\tau$ if $\tau\notin(\sigma)$, we may assume $a_{\sigma^i,\sigma^j}=1$ if $i+j<|\sigma|=n$ and $a_{\sigma^i,\sigma^j}=a$ if $i+j\geqq n$, where a is a unit element in $\mathfrak{O}$. It is clear that a is an element of the (σ)-fixed subfield $L_{(\sigma)}$ of L. Since $\bar{\mathfrak{N}}=\sum(1-\bar{u}_\sigma)\bar{\mathfrak{O}}$, $(1-u_\sigma)\in\mathfrak{N}$. $(1-u_\sigma)(1+u_\sigma+u_{\sigma^2}+\cdots+u_{\sigma^{-1}})=1-a\in\mathfrak{N}$. Hence $1-a\in\mathfrak{N}\cap\bar{\mathfrak{O}}_{(\sigma)}=\mathfrak{P}_{(\sigma)}$. Furthermore, every one-sided ideal in Λ is a two-sided ideal and a power of $\mathfrak{N}$ by [2], Theorem 3.11. Since $(1-u_\sigma)\Lambda\not\subseteq\mathfrak{P}\Lambda$, $(1-u_\sigma)\Lambda\supseteqq\mathfrak{P}\Lambda$. Put $\mathfrak{P}=(\pi)$. Then $\pi=(1-u_\sigma)\sum u_\tau x_\tau=\sum u_\rho(x_\rho-x_{\sigma^{-1}\rho}a_{\sigma,\sigma^{-1}\rho})$. Hence, $x_1-x_{\sigma^{-1}}a=\pi$, $x_1=x_\sigma=x_{\sigma^2}=\cdots=x_{\sigma^{-1}}$. Therefore, $x_1(1-a)=\pi$. However, $(1-a)\equiv 0$ $(mod\,\mathfrak{P}_{(\sigma)})$. Therefore, $\mathfrak{P}$ is unramified over $\mathfrak{P}_{(\sigma)}$ which implies $|\sigma|=1$. Hence $V=(1)$, which has proved the theorem.

Corollary 3. *Let $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$. Then Λ is hereditary if and only if $\Lambda/P\Lambda$ is sime-simple, where $P=\Pi\mathfrak{P}_i$.*

Proof. It is clear from Theorems 1 and 2 and the proof of Proposition 1.

Proposition 2. *Let* $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$ *and* t *the ramification index of a maximal order* Ω *in* $\Lambda K: (N(\Omega)^t=\mathfrak{p}\Omega)$. *We assume that* $R/\mathfrak{p}$ *is perfect. If* Λ *is a hereditary order of rank* r, *then the ramification index of* $\mathfrak{P}$ *is equal to* rt, *where* $N(\Omega)$ *means the radical of* Ω.

Proof. If Λ is hereditary, then $N(\Lambda)=P\Lambda$ by Corollary 3. Hence, $N(\Lambda)^e=\mathfrak{p}\Lambda$. Therefore, $e=rt$ by [7], Theorem 6.1.

Corollary 4. *Let* $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$ *be a hereditary order. Then* $\Lambda\approx\Gamma=(b_{\sigma,\tau}, G, \mathfrak{O})$ *if and only if* $\Lambda K\approx\Gamma K$.

Proof. Since Λ is hereditary, $\mathfrak{P}$ is tamely ramified. If $\Lambda K\approx\Gamma K$, then $\Lambda\approx\Gamma$ by Proposition 2 and [8], Corollary 4.3.

Corollary 5. *Let* $\Lambda=(a_{\sigma,\tau}, G, \mathfrak{O})$ *and* e *the ramification index of* $\mathfrak{P}$ *over* $\mathfrak{p}$. *Then* Λ *is a hereditary order of rank* e *if and only if* $(e, p)=1$ *and a maximal order in* ΛK *is unramified.*

Corollary 6. *We assume* $\Lambda=(a_{\sigma,\tau}, H, \mathfrak{O})$ *is hereditary and a maximal order in* ΛK *is unramified. Then* Λ *is a minimal hereditary order*[3].

Proof. Let Ω be a maximal order in ΛK. Put $\Omega/N(\Omega)=\Delta_m$ and $[\Delta : R/\mathfrak{p}]=s$, where Δ is a division ring. Since $N(\Omega)^i/N(\Omega)^{i+1}\approx\Omega/N(\Omega)$, we obtain $m^2s=[\Omega/\mathfrak{p}\Omega : R/\mathfrak{p}]=[\Lambda/\mathfrak{p}\Lambda : R/\mathfrak{p}]=|H|^2$. The ranker of $\Lambda\leqslant m$ by [8], Corollary to Lemma 2.5. Hence $r=|H|=m\sqrt{s}\geqslant r\sqrt{s}$ by Proposition 2. Therefore, $s=1$ and $m=|H|=r$. Hence, Λ is minimal by [8], Corollary to Lemma 2.5.

REMARK 1. If R is complete and $R/\mathfrak{p}$ is finite, then we obtain, as well known (cf. [6]), that the ramification index of a maximal order in $\Sigma=(a_{\sigma,\tau}, G, L)$ is equal to the index of Σ.

Finally we shall generalize Corollary 2.

The following lemma is well known. However we shall give a proof for a completeness, (cf. [11], Theorem 1).

Lemma 4. *Let* K *be a commutative ring and* G *a finite group which operates on* K *trivially.* $\{a_{\sigma,\tau}\}$ *is a factor set in the unit elements of* K. *Then a generalized group ring* $(a_{\sigma,\tau}, G, K)$ *is separable over* K *if and only if* $Kn=K$, *where* $n=|G|$.

Proof. Let ψ be a K-homomorphism of Λ to $\Lambda\otimes\Lambda^*=\Lambda^e$:

$$\psi(u_\sigma) = \Sigma u_\tau\otimes u_\rho^* k(\sigma, \tau, \rho), \qquad k(\sigma, \tau, \rho)\in K.$$

Then ψ is left Λ^e-homomorphic if and only if

3) See the definition in [8], § 2.

$$(6)\quad \begin{aligned} a_{\eta,\tau}k(\sigma,\tau,\rho) &= a_{\eta,\rho}k(\eta\sigma,\eta\tau,\rho)\\ a_{\rho,\eta}k(\sigma,\tau,\rho) &= a_{\sigma,\eta}k(\sigma\eta,\tau,\rho\eta) \quad \text{for any } \eta\in G. \end{aligned}$$

From (6) we have $k(1,\tau,\rho)=a_{\rho,\tau}^{-1}k(\rho\tau,\rho\tau,\rho\tau)$. If Λ is separable over K, then there exists a Λ^e-homomorphism ψ of Λ to Λ^e such that $\varphi\psi=I$, where $\varphi:\Lambda^e\to\Lambda$; $\varphi(x\otimes y^*)=xy$. Hence $1=\varphi\psi(1)=\sum u_{\tau,\rho}a_{\tau,\rho}k(1,\tau,\rho)=u_1(\sum_{\tau\rho=1} a_{\tau,\rho}a_{\rho,\tau}^{-1}k(1,1,1)$. If we replace ρ, σ and τ by η^{-1}, η and η^{-1} in the relation of factor sets, then we have $a_{\eta,\eta^{-1}}=a_{\eta^{-1},\eta}$, where we assume $a_{\eta,1}=a_{1,\eta}=1$. Hence $1=nk(1,1,1)$. The converse is given by [11], Theorem 1. (cf. the proof of Proposition 1).

Proposition 3. *We assume that $\Lambda=(a_{\sigma,\tau},G,\mathfrak{O})$ is an order in a matric K-algebra over K and $R/\mathfrak{p}$ is not necessarily perfect. Then Λ is hereditary if and only if $\mathfrak{P}$ is tamely ramified. In this case the rank of Λ is equal to the ramification index of $\mathfrak{P}$.*

Proof. We assume that Λ is hereditary. Since $\{a_{\sigma,\tau}\}$ is similar to the unit factor set in L, $\Lambda_H=(a_{\sigma,\tau},H,\mathfrak{O})$ is in $(K)_{|H|}$. We know similarly to the proof of Corollary 2 that $N(\Lambda_H)=\mathfrak{p}\Lambda_H$. Hence, $\bar{\Lambda}_H=\bar{\Lambda}_H/\mathfrak{p}\Lambda_H=\bar{\mathfrak{O}}+\bar{u}_\sigma\bar{\mathfrak{O}}+\cdots+\bar{u}_\rho\bar{\mathfrak{O}}$ is semi-simple. However, since $\Omega/N(\Omega)=(R/\mathfrak{p})_{|H|}$ for a maximal order Ω in $(K)_{|H|}$, $\bar{\Lambda}=\Sigma(R/\mathfrak{p})_{m_i}$ by [7], Theorem 4.6. Hence, $\bar{\Lambda}$ is separable. Therefore, $(|H|,p)=1$ by Lemma 4.

3. Hereditary orders in a generalized quaternions

Finally, we shall determine all the hereditary orders in a generalized quatenions. Let Z be the ring of integers and K the field of rationals. Let d be an integer which is not divided by any quadrate and $L=K(\sqrt{d})$. Then the Galois group $G=\{1,g\}$ and $(\sqrt{d})^g=-\sqrt{d}$. For any integer a we have $\Sigma=(a,G,L)=K+Kg+K\sqrt{d}+Kg\sqrt{d}$ with relations $g^2=a$, $(\sqrt{d})^2=d$, and $g\sqrt{d}=-\sqrt{d}g$. We have determined all hereditary orders in [9], Theorem 1.2 in the case $a=-1$.

We use the same argument here as that in [9], §1.

First we shall determine the types of maximal orders over $Z_{\mathfrak{p}}$.

Proposition 4. *Let R be the ring of $\mathfrak{p}$-adic integers, $L=K(\sqrt{d})$ and $\Lambda=(a,G,\mathfrak{O})$. We denote the radical of Λ by $\mathfrak{N}$ and $\Lambda/\mathfrak{N}$ by $\bar{\Lambda}$. Then*

1) *If $\mathfrak{p}=2$, $d\equiv 1$ (mod 4), then Λ is a maximal order such that $\bar{\Lambda}=(R/2)_2$.*

2) *If $\mathfrak{p}=2$, $d\equiv 2,3$ (mod 4), then Λ is not hereditary.*

3) *If $\mathfrak{p}\neq 2$, $d\not\equiv 0$ (mod $\mathfrak{p}$), then Λ is a maximal order such that $\bar{\Lambda}=(R/\mathfrak{p})_2$.*

4) *If $\mathfrak{p}\neq 2$, $d\equiv 0$ (mod $\mathfrak{p}$),*

a) $(a/\mathfrak{p})$[4] $=1$, *then* Λ *is a herediary order of rank two.*

b) $(a/\mathfrak{p})=-1$, *then* Λ *is a unique maximal order.*

Proof. We shall consider the following three cases.

1) $H=1$. Then i) $\mathfrak{p}=\mathfrak{P}_1\mathfrak{P}_2$ and $S=H$, ii) $\mathfrak{p}=\mathfrak{P}$ and $S=G$. Since $\mathfrak{P}$ is unramified, Λ is maximal order by Theorem 1. In the case i) $\mathfrak{O}/\mathfrak{p}\mathfrak{O}=\mathfrak{O}/\mathfrak{P}_1+\mathfrak{O}/\mathfrak{P}_2$, and Λ is a maximal order such that $\Lambda/\mathfrak{p}\Lambda=(R/p)_2$. The case ii) $\Lambda/\mathfrak{p}\Lambda=\mathfrak{O}/\mathfrak{P}+g\mathfrak{O}/\mathfrak{P}$. Since $G=S$, $\Lambda/\mathfrak{p}\Lambda$ is not commutative and hence, Λ is not a unique maximal.

2) $G=S=H$, $\mathfrak{p}=2$ *and* $a\equiv 1$ $(mod\ 2)$. In this case 2 is remified and hence, Λ is not hereditary by Theorem 3.

3) $G=S=H$, *and* $\mathfrak{p}=2$. Then $\mathfrak{p}=\mathfrak{P}^2$ and $\Lambda/\mathfrak{P}\Lambda=R/\mathfrak{p}+(R/\mathfrak{p})g$. Since $\mathfrak{P}$ is tamey ramefied, $\mathfrak{P}\Lambda=\mathfrak{N}$ by the remark before Corollary 1, and Λ is hereditary. Let $\mathfrak{A}$ be a two-sided ideal in $\bar{\Lambda}$. If $\mathfrak{A}$ is proper, then $\mathfrak{A}=(1+\bar{y}\bar{g})R/\mathfrak{p}$ and $\bar{a}\bar{y}^2=1$ for some $\bar{y}\in\bar{\mathfrak{O}}=R/\mathfrak{p}$, and conversely. Therefore, if $(a/\mathfrak{p})=1$ then Λ is a hereditary order of rank 2 and if $(a/\mathfrak{p})=-1$, then Λ is a unique maximal order. The proposition is trivial from the well known facts of quadratic field.

If we set $g=i$ and $\sqrt{d}=j$, then $\Sigma=(a, G, L)$ is a generalized quaternions over the field K of rationals. For any element $x=x_1+x_2i+x_3j+x_4ij$ we define

$$N(x) = x_1^2-ax_2^2-dx_3^2+adx_4^2\,.$$

Let Ω be a maximal order over R with basis u_1, u_2, u_3 and u_4. We call an element $x=\Sigma x_iu_i$ in Ω normalized if $(x_1, \cdots, x_4)=1$.

We note that if Σ contains at least two maximal orders, then $\hat{\Sigma}$ is a matrix ring over $\hat{K}$ where $\wedge$ means the completion with respect to $\mathfrak{p}$, (cf. [9], Lemma 1.4).

In order to use the same argument as in the proof of [9], Theorem 1.2 we need

Lemma 6. 1) *If either* $\mathfrak{p}=2$, $d\equiv 3$ $(mod\ 4)$ *and* $a\equiv 1$ $(mod\ 4)$ *or* $\mathfrak{p}=2$, $d\equiv 2$ $(mod\ 4)$, *and* $a\equiv 1$ $(mod\ 8)$, *then there exists a maximal order* Ω *such that* $\bar{\Omega}=(R/2)_2$. 2) *If* $\mathfrak{p}=2$, $d\equiv 2$ $(mod\ 4)$, $a\equiv 1$ $(mod\ 4)$ *and* $a\not\equiv 1$ $(mod\ 8)$, *then there exists a unique maximal order.* 3) *If* $\mathfrak{p}\neq 2$, $d\equiv 0$ $(mod\ \mathfrak{p})$ *and* $(a/\mathfrak{p})=1$, *then there exists a maximal order* Ω *such that* $\bar{\Omega}=(R/\mathfrak{p})_2$, *where* $\bar{\Omega}$ *means the factor ring of* Ω *modulo its radical.*

Proof. Let $\Omega=\mathfrak{O}+(1/2)(1+g)\mathfrak{O}=R+Rj+R1/2(1+i)+R(1/2)(j+ij)$, where $i=g$ and $j=\sqrt{d}$. We denote $(1/2)(1+i)$ and $(1/2)(j+ij)$ by h and l. Then we obtain by the direct computations that

4) Legendre's symbol,

$$(7)\qquad \begin{aligned} jh &= i-l,\ hj = l,\ jl = d(1-h),\ lj = dh,\ hl = l+jr,\ lh \\ &= -rj,\ h^2 = h+r \ \ and\ \ l^2 = dr, \end{aligned}$$

where $a=1+4r$, $r\in R$.

1) $d\equiv 3 \pmod 4$. Let $N(\Omega)$ be the radical of Ω and $\bar{x}=\bar{x}_1+\bar{x}_2 j+\bar{x}_3 h+\bar{x}_4 l\in N(\Omega)/2\Omega$. Then $\bar{x}j+j\bar{x}=\bar{x}_4\bar{d}+\bar{x}_3 j$. If $x_3\not\equiv 0 \pmod 2$, then we may assume $1+j\in N(\Omega)$. Then $0\equiv(1+j)l+l(1+j)\equiv d \pmod 2$, which is a contradiction. Hence, we know $N(\Omega)=2\Omega$ by the similar argument for x_1, x_2. Since $\Omega/N(\Omega)$ is not commutative by (7), $\Omega/N(\Omega)=(R/2)_2$ and Ω is a maximal order (cf. [9], Lemma 1.3).

2) $d\equiv 2 \pmod 4$. From (7) we obtain $N(\Omega)=\Lambda j$. If $r\equiv 0 \pmod 2$, then $\Omega/N(\Omega)=(R/2)h+(R/2)(1+h)$. Hence Ω is a hereditary order of rank two. Let $\Omega_0=R+Rj+Rh+R(1/2)$. It is clear that $\Omega_0\supsetneqq\Lambda$ and Ω_0 is a ring. Hence Ω_0 is a maximal order by [7], Theorems 1.7 and 3.3. If $r\not\equiv 0$ (*mod* 2), then $\Omega/N(\Omega)$ is a field and hence Ω is a unique maximal order.

3) In this case Λ is hereditary. Let $\Omega=R+Ri+Rj+R(1/p)(j+yij)$, where $ay^2=1+px$, $x\in R$. It is clear that $\Omega\supsetneqq\Lambda$. We shall show that Ω is a ring. $((1/p)(j+yij))^2=(d/p)x\in\Omega$, and $(1/p)(j+yij)i=-(x/y)j-(1/yp)(j+yij)\in\Omega$, and $(1/p)(j+yij)j=(d/p)(1+ky)\in\Omega$. Therefore, Ω is a maximal order as above.

Next, we consider a case of $a\not\equiv 1 \pmod 4$ and $\mathfrak{p}=2$.

Lemma 7. *We consider the following conditions*

i) $a\equiv 3$ (*mod* 8), $d\equiv 2$ (*mod* 4), *but* $d\not\equiv 2$ (*mod* 8).

ii) $a\equiv 3$ (*mod* 8), *and* $d\equiv 2$ (*mod* 8).

iii) $a\equiv 7$ (*mod* 8), *and* $d\equiv 2$ (*mod* 4), *but* $d\not\equiv 2$ (*mod* 8).

iv) $a\equiv 7$ (*mod* 8), *and* $d\equiv 2$ (*mod* 8).

v) $a\not\equiv 1$ (*mod* 4), *and* $d\equiv 3$ (*mod* 4).

If one of i) *and* iv) *is satisfied, then there is a maximal order* Ω *such that* $\Omega/N(\Omega)=(R/2)_2$. *If one of* ii), iii) *and* v) *is satisfied, then there exists a unique maximal order.*

Proof. We shall show this lemma by a direct computation. Thus, we give here only a sketch of the proof.

Put $i=g$, $j=\sqrt{d}$ and $H=1/2(1+i+j)$, $L=1/2(i+i+ij)$. Let $\Lambda=R+Ri+RH+RL$. If we set $a=1+2r$, $d=2+4k$ where $r\equiv 1$ (*mod* 4), $k\not\equiv 0$ (*mod* 2), we have

$$(8)\qquad \begin{aligned} &i^2=1+2r,\ H^2=k+(1+r)/2+H,\ L^2=-(1/2)(1+r)-(1+2r)k+L, \\ &iH=L+r,\ Hi=1+r+i-L,\ iL=-ri+(1+2r)H,\ Li=1+2r \\ &+(1+r)i-(1+2r)H.\ LH=r+((1+r)/2+k)i-rH+L,\ \ and \end{aligned}$$

$HL=-(k+(1+r)/2)i+(1+r)H.$

In cases i) and iv) we can show directely that $N(\bar{\Lambda})=\bar{\Lambda}(\bar{i}+\bar{1})$ and $\bar{\Lambda}/\bar{\Lambda}(1+i)\approx(R/2)\bar{H}\oplus(R/2)(\bar{1}+\bar{H})$, $\bar{H}(\bar{1}+\bar{H})=\bar{0}$, where $\bar{\Lambda}=\Lambda/2\Lambda$. Since $(1-i)(1+i)=1-a=-2r$, $r\not\equiv 0\ (mod\ 2)$, $\Lambda(1+i)\supseteq 2\Lambda$. Hence $N(\Lambda)=\Lambda(1+i)$, which implies that Λ is a hereditary order of rank two. Therefore, there exists a maximal order as in the lemma.
In cases ii) and iii) we obtain similarly that $\Lambda/\Lambda(1+i)\approx(R/2)\bar{H}+(R/2)(\bar{1}+\bar{H})$ *and* $\bar{H}^2=\bar{1}+\bar{H}$, $(\bar{1}+\bar{H})^2=\bar{H}$, $\bar{H}(\bar{1}+\bar{H})=\bar{1}$. Hence, Λ is a unique maximal order.
In case v) we put $t=1/2(1+i+j+ij)$ and $\Lambda=R+Ri+Rj+Rt$. Then by the same argument in [9], Lemma 1.3 we can show that $N(\Lambda)=\Lambda(1+i)$ and $\Lambda/\Lambda(1+i)$ is a field. Hence, Λ is a unique maximal order.

From Proposition 4, Lemmas 6 and 7 and the proof of [9], Theorem 1.2 we have

Theorem 4. *Let R be a ring of $\mathfrak{p}$-adic integers, K the field of rationals and $L=K(\sqrt{d})$. For a unit element a in R, $\Sigma=(a, G, L)$ is a generalized quaternions and $\Lambda=(a, G, \mathfrak{O})$. Then every hereditary order over R in Σ is isomorphic to one of the following*:

1) Λ (*unique maximal*) *if* $\mathfrak{p}=2$, $d\equiv 0\ (mod\ \mathfrak{p})$, $(a/\mathfrak{p})=-1$.
2) $\Omega_1=R+R\sqrt{d}+R(1/2)(1+g)+(1/2)(\sqrt{d}+g\sqrt{d})$
(*unique maximal*) *if* $\mathfrak{p}=2$, $d\equiv 2\ (mod\ 4)$, $a\equiv 1\ (mod\ 4)$ *and* $a\not\equiv 1\ (mod\ 8)$.
3) Λ (*maximal*), $\Lambda\cap\alpha^{-1}\Lambda\alpha$
if either a) $\mathfrak{p}=2$, $d\equiv 1\ (mod\ 4)$ *or*
b) $\mathfrak{p}\neq 2$, $d\not\equiv 0\ (mod\ \mathfrak{p})$.
4) Ω (*maximal*), $\Gamma_1=R+Rg+RH+RL$,
if one of i) *and* iv) *in Lemma* 8 *is valid.*
5) Γ_1 (*unique maximal*)
if one of ii), iii) *and* iv) *in Lemma* 8 *is valid.*
6) $\Omega_2=R+Rg+R\sqrt{d}+Rt$ (*unique maximal*)
if $\mathfrak{p}=2$, $d\equiv 3\ (mod\ 4)$, *and* $a\not\equiv 1\ (mod\ 4)$.
7) $\Omega_3=R+R\sqrt{d}+R(1/2)(1+g)+R(1/4)(\sqrt{d}+g\sqrt{d})$
(*maximal*),
$\Gamma_2=R+R\sqrt{d}+R(1/2)(1+g)+R(1/2)(\sqrt{d}+g\sqrt{d})$
if $\mathfrak{p}=2$, $d\equiv 0\ (mod\ 4)$, *and* $a\equiv 1\ (mod\ 8)$.
8) Ω_1 (*maximal*), $\Omega_1\cap\alpha^{-1}\Omega\alpha$
if either a) $\mathfrak{p}=2$, $d\equiv 3\ (mod\ 4)$ $a\equiv 1\ (mod\ 4)$ *or*
b) $\mathfrak{p}=2$, $d\equiv 2\ (mod\ 4)$ *and* $a\equiv 1\ (mod\ 8)$.
9) $\Omega_4=R+Rg+R\sqrt{d}+R(1/p)(\sqrt{d}+yg\sqrt{d})$ (*maximal*),
Λ *if* $\mathfrak{p}\neq 2$, $d\equiv 0\ (mod\ \mathfrak{p})$ *and* $(a/\mathfrak{p})=1$.

Where $\mathfrak{O}$ means the integral closur of R in L and α is a normalized element with respect to the basis of a maximal order and $N(\alpha)=pq$, $(p, q)=1$ and $ay^2\equiv 1$ (mod $\mathfrak{p}$), $H=(1/2)(1+g\sqrt{d})$, $L=(1/2)(1+\sqrt{d}+g\sqrt{d})$, $t=\frac{1}{2}(1+g+\sqrt{d}+g\sqrt{d})$, and $\mathfrak{p}=(p)$.

REMARK 2. A maximal order Ω in 4) is any ring which contains properly Λ.

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