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ADDENDUM TO "AMITSUR'S COMPLEX FOR PURELY INSEPARABLE FIELDS"

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Introduction

We begin this note by pointing out that a few modifications in some of the notations and arguments of [13] will make these fit in more closely with results in the literature. We also complete the results of [13] in several points. In particular we point out that the spectral sequence used in [13] is not quite a genuine generalization of the Hochschild-Serre spectral sequence in Galois cohomology. However with a slightly different spectral sequence the results of [13] can also be obtained and we shall show in section 2 that this is indeed a genuine generalization of the Hochschild-Serre sequence for Galois cohomology. In section 3 we shall use some of the results of [13] to derive an exact sequence complementary to that of Proposition 7. 8 of [13] from which we deduce the following result first pointed out to us by S. Shatz: Let C be a field, C_s its separable algebraic closure and $\hat{C}$ its algebraic closure. Then if λ is the lift map [2, Def. 2. 3.], we have that $\lambda: H^r(C_s/C)\rightarrow H^r(\hat{C}/C)$ is an isomorphism for $r=1, 2, \cdots$.

1. Notations and preliminary results

Throughout we use the notations and definitions of [13] with the following modifications: The complex $\mathfrak{C}(F/C)$ is now defined by $\mathfrak{C}^n(F/C)=(F^{n+1})^*$ $n=0, 1, 2, \cdots$. Thus for $n\neq 0$, $H^n(F/C)$ as defined in [13] is the n^{th} cohomology group of $\mathfrak{C}(F/C)$ as defined here and the only difference is in $H^0(F/C)$. We also carry out a corresponding modification in the definition of the double complex $\mathfrak{C}(K, F/C)$ of [13, §4]; now $E_0^{m,n}=(K^{m+1}\otimes F^{n+1})^*$ if $m, n\geqq 0$ and 1 otherwise. The two derivations $\Delta_F^{m,n}: E_0^{m,n}\rightarrow E_0^{m,n+1}$, $\Delta_K^{m,n}: E_0^{m,n}\rightarrow E_0^{m+1,n}$ are defined just as in [2 and 13] by $\Delta_F^{m,n}=\sum_1^{n+2}(-1)^{m+i}(1\otimes\mathcal{E}_i)$ and $\Delta_K^{m,n}=\sum_1^{n+2}(-1)^{i+1}(\mathcal{E}_i\otimes 1)$, written additively

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for the sake of convenience. We shall use the notation Tot $\mathfrak{C}(K, F/C)$ for the canonical associated single complex of $\mathfrak{C}(K, F/C)$, whose cohomology groups we denote, as in [13], by $H(K, F/C)$. Adhering to the notation of [13], we find for the two spectral sequences of $\mathfrak{C}(K, F/C)$

$$'E_1^{m,n} = H^n(K^{m+1}\otimes F/K^{m+1})\,;\ ''E_1^{m,n} = H^m(K\otimes F^{n+1}/F^{n+1})\,.$$

We shall have occasion to use the following two hypotheses

$$(e_{K,m}):\ 1 \to (K^{m+1})^* \to (K^{m+1}\otimes F)^* \xrightarrow{\Delta_F^{m,0}} (K^{m+1}\otimes F^2)^* \quad \text{is exact}$$

$$(e_{F,n}):\ 1 \to (F^{n+1})^* \to (K\otimes F^{n+1})^* \xrightarrow{\Delta_K^{0,n}} (K^2\otimes F^{n+1})^* \quad \text{is exact,}$$

where in each case the first mappings are given by $x \to x\otimes 1$, x in $(K^{m+1})^*$ and $y \to 1\otimes y$, y in $(F^{n+1})^*$ respectively.

Lemma 1.1. *If F is a faithfully flat C-algebra $(e_{K,m})$, $m=0, 1, 2, \cdots$ holds. If K is a faithfully flat C-algebra $(e_{F,n})$, $n=0, 1, 2, \cdots$ holds. If $(e_{K,m})$ holds, then $'E_1^{m,0}\cong(K^{m+1})^*$; if $(e_{F,n})$ holds, then $''E_1^{0,n}\cong(F^{n+1})^*$.*

Proof. If F is a faithfully flat C-algebra then by [5, Prop. 5 p. 48], $K^{m+1}\otimes F$ is a faithfully flat K^{n+1}-algebra. But then, taking into account our slight change in numbering, Lemma 3.1 of [13] asserts precisely that $(e_{K,m})$ is exact. The rest of the proof is then clear.

Lemma 1.2. *If F is a K-algebra $(e_{F,n})$ hold for $n=0, 1, 2, \cdots$. Furthermore, $''E_1^{m,n}=0$ for $m>0$, $''E_1^{0,n}\cong(F^{n+1})^*$ and the injection map ψ of $''E_1^{0,n}$ into Tot $\mathfrak{C}(K, F/C)$ induces an isomorphism of cohomology ψ^*: $H^n(F/C)\to H^n(K, F/C)$.*

Proof. Following [2, Theorem 2.9] we define a C-algebra homomorphism $\kappa: K^{m+1}\otimes F^{n+1}\to K^m\otimes F^{n+1}$,[2] $m, n=0, 1, 2, \cdots$ by $\kappa(k_1\otimes\cdots\otimes k_{m+1}\otimes f_1\otimes\cdots\otimes f_{n+1})=k_1\otimes\cdots\otimes k_{m+1}f_1\otimes f_2\otimes\cdots\otimes f_{n+1}$. It is shown in [2, Theorem 2.9] that κ is a contracting homotopy for $\mathfrak{C}(K\otimes F^{n+1}/F^{n+1})$. Hence it follows that $''E_1^{m,0}=0$ if $m>0$. Furthermore, $\kappa(1\otimes y)=y$ for y in F^{n+1}, so that $y\to 1\otimes y$ is a monomorphism. Moreover, if u in $(K\otimes F^{n+1})^*$ is such that $\Delta_K^{0,n}(u)=1$, an easy computation shows that $u=1\otimes\kappa(u)$ so that $(e_{F,n})$ holds and $''E_1^{0,n}\cong(F^{n+1})^*$. The last assertion of the lemma is then a direct consequence of [9, Theorem 4.8.1 p. 89].

REMARK. Lemma 1.2 is the main difference between the results of [13] and this note. With the double complex of [13], in case F is a K-algebra, one has $H^n(K/C)\cong H^n(K, F/C)$ [13, Lemma 4.2].

2) For notational convenience we set $K^0=C$.

With our definition of $\mathfrak{C}(K, F/C)$ some of the exact sequences of [13] are direct consequences of standard theorems on spectral sequences and it is also easier to compute some of the maps explicitly. We begin with

Lemma 1. 3. *Let* $E_2^{p,q} \underset{p}{\Longrightarrow} H^n$ *be a spectral sequence such that* $E_2^{p,q}=0$ *if either* p *or* q *is less than* 0 *and such that* $E_2^{1,1}=E_2^{2,1}=0$ *then there are maps to make*

$$E_2^{0,1} \to E_2^{2,0} \to H^2 \to E_2^{0,2} \to E_2^{3,0} \to H^3$$

exact.

Proof. We apply [6, Prop. 5. 7 p. 326] with $r=n=p=k=2$ to find that there is an exact sequence

$$E_2^{2,0} \to H^2 \to E_2^{0,2}.$$

Next [6, Prop. 5. 9 p. 327] with $p=0$, $q=r=2$, $s=3$ yields an exact sequence

$$H^2 \to E_2^{0,2} \to E_2^{3,0}.$$

Finally we use [6, Prop. 5. 9(a) p. 328] first with $p=s=3$, $q=0$, $r=2$ and then with $p=s=2$, $q=0$, $r=2$ to obtain two exact sequences

$$E_2^{0,2} \to E_2^{3,0} \to H^3 \quad \text{and}$$
$$E_2^{0,1} \to E_2^{2,0} \to H^2.$$

From the cited propositions it is clear that whenever in these four exact sequences two maps have the same domain and range, they are equal. Hence by composing all these sequences we obtain our result. Of course, it is easy to give a direct proof of the lemma based solely on the usual elementary properties of spectral sequences.

We now begin to recover some of the exact sequences of [13].

Lemma 1. 4. *If* $H^1(K^{m+1} \otimes F/K^{m+1})=0$, $m=0, 1, 2$ *and* $(e_{K,m})$ $m=1, 2, 3, 4$, *holds then there are homomorphisms to make*

$$\begin{aligned} 0 \to H^2(K/C) \to H^2(K, F/C) \to H^2(K \otimes F/K)^{0\ 3)} \to \\ H^3(K/C) \to H^3(K, F/C) \end{aligned}$$

exact.

3) As in [2, p. 16] we set $H^n(K \otimes F/K)^0 = \mathrm{Ker}\,(d_1^{0,n} : {}'E_1^{0,n} \to {}'E_1^{1,n})$ i.e. the elements in $H^n(K \otimes F/K)$ for which $\varepsilon_1^*(z)=\varepsilon_2^*(z)$ where ε_i^* are the mappings on $H^n(K \otimes F/K) \to H^n(K^2 \otimes F/K^2)$ induced by the $\varepsilon_i : K \to K^2$.

Proof. By hypothesis $'E_1^{m,1}=H^1(K^{m+1}\otimes F/K^{m+1})=0$, $m=0,1,2$. Hence Lemma 1.3 is applicable and yields the exact sequence

$$0\to 'E_2^{2,0}\to H^2(K, F/C)\to 'E_2^{0,2}\to 'E_2^{3,0}\to H^3(K, F/C).$$

Since $(e_{K,m})$ holds for $m=1,2,3,4$, Lemma 1.1 shows that $'E_1^{m,0}\cong(K^{m+1})^*$ for $m=1,2,3,4$. Hence, clearly, $'E_2^{m,0}\cong H^m(K/C)$, $m=2,3$. Finally $'E_1^{0,2}=H^2(K\otimes F/K)$ so that $'E_2^{0,2}=H^2(K\otimes F/K)^0$ by definition completing the proof of the lemma.

REMARK. If $''E_2^{1,0}=''E_2^{2,0}=''E_2^{1,1}=0$, for which $H^1(K\otimes F/F)=H^2(K\otimes F/F)=H^1(K\otimes F^2/F^2)=0$ is sufficient, since these are $''E_1^{1,0}$, $''E_1^{2,0}$, $''E_1^{1,1}$ respectively, it is readily verified that the injection $\psi:(F^{n+1})^*\to ''E_1^{0,n}\to \mathrm{Tot}\,\mathfrak{C}(K, F/C)$ induces an isomorphism $\psi^*: H^2(F/C)\cong H^2(K, F/C)$. If $''E_2^{2,0}=''E_2^{3,0}=''E_2^{1,1}=''E_2^{2,1}=''E_2^{1,2}=0$ for which $H^2(K\otimes F/F)=H^3(K\otimes F/F)=H^1(K\otimes F^2/F^2)=H^2(K\otimes F^2/F^2)=H^1(K\otimes F^3/F^3)=0$ is sufficient, since these are $''E_1^{2,0}$, $''E_1^{3,0}$, $''E_1^{1,1}$, $''E_1^{2,1}$, $''E_1^{1,2}$ respectively, it is readily verified that $\psi^*: H^3(F/C)\to H^3(K, F/C)$ is an isomorphism. Combining this with Lemma 1.3 we recover the long exact sequence of [13, Prop. 5.3].

The maps $'E_2^{n,0}\to H^n(K, F/C)$ and $H^n(K, F/C)\to 'E_2^{0,n}$ that occur in the above sequences are the edge homomorphisms of the spectral sequence $'E$. We next wish to compute these explicitly in case F is a K-algebra and $(e_{K,m})$ holds. We begin by defining a chain map of $\mathrm{Tot}\,\mathfrak{C}(K, F/C)$ to $\mathfrak{C}(F/C)$ in case F is a K-algebra: Let $\theta^{m+1,n+1}: K^{m+1}\otimes F^{n+1}\to F^{m+n+1}$ be the C-algebra homomorphism defined by $\theta^{m+1,n+1}(k_1\otimes\cdots\otimes k_{m+1}\otimes f_1\otimes\cdots\otimes f_{n+1})=(k_1\cdot 1)\otimes\cdots\otimes(k_n\cdot 1)\otimes k_{n+1}f_1\otimes\cdots\otimes f_{n+1}$ where 1 is the identity element of F. A routine computation then shows that for any u in $(K^{m+1}\otimes F^{n+1})^*$, $\theta^{m+2,n+1}(\Delta_K^{m+1,n+1}(u))\cdot\theta^{m+1,n+2}(\Delta_F^{m+1,n+1}(u))=\Delta_F(\theta^{m+1,n+1}(u))$ where Δ_F is the derivation of $\mathfrak{C}(F/C)$. We therefore define φ: $\mathrm{Tot}\,\mathfrak{C}(K, F/C)\to\mathfrak{C}(F/C)$ by $\varphi(\Pi_{i+j=m+2}u_{i,j})=\Pi\theta^{i,j}(u_{i,j})$ with $u_{i,j}$ in $(K^i\otimes F^j)^*$ and clearly have that φ is a homomorphism of complexes.

Lemma 1.5. *If F is a K-algebra, the homomorphism $\varphi^*: H^n(K, F/C)\to H^n(F/C)$ is the inverse of the isomorphism $\psi^*: H^n(F/C)\to H^n(K, F/C)$ of Lemma 1.2.*

Proof. By Lemma 1.2, ψ^* is induced by the map of complexes $\psi:\mathfrak{C}(F/C)\to\mathfrak{C}(K, F/C)$ which sends a unit v in F^{n+1} to the element $1\otimes v$ in $E_0^{0,n}$. But then clearly $\varphi\psi$ is the identity map on $\mathfrak{C}(F/C)$ and thus $\varphi^*\psi^*$ is the identity map on $H^n(F/C)$. Since ψ^* is an isomorphism, the result follows.

We recall, next, that if F is a K-algebra, the C-algebra homomorphism $K \to K\cdot 1 \subseteqq F$ induces an algebra homomorphism $K^{n+1} \to F^{n+1}$ which in turn induces a homomorphism $\lambda: H^n(K/C) \to H^n(F/C)$ called the lift map [2, Definition 2.3]. Furthermore the C-algebra homomorphism $F \to K\otimes F$ defined by $x \to 1\otimes x$ for x in F is always defined and induces a map $\rho: H^n(F/C) \to H^n(K\otimes F/K)$ which is called the restriction map [2, Definition 2.1].

Proposition 1.6. *Let F be a K-algebra such that $(e_{K,m})$ holds. Then $'E_1^{m,0} \simeq (K^{m+1})^*$, the isomorphism being given by $x \to x\otimes 1$ for x in $(K^{m+1})^*$. Composing the isomorphism $\psi^*: H^n(F/C) \to H^n(K, F/C)$ with the edge homomorphism $H^n(K, F/C) \to {}'E_2^{0,n} = H^n(K\otimes F/K)^0$ yields the restriction map $\rho: H^n(F/C) \to H^n(K\otimes F/K)$. Composing the edge homomorphism $'E_2^{m,0} \simeq H^m(K/C) \to H^m(K, F/C)$ with the isomorphism $\varphi^*: H^m(K, F/C) \to H^m(F/C)$ yields the lift map $\lambda: H^m(K/C) \to H^m(F/C)$.*

Proof. The first assertion is already contained in Lemma 1.1. By [11, Theorem 8.1 p. 346] the edge homomorphism $\eta^*: H^n(K, F/C) \to {}'E_2^{0,n}$ is induced by the mapping of complexes $\eta: \mathfrak{C}(K, F/C) \to \mathfrak{C}(K, F/C)/F'(\mathfrak{C}(K, F/C))$ where F' is the first filtration, $\sum_{i>0} E_0^{i,j}$ in this case. Thus a cochain $u^{0,n}\cdot u^{1,n-1}\cdots u^{n,0}$ is mapped to the class of $u^{0,n}$, where $u^{i,j}$ lies in $(K^{i+1}\otimes F^{j+1})^*$. Thus if u in $(F^{n+1})^*$ is a cocycle, $\eta^*\varphi^*u$ is simply the class of $1\otimes u$ in $H^n(K\otimes F/K)$ which proves the second assertion. For the last assertion, we again note by [11, Theorem 8.1 p. 346] that the edge homomorphism $\xi^*: {}'E_2^{n,0} \to H^n(K, F/C)$ is precisely the map on cohomology induced by the inclusion $\xi: {}'E_1^{n,0} \to \operatorname{Tot}\mathfrak{C}(K, F/C)$. Hence, if u is a cocycle in $(K^{n+1})^*$, since $\theta^{n+1,1}(u)$ is just $u\cdot 1$, 1 the unit element of F, it is clear that $\varphi(u) = u\cdot 1$. Thus $\varphi\xi(u) = u\cdot 1$ so that $\varphi^*\xi^*$ on the class of u is λ on the class of u.

Corollary 1.7. *Let F be a K-algebra and suppose in addition that $(e_{K,m})$ holds and that $H^1(K^{m+1}\otimes F/K^{m+1}) = 0$, $m = 0, 1, 2$. Then we have an exact sequence*

$$0 \to H^2(K/C) \xrightarrow{\lambda} H^2(F/C) \xrightarrow{\rho} H^2(K\otimes F/K)^0 \to H^3(K/C) \xrightarrow{\lambda} H^3(F/C).$$

Proof. By Lemmas 1.4 and 1.2 we see that we have this exact sequence with the maps not identified. But by the proof of Lemmas 1.2 and 1.3 it is clear that the labelled maps all arise from edge homomorphisms and an appeal to Proposition 1.6 completes the task.

We end this section by pointing out how the spectral sequence $'E$ as defined here yields a proof of an analogue of Proposition 4.1 and Theorem 4.3 of [13]. Under the hypotheses stated there it follows that $'E_2^{m,n} = 0$ if m and n is nonzero. But then it is clear that $'E_\infty^{0,n} =$

Ker $(\delta: {}'E_2^{0,n} \to {}'E_2^{n+1,0})$, ${}'E_\infty^{n,0} = {}'E_2^{n,0}/\delta' E_2^{0,n-1}$ with δ the appropriate derivation, ${}'E_\infty^{m,n}=0$, m and n nonzero. But then it is clear that ${}'E_2^{0,n} = {}'E_1^{0,n} = H^n(K\otimes F/K)$ and that there is an exact sequence

$$\cdots {}'E_2^{n,0} \to H^n(K, F/C) \to {}'E_2^{0,n} \to {}'E_2^{n+1,0} \to \cdots$$

with the first two arrows being edge homomorphisms. In Proposition 4.1 of [13] F is faithfully flat over C, thus Lemma 1.1 shows that ${}'E_2^{n,0} \cong H^n(K/C)$ so that we have an exact sequence

$$\cdots H^n(K/C) \to H^n(K, F/C) \to H^n(K\otimes F/K) \to H^{n+1}(K/C) \to \cdots$$

Now if F is a K-algebra, Proposition 1.6 and Lemma 1.2 yield the exact sequence

$$\cdots H^n(K/C) \xrightarrow{\lambda} H^n(F/C) \xrightarrow{\rho} H^n(K\otimes F/K) \to H^{n+1}(K/C) \to \cdots$$

valid, of course, only under the hypotheses of Theorem 4.3 of [13].

2. Hochschild-Serre spectral sequence in Galois cohomology

In this section we shall show that the spectral sequence $'E$ reduces to one of the Hochschild-Serre spectral sequences, as given in [10], for Galois cohomology in case $F \supset K \supset C$ are normal separable field extensions.

Let C be a field and K a possibly infinite-dimensional, normal separable extension field of C with Galois group $\mathfrak{G}$. We consider $\mathfrak{G}$ as a topological group with the usual Krull topology and shall use $\mathfrak{N}_\alpha$ to denote the family of closed and open normal subgroups of finite index which correspond to finite normal extension fields of C in the Galois correspondence. If A is any discrete $\mathfrak{G}$-module on which $\mathfrak{G}$ operates continuously, equivalently $A = \bigcup A^{\mathfrak{N}_\alpha}$ [7, p. 3; 14, p. I-8], we denote by $C_c^n(\mathfrak{G}, A)$ the group of continuous homogeneous n-cochains on $\mathfrak{G}$ with values in A. It is known, and easy to verify, that $C_c^n(\mathfrak{G}, A) = \lim_{\substack{\to \\ \alpha}} C^n(\mathfrak{G}/\mathfrak{N}_\alpha, A^{\mathfrak{N}_\alpha})$, where the latter are the ordinary homogeneous cochains on the finite group $\mathfrak{G}/\mathfrak{N}_\alpha$ [14, p. I-9]. $C_c^n(\mathfrak{G}, A)$ is a complex under the usual derivation whose cohomology groups we denote by $\dot{H}^n(\mathfrak{G}, A)$. Note that $\dot{H}^n(\mathfrak{G}, A) \cong \lim_{\substack{\to \\ \alpha}} H^n(\mathfrak{G}/\mathfrak{N}_\alpha, A^{\mathfrak{N}_\alpha})$.

The action of $\mathfrak{G}$ on K clearly makes, for any extension field F of C, $(K\otimes F^{n+1})^*$ into a discrete $\mathfrak{G}$-module on which $\mathfrak{G}$ operates continuously. We define a map

$$\nu: (K^{m+1}\otimes F^{n+1})^* \to C_c^m(\mathfrak{G}, (K\otimes F^{n+1})^*) \quad \text{by}$$
$$\nu(\sum k_1\otimes \cdots \otimes k_{m+1}\otimes f)(\sigma_1, \sigma_2, \cdots, \sigma_{m+1}) = \sum (\Pi\sigma_i(k_i)\otimes f)$$

with f in F^{n+1}. Then

Lemma 2.1. *ν is an isomorphism of the complex $\mathfrak{C}(K\otimes F^{n+1}/F^{n+1})$ with $C_c(\mathfrak{G}, (K\otimes F^{n+1})^*)$.*

Proof. This is a very slight recasting of [12, Lemma 2.2]. It is shown there that if $K_{\mathfrak{N}}$ is the fixed field of $\mathfrak{N}$, ν yields a complex isomorphism

$$\mathfrak{C}(K_{\mathfrak{N}}\otimes F^{n+1}/F^{n+1}) \xrightarrow{\nu_{\mathfrak{N}}} C(\mathfrak{G}/\mathfrak{N}, (K_{\mathfrak{N}}\otimes F^{n+1})^*).$$

If $\mathfrak{N}'\subset\mathfrak{N}$ so that $K_{\mathfrak{N}'}\supset K_{\mathfrak{N}}$, it is readily verified that

$$\begin{array}{ccc} \mathfrak{C}(K_{\mathfrak{N}}\otimes F^{n+1}/F^{n+1}) & \xrightarrow{\nu_{\mathfrak{N}}} & C(\mathfrak{G}/\mathfrak{N}, (K_{\mathfrak{N}}\otimes F^{n+1})^*) \\ \downarrow & & \downarrow \\ \mathfrak{C}(K_{\mathfrak{N}'}\otimes F^{n+1}/F^{n+1}) & \xrightarrow{\nu_{\mathfrak{N}'}} & C(\mathfrak{G}/\mathfrak{N}', (K_{\mathfrak{N}'}\otimes F^{n+1})^*) \end{array}$$

is commutative where the left vertical arrow is the obvious one induced by the inclusion $K_{\mathfrak{N}}\subset K_{\mathfrak{N}'}$ and the right vertical map is the usual cochain map leading to the lift map in the cohomology of finite groups (remembering that $\mathfrak{G}/\mathfrak{N}\cong(\mathfrak{G}/\mathfrak{N}')/(\mathfrak{N}/\mathfrak{N}')$). Now it is clear that $\lim_{\rightarrow}\mathfrak{C}(K_{\mathfrak{N}}\otimes F^{n+1}/F^{n+1})=\mathfrak{C}(K\otimes F^{n+1}/F^{n+1})$ and as noted above $C_c(\mathfrak{G}, (K\otimes F^{n+1})^*)=\lim_{\rightarrow}C(\mathfrak{G}/\mathfrak{N}, (K_{\mathfrak{N}}\otimes F^{n+1})^*)$ so that the result follows.

It is easily verified that a function from $\mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}$ to any discrete $\mathfrak{G}$-module on which $\mathfrak{G}$ acts continuously is continuous if and only if it is locally constant and finite-valued [14, p. I-8]. Using this it is easy to see that $C_c^n(\mathfrak{G},.)$ is an exact functor on the category of discrete $\mathfrak{G}$-modules on which $\mathfrak{G}$ acts continuously. Now we make $C_c^m(\mathfrak{G}, (K\otimes F^{n+1})^*)$ into a double complex by using the derivation induced by $(-1)^{m-1}\Delta_F$ as well as the standard one on cochains. It is then clear that ν yields an isomorphism of double complexes

$$\nu: \mathfrak{C}(K, F/C) \rightarrow C_c(\mathfrak{G}, (K\otimes F^{n+1})^*).$$

By [6, Theorem 7.2, p. 68] $\nu^*('E_1^{m,n})=C_c^m(\mathfrak{G}, H^n(K\otimes F/K))$ and thus $\nu^*('E_2^{m,n})=H^m(\mathfrak{G}, H^n(K\otimes F/K))$. Invoking Lemma 1.2 we have

Theorem 2.2. *Let C be a field, $F\supset K\supset C$ extension fields, with K normal separable over C with Galois group $\mathfrak{G}$. Then there is a spectral sequence*

$$\dot{H}^m(\mathfrak{G}, H^n(K\otimes F/K)) \underset{m}{\Longrightarrow} H^r(F/C),$$

where $H(K\otimes F/K)$ has a $\mathfrak{G}$-structure via the action of $\mathfrak{G}$ on K.

Corollary 2. 3. *With the same hypotheses as in Theorem 2. 2 suppose in addition that F is also normal separable with Galois group $\mathfrak{M}$ and that $\mathfrak{H}$ is the normal subgroup of $\mathfrak{M}$ leaving K elementwise fixed. Then the spectral sequence $'E$ is isomorphic to one of the Hochschild-Serre sequences* [10, Proposition 7]

$$\dot{H}^m(\mathfrak{M}/\mathfrak{H}, \dot{H}^n(\mathfrak{H}, F^*)) \underset{m}{\Longrightarrow} \dot{H}^r(\mathfrak{M}, F^*).$$

NOTE. In the infinite case we take the obvious generalization of the usual Hochschild-Serre spectral sequence.

Proof. As already noted we have an isomorphism of double complexes

$$\nu: \mathfrak{C}(K, F/C) \to C_c(\mathfrak{G}, (K\otimes F^{n+1})^*)$$

where $\mathfrak{G}\cong\mathfrak{M}/\mathfrak{H}$. Suppose first that $[F:C]<\infty$. Then it is shown in [2, Lemma 5. 5] that the map Θ, defined by $\Theta(\sum k\otimes f_1\otimes \cdots \otimes f_{n+1})(\sigma_1, \sigma_2, \cdots, \sigma_{n+1}) = \sum k\Pi\sigma_i(f_i)$, k in K, f_i in F, σ_i in $\mathfrak{M}$, yields an isomorphism of $(K\otimes F^{n+1})^*$ with $\mathrm{Hom}_{C(\mathfrak{H})}(\overset{n+1}{\underset{1}{\otimes}} C(\mathfrak{M}), F^*)$; the latter in the notation of [10] is simply $M_n^{\mathfrak{H}}$ with $M_n=\mathrm{Hom}_C(\overset{n+1}{\underset{1}{\otimes}} C(\mathfrak{M}), F^*)$. Moreover it is readily verified as in [12] that Θ is an isomorphism of the complex $C(K\otimes F/K)$ with the complex structure of $M^{\mathfrak{H}}$ as defined in [10]. Now if Φ is a function in $M_n^{\mathfrak{H}}$, an $\mathfrak{M}/\mathfrak{H}$ structure on $M_n^{\mathfrak{H}}$ is defined in [10] by setting $(\bar{\tau}\Phi)(\sigma_1, \cdots, \sigma_{n+1})=\tau\Phi(\tau^{-1}\sigma_1, \cdots, \tau^{-1}\sigma_{n+1})$ for $\tau, \sigma_1, \cdots, \sigma_{n+1}$ in $\mathfrak{M}$. But then $\Theta(\tau(\sum k\otimes f_1\otimes \cdots \otimes f_{n+1}))=\Theta(\sum\tau(k)\otimes f_1\otimes \cdots \otimes f_{n+1})=\Phi$ with $\Phi(\sigma_1, \cdots, \sigma_{n+1})=\sum\tau(k)\Pi\sigma_i(f_i)$, and $[\bar{\tau}\Theta(\sum k\otimes f_1\otimes\cdots\otimes f_{n+1})]\ (\sigma_1, \cdots, \sigma_{n+1})= \tau\Theta(\sum k\otimes f_1\otimes\cdots\otimes f_{n+1})\ (\tau^{-1}\sigma_1, \cdots, \tau^{-1}\sigma_{n+1})=\sum\tau(k)\Pi\sigma_i(f_i)$ so that Θ is an isomorphism even of $\mathfrak{M}/\mathfrak{H}$ complexes. We thus have shown that in the finite case $\mathfrak{C}(K, F/C)$ is isomorphic as a double complex to the double complex constructed in [10, Proposition 7]. In the infinite case it is clear that $(K\otimes F^{n+1})^*\cong\lim_{\rightarrow}\mathrm{Hom}_{C(\mathfrak{H}_i)}(\overset{n+1}{\underset{1}{\otimes}} C(\mathfrak{M}_i), F_i^*)$ where $F_i\supset K_i$ are the normal separable finite extensions of C with Galois groups $\mathfrak{M}_i$ and $\mathfrak{H}_i$, whose union is F and K respectively. Thus $E_0^{m,n}\cong C_c^m(\mathfrak{G}, \lim_{\rightarrow}M_i^{\mathfrak{H}_i})$. Now since the homology functor commutes with the direct limit functor [6, Proposition 9. 3*, p. 100], we see by repeating the proof of Proposition 7 of [10] that

$$'E_2^{m,n} \cong \dot{H}^m(\mathfrak{M}/\mathfrak{H}, \dot{H}^n(\mathfrak{H}, F^*)).$$

Finally, as in the proof of Theorem 2. 2, $H^r(F/C)\cong\dot{H}^r(\mathfrak{M}, F^*)$ so that the Corollary is proved.

3. Inseparable extensions

In this last section we find an exact sequence which complements that of section 7 of [13]: Indeed, let F be an algebraic field extension of the field C and let K be the maximal separable subfield of F. In [13] some relations between $H^n(K/C)$ and $H^n(F/C)$ were found by using a map on cohomology which resulted from raising to p^{th} powers. In the present section we shall use the lift map $\lambda: H^n(K/C) \to H^n(F/C)$. Unfortunately, we only obtain results in the special case $F = K \otimes F_i$ with F_i purely inseparable over C.

Lemma 3.1. *Let C be a field, K an algebraic separable extension field of C, F_i a purely inseparable extension field of C, and let $F = K \otimes F_i$ be an extension field of K.*[4] *Then ${}'E_1^{m,n} = 0$ if $n \neq 0$ or 2, where ${}'E_1^{m,n}$ still refers to the first spectral sequence of the double complex $(K^{m+1} \otimes F^{n+1})^*$, $m, n = 0, 1, 2, \cdots$.*

Proof. Since K is separable algebraic we may write $K = \bigcup K_\alpha$ and $F_i = \bigcup F_{\alpha i}$ with K_α a finite separable extension field and $F_{\alpha i}$ a purely inseparable extension field of finite exponent. Then $F = \bigcup F_\alpha$ with $F_\alpha = K_\alpha \otimes F_{\alpha i}$. By definition, ${}'E_1^{m,n} = H^n(K^{m+1} \otimes F/K^{m+1})$. Now as was noted in [13, Proof of Corollary 3.5] $H^n(K^{m+1} \otimes F/K^{m+1}) = \lim_{\to} H^n(K_\alpha^{m+1} \otimes F_\alpha / K_\alpha^{m+1})$, so that it is sufficient to prove the lemma in case $[K:C] < \infty$ and F has finite exoponent. In that case it is well known that K^{m+1} is a direct sum of fields, $K^{m+1} = \sum_\oplus L_j$, each of which is a K-algebra. Consequently, $\mathfrak{C}(K^{m+1} \otimes F/K^{m+1})$ is isomorphic to the direct product of the complexes $\mathfrak{C}(L_j \otimes F/L_j)$. Now the maps $h: L_j \otimes F = L_j \otimes K \otimes F_i \to L_j \otimes F_i$ and $g: L_j \otimes F_i \to L_j \otimes F$ defined by $h(l \otimes k \otimes f) = lk \otimes f$ and $g(l \otimes f) = l \otimes 1 \otimes f$ for l in L_j, k in K, and f in F_i are clearly L_j-algebra homomorphisms. They thus give rise to induced homomorphisms $h^*: H^n(L_j \otimes F/L_j) \to H^n(L_j \otimes F_i/L_j)$ and $g^*: H^n(L_j \otimes F_i/L_j) \to H^n(L_j \otimes F/L_j)$. Since hg is an L_j-algebra homomorphism of $L_j \otimes F_i$ to $L_j \otimes F_i$ and gh is an L_j-algebra homomorphism of $L_j \otimes F$ to $L_j \otimes F$ it follows from [2, Lemma 2.7] that hg and gh are homotopic to the identity so that $h^* g^* = g^* h^* = 1$ and h^*, g^* are isomorphisms. As already noted, $L_j \otimes F_i$ is a field and clearly a purely inseparable extension of L_j. But then by [13, Theorem 6.1] $H^n(L_j \otimes F_i/L_j) = 0$ $n \neq 0, 2$,[5] completing the proof.

Theorem 3.2. *Let C be a field, K a separable algebraic extension field of C, F_i a purely inseparable extension field of C, and let $F = K \otimes F_i$.*

4) If $[K:C]$ and $[F_i:C]$ are finite then it is shown in [1, Theorem 2.31] that $F = K \otimes F_i$ is a field. The general case follows immediately from this.

Then there is an exact sequence

$$\cdots {}'E_2^{r-3,2} \to H^r(K/C) \xrightarrow{\lambda} H^r(F/C) \to {}'E_2^{r-2,2} \to \cdots$$

with $'E_2^{m,n}$ *denoting the* E_2 *term of the first spectral sequence of* $\mathfrak{C}(K, F/C)$.

Proof. By Lemma 3.1, $'E_2^{m,n}=0$ if $n\neq 0, 2$. By [9, Theorem 4.6.2, p. 85] with $n=r=2$, there is an exact sequence

$$\cdots {}'E_2^{r-3,2} \to {}'E_2^{r,0} \xrightarrow{\xi^*} H^r(K, F/C) \to {}'E_2^{r-2,2} \to \cdots$$

with ξ^* the edge homomorphism. By Lemma 1.1 we have an isomorphism of $(K^{r+1})^*$ with $'E_1^{r,0}$ and consequently $'E_2^{r,0}\cong H^r(K/C)$. By Lemma 1.2, $H^r(K, F/C)\cong H^r(F/C)$ and then an appeal to Proposition 1.6 yields the desired result.

Corollary 3.3. *If K is normal separable (possibly infinite dimensional) with Galois group* $\mathfrak{G}$, *there is an exact sequence*

$$\cdots \dot{H}^{r-3}(\mathfrak{G}, H^2(K\otimes F/K)) \to H^r(K/C) \xrightarrow{\lambda} H^r(F/C)$$
$$\to \dot{H}^{r-2}(\mathfrak{G}, H^2(K\otimes F/K)) \to \cdots$$

Proof. This follows immediately from Theorems 2.2 and 3.2.

Corollary 3.4. *Let* $\hat{C}$ *be an algebraic closure of the field C and let* C_s *be the separable algebraic closure of C in* $\hat{C}$. *Then* $\lambda: H^r(C_s/C)\to H^r(\hat{C}/C)$ *is an isomorphism for* $r=0, 1, 2, \cdots$.

Proof. Let C_i be the maximal purely inseparable extension field of C in $\hat{C}$, i.e. $C_i=C^{p^{-\infty}}$ in the notation of [4]. It is then well known that $\hat{C}\cong C_s\otimes C_i$, although this doesn't seem to have been noted in the literature explicitly. However, by [4, Proposition 1, p. 127 and Theorem 2, p. 119] it follows that C_i and C_s are linearly disjoint over C so that $C_i\otimes C_s$ is isomorphic to a subfield $\tilde{C}$ of C. Furthermore, if N is any finite normal subextension of $\hat{C}$, N_i its maximal purely inseparable subfield, and N_s its maximal separable subfield it follows by [15, Corollary 3, p. 74] and [1, Theorem 2.31, p. 34 and Lemma 7, p. 102] that $N=N_sN_i$. Thus $\tilde{C}=\hat{C}$.

Hence Corollary 3.3 with $K=C_s$, $F=\hat{C}$ is applicable. Now since $\hat{C}\cong C_s\otimes C_i$ it is clear that there is an isomorphism of $\hat{C}$ into $C_s\otimes\hat{C}$ which

5) As we noted earlier the present H^0 and that of [13] are different which explains why in [13], $H^0(L_j\otimes F_i/L_j)=0$, but not here.

sends C_s identically onto $C_s\otimes 1$. There also is of course a $C_s\otimes 1$-algebra homomorphism of $C_s\otimes\hat{C}$ to $\hat{C}$ given by the usual contraction. Hence, $H^2(C_s\otimes\hat{C}/C_s)\cong H^2(\hat{C}/C_s)$ [2, Theorem 2.8]. Finally it follows by [12, Corollary 3.17] that $H^2(\hat{C}/C_s)$ is the Brauer group of central simple C_s-algebras. Since C_s is separably algebraicly closed and a central simple algebra always has a separable splitting field, it follows that this Brauer group, and hence $H^2(\hat{C}/C_s)$, is 0. Corollary 3.3 completes the proof.

REMARKS. In a letter to the authors the Corollary has also been proved by S. Shatz using the results of [13] on inseparable fields and a spectral sequence from [3]. The fact that $H^r(\hat{C}/C_s)=0$ for $r>0$ is a very special case of Theorem 4.9 p. 93 of [3].

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