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UNSTABLE HOMOTOPY GROUPS OF UNITARY GROUPS
(odd primary components)

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1. Introduction

The purpose of this paper is to prove the following

Theorem. *For each odd prime* p,

$$^{p}\pi_{2n+2k-3}(U(n)) = Z_{p^N}$$

for $k \leqq p(p-1)$, $n > k$ *and* $n+k \equiv 0 \mod p$, *where* $N = \min\left(\left[\frac{k-1}{p-1}\right], \nu_p(n+k)\right)$ *and* $\nu_p(x)$ *is the highest exponent of* p *dividing the integer* x.

This theorem contains one of the result of [5] as a special case. We shall use the following well-known isomorphism.

$$\begin{aligned}\pi_{2n+2k-3}(U(n)) &\approx \pi_{2n+2k-2}(EP_{n+k}/EP_n) \quad \text{for} \quad n \geqq k-2 \ [8] \\ &\approx \pi_{2n+2k-2}(E(P_{n+k,k})) \\ &\approx \pi_{2n+2k-3}(P_{n+k,\,k}) \quad \text{for} \quad n>k \ [4],\end{aligned}$$

where E is the suspension, P_m $(m-1)$ complex dimensional projective space, EP_{n+k}/EP_n or $P_{n+k,\,k}$ the space obtained from EP_{n+k} or P_{n+k} by smashing the subcomplex EP_n or P_n to a point.

In §2 we recall some material from the homotopy theory of the sphere and the K-theory, and deduce some results which are used in §3. In §3 we prove the Theorem.

2. Preliminary material

2.1. Denote by $\alpha_{n+k,\,r}$ the coefficient of x^{n+k-1} in $(e^x-1)^{n+k-r}$ for $1 \leqq r \leqq t$. For any non zero rational number x, if $x = p^r \cdot q^s \cdots$ is the factorization of x into prime powers, we define $\nu_p(x) = r$. By (5.3), (5.4), (6.4) and (6.5) in [1], if $\nu_p(\alpha_{n+k,\,r}) \geqq 0$ for $1 \leqq r \leqq t$ and a fixed prime p, then we have that $\nu_p(\alpha_{n+k,\,t+1}) \geqq 0$ with the exceptional case $t = s(p-1)$, and in this case, $\nu_p(\alpha_{n+k,\ t+1})\geqq 0$ if and only if $\nu_p(n+k)-\nu_p(s)-s\geqq 0$.

2.2. In the present work we discuss only such finite CW-complexes K that consisting only of even dimensional cells, at most one for each even dimension. So we make this assumption without any more comments. Then $H^n(K, Z)=Z$ or 0, and the n-cell e_n, if it exists, is the generator and, for any coefficient group G, the element αe_n of $H^n(K, G)$ determines uniquely $\alpha\in G$, we shall identify $\alpha\cdot e_n$ and α as our convention.

Now consider two finite CW-complexes X and X'. If a mapping $f: X'\to X$ induces isomorphisms $f^*: H^*(X, Z_p)\overset{\approx}{\to} H^*(X', Z_p)$ for a fixed prime p, then we have that

(i) it induces the isomorphism $f_p^!: K(X)\otimes Z_p\to K(X')\otimes Z_p$,

and

(ii) $\nu_p\,\mathrm{ch}_n(\lambda)=\nu_p\,\mathrm{ch}_n(f^!\cdot\lambda)$ for any λ of $K_c(X)$.

Proof. Since $H^{2n+1}(X, Z)=H^{2n+1}(X', Z)=0$ for each n, using 2.1 in [2] we have that

$$H^{2n}(X, Z)\cong K_{2n}(X)/K_{2n+1}(X)\,,\quad K_{2n-1}(X)=K_{2n}(X)\,,$$

and

$$H^{2n}(X', Z)\cong K_n(X')/K_{2n+1}(X'),\quad K_{2n-1}(X')=K_{2n}(X')\,,$$

where $K_m(X)=\ker[K(X)\to K(X^{m-1})]$, X^{m-1} is the $(m-1)$-skeleton of X, and for $K_m(X')$ we make the same convention. Then f^* induces the isomorphism $\bar{f}_p: H^n(X, Z)\otimes Z_p\to H^n(X', Z)\otimes Z_p$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0\to & K_{2n+1}(X)\otimes Z_p & \to & H^{2n}(X, Z)\otimes Z_p & \to & K_{2n}(X)\otimes Z_p & \to 0 \\ & \downarrow \bar{f}^{n+1} & & \downarrow \bar{f} & & \downarrow \bar{f}^n & \\ 0\to & K_{2n+1}(X')\otimes Z_p & \to & H^{2n}(X', Z)\otimes Z_p & \to & K_{2n}(X')\otimes Z_p & \to 0\,, \end{array}$$

where the horizontal sequences are exact. If $\bar{f}^{n+1}$ and $\bar{f}$ are isomorphisms then $\bar{f}^n$ is an isomorphism. By descending induction on n we complete the proof of (i). The relation (ii) follows from the naturality of ch and that $f^*e_n\not\equiv 0 \bmod p$.

2.3. In a complex of two cells $X=S^{2m}\bigcup_f e^{2m+2s(p-1)}$ $(1\leqq s\leqq p)$ where f belongs to an element of the p-primary component of the stable homotopy group of the sphere, by (3.13) in [7] III, Theorem 4, Lemma 3 in [6], Theorem 1 in [3], 2.2 above, and (4.13) in [7] IV, we have that for any bundle λ of $K_c(X)$, $\nu_p(\mathrm{ch}_{m+s(p-1)}(\lambda))\geqq 0$ if and only if f is inessential.

2.4. Take the stunted projective space $P_{n+k,\ k}$ such that $k\leqq p(p-1)$.

By (4.13) in [7] IV there exists a CW-complex $P'_{n+k,k}$ consisting of one cell for each degree $2s$, $n\leqq s\leqq n+k-1$, and a mapping $f:P'_{n+k,k}\to P_{n+k,k}$ such that f induces isomorphisms $f^*:H^*(P_{n+k,k},Z_p)\to H^*(P'_{n+k,k},Z_p)$ and the order of the homotopy boundary of each cell of $P'_{n+k,k}$ is a power of p. Then the complex $P'_{n+k,k}$ has the following cell structure.

$$P'_{n+k,k}=[\bigvee_{i=0}^{l}(S^{2n+2i}\cup e^{2n+2i+2(p-1)}\cup\cdots\cup e^{2n+2i+2q(p-1)}]$$
$$\vee[\bigvee_{j=l+1}^{p-2}(S^{2n+2j}\cup e^{2n+2j+2(p-1)}\cup\cdots\cup e^{2n+2j+2(q-1)(p-1)}],$$

where we denote by $\vee$ the union with a single common point and set $k=q(p-1)+l+1$ for $0\leqq l\leqq p-2$ and $q<p$. Using the formula in §1 and $\mathcal{C}$-theory (Serre) we have

$${}^p\pi_{2n+2k-3}(U(n))\approx{}^p\pi_{2n+2k-3}(S^{2n+2l}\cup\cdots\cup e^{2n+2l+2q(p-1)}).$$

2.5. Let ξ be the dual bundle to the canonical line bundle over P_{n+k}. It is well-known that $\tilde{K}(P_{n+k})$ is a truncated polynomial ring over the integer with the generator $\tilde{\xi}=\xi-1$ and a single relation $\tilde{\xi}^{n+k}=0$.

Consider the following exact sequence

$$0\to\tilde{K}(P_{n+k,k})\xrightarrow{p^!}\tilde{K}(P_{n+k})\xrightarrow{i^!}\tilde{K}(P_n)\to 0,$$

where $i^!$ and $p^!$ are induced by the injection and the projection respectively. Define the elements of $\tilde{K}(P_{n+k,k})$ by $p^!\xi_i=\tilde{\xi}^i$ $n\leqq i\leqq n+k-1$ It is well-known that $H^*(P_{n+k,k})$ is a Z-module with generators $x_n,\cdots,x_{n+k-1}$, where $p^*x_i=x^i$ $n\leqq i\leqq n+k-1$, and x is the chern class of $\tilde{\xi}$. Then $\pm\alpha_{n+k,r}=\mathrm{ch}_{n+k-1}(\xi_{n+k-r})$ for $1\leqq r\leqq t$.

Now we suppose that under the condition $\nu_p(\alpha_{n+k,r})\geqq 0$ for $1\leqq r\leqq t$ and $t=s(p-1)$ $(s<p)$ the homotopy boundary of the $2(n+k-1)$-cell in $P'_{n+k,s(p-1)+1}$ is deformable into its $2(n+k-s(p-1)-1)$-skeleton. Then we may regard a complex $S^{2(n+k-s(p-1)-1)}\cup e^{2(n+k-1)}$ as a subcomplex of $P'_{n+k,s(p-1)+1}$ up to homotopy equivalence. Denote by P'' the complex obtained from $P'_{n+k,s(p-1)+1}$ by smashing the subcomplex $S^{2(n+k-s(p-1)-1)}\cup e^{2(n+k-1)}$, say $S\cup e$, to a point. The commutative diagram

$$\begin{array}{ccccccc}
0\to\tilde{K}(P'')&\to&\tilde{K}(P_{n+k,s(p-1)+1})&&\to&\tilde{K}(S\cup e)\to 0\\
\downarrow\mathrm{ch}_{n+k-1}&&\downarrow\mathrm{ch}_{n+k-1}&&&\downarrow\mathrm{ch}_{n+k-1}\\
0&\to&H^{2(n+k-1)}(P'_{n+k,s(p-1)+1},Q)&\xrightarrow{\approx}&&H^{2(n+k-1)}(S\cup e,Q)
\end{array}$$

shows that

$$\nu_p(\mathrm{ch}_{n+k-1}\tilde{K}(P'_{n+k,s(p-1)+1}))\geqq 0$$

if and only if

$$\nu_p(\mathrm{ch}_{n+k-1}\tilde{K}(S^{2(n+k-s(p-1)-1)}\cup e^{2(n+k-1)}))\geqq 0\,.$$

On the other hand by 2.2 we see that

$$\nu_p(\mathrm{ch}_{n+k-1}\tilde{K}(P_{n+k,\,s(p-1)+1}))\geqq 0$$

if and only if

$$\nu_p\mathrm{ch}_{n+k-1}\tilde{K}(P'_{n+k,\,s(p-1)+1}))\geqq 0\,.$$

Then 2.1 and 2.3 show that the homotopy boundary $\beta e^{2(n+k-1)}$ in $P'_{n+k,\,s(p-1)+1}$ is trivial if and only if $\nu_p(n+k)-s\geqq 0$.

3. Proof of the Theorem

Consider a CW-complex $X=S\cup e_1\cup e_2\cup\cdots\cup e_m$, where S is an N-sphere, N even, e_i $(1\leqq i\leqq m)$ are $(N+2i(p-1))$-cells and $m<p$. Through out this section we denote by $\pi(K)$ the p-primary component of $(N+2q(p-1)-1)$-th homotopy group of K and suppose $N>2q(p-1)$. Later in this section we prove the following

Proposition 3.1. *If, for a generator S of the group $H^N(X, Z_p)$, $\mathfrak{P}_p^i S\neq 0$ for $1\leqq i\leqq m$, and $m<q<p$, then we have*

$$\pi(X)=Z_{p^{m+1}}$$

From this Proposition follows the

Proposition 3.2. *For $m=q$, if the homotopy boundary of the cell e_q in the complex X, say α, is deformable into the N-skeleton S (then $S\underset{\alpha}{\cup} e_q$ can be regarded as a subcomplex of X up to homotopy equivalence), and if $\mathfrak{P}_p^i S\neq 0$ for $1\leqq i\leqq q-1$, then we have that*

$$\pi(X)=\begin{cases} Z_{p^{q-1}} & \text{if the } p\text{-primary component of } \alpha \text{ is not zero}\\ Z_{p^q} & \text{if the } p\text{-primary component of } \alpha \text{ is zero.}\end{cases}$$

Proof. If the p-primary component of α is not zero we have $\pi(S\underset{\alpha}{\cup} e_q)=0$. Consider the following exact sequence

$$0\to\pi(Z\cup e_q)\to\pi(X)\to\pi(X, S\cup e_q)\to 0\,.$$
$$\approx$$
$$\pi(X/S\underset{\alpha}{\cup} e_q)$$

By the Adem relation we see easily that the complex $X/S\underset{\alpha}{\cup} e_q$ satisfies the condition of 3.1 for $q-1$. Then by 3.1 we have $\pi(X)=Z_{p^{q-1}}$. If the p-primary component of α is zero, we have

$$\begin{aligned}\pi(X) \approx \pi((S\cup e_1 \cup\cdots\cup e_{q-1})\vee S_q) &\cong \pi(S\cup e_1 \cup\cdots\cup e_{q-1}) \\ &= Z_{p^q},\end{aligned}$$

where S_q is the $(N+2q(p-1))$-sphere.

Now we state Proposition 3.3, by which and by 2.5, the proof of the Theorem are completed because the conditions about $\mathfrak{P}_p^i$ are easily checked from the known cohomological structure about the complex projective space.

Proposition 3.3. *For $m=q$, if the homotopy boundary βe^q in X is deformable into the $(N+2(q-s-1)(p-1))$-skeleton and not deformable into $(N+2(q-s-2)(p-1))$-skeleton (the complex $S\cup e_1 \cup\cdots\cup e_{q-s-1}\cup e_q$ can be regarded as a subcomplex of X) and $\mathfrak{P}_p^i\ S\neq 0$ for $1\leqq i\leqq q-1$, then we have*

$$\pi(X) = Z_{p^s}.$$

To prove the Propositions 3.1 and 3.3 we use the following

Lemma. *In a complex $S^N \underset{\alpha}{\cup} e^{N+2(p-1)}$, $N>2s(p-1)$, if the p-primary component of α is not zero, then we have*

$${}^p\pi_{N+2s(p-1)-1}(S^N \underset{\alpha}{\cup}{}^{N+2(p-1)}) = Z_{p^2} \quad \textit{for} \quad 2\leqq s\leqq p-1.$$

Proof of 3.1. We prove this proposition by induction on m. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
0 & & \pi(S\cup e_1\cup\cdots\cup e_{m-1}) & & \pi(S_1\cup\cdots\cup e_m) & & \\
\downarrow & & \downarrow & & \approx & & \\
\pi(S) & \xrightarrow{i_1} & \pi(X) & \xrightarrow{p_1} & \pi(X, S) & \longrightarrow & 0 \\
\downarrow i & & \downarrow & & \downarrow & & \\
\pi(S\cup e_1) & \xrightarrow{i_2} & \pi(X) & \xrightarrow{p_2} & \pi(X, S\cup e_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow\approx & & \\
\pi(S_1) & & \pi(S_m) & & \pi(S_2\cup\cdots\cup e_m), & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where $S_i\cup e_{i+1} \cup\cdots\cup e_m$ denotes the complex obtained form the complex $S\cup e_1 \cup\cdots\cup e_m$ by smashing a subcomplex $S\cup e_1 \cup\cdots\cup e_{i-1}$ to a point. Two vertical and horizontal sequences are exact. By the Adem relation we see easily that the complexes $S_1\cup\cdots\cup e_m$ and $S_2\cup\cdots\cup e_m$ satisfy the conditions of 3.1 for $m-1$ and $m-2$ respectively. Hence $\pi(S_1\cup\cdots\cup e_m)$

$=Z_{p^m}$ and $\pi(S_2 \cup \cdots \cup e_m)=Z_{p^{m-1}}$ by induction hypothesis. The middle vertical exact sequence takes the form

$$0 \to Z_{p^m} \to \pi(X) \to Z_p \to 0\,.$$

Therefore $\pi(X)=Z_{p^{m+1}}$ or $Z_{p^m} \oplus Z_p$.

If we suppose that $\pi(X)=Z_{p^m}\oplus Z_p$, the exactness of the upper horizontal sequence shows that i_1-image must be the second direct factor, which is impossible because

$$\begin{aligned} i_1(\pi(S)) &= i_2 \circ i(\pi(S)) \\ &= i_2(p\pi(S \cup e_1))) \\ &= pi_2(\pi(S \cup e_1))\,. \end{aligned}$$

Then $\pi(X)=Z_{p^{m+1}}$. q. e. d.

Proof of 3. 3. Put $S \cup e_1 \cup \cdots \cup e_{q-s-1} \cup e_q / S \cup e_1 \cup \cdots \cup e_{i-1} = Y_i$ for $0 \leqq i \leqq q-s-1$ $(Y_0 = S \cup e_1 \cup \cdots \cup e_{q-s-1} \cup e_q)$. By decending induction on i, we shall prove that

$$(*) \qquad \pi(Y_i) = 0 \quad \text{for} \quad 0 \leqq i \leqq q-s-1\,.$$

By the assumption of the proposition we have $(*)$ for $i=q-s-1$. Assume that $(*)$ is true for $0 \leqq k < i \leqq q-s-1$ and consider the following commutative diagram

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ \pi(S_k) & \to & \pi(Y_k) & \to & \pi(Y_{k+1}) \\ \downarrow & & \| & & \\ \pi(S_k \cup e_{k+1}) & \to & \pi(Y_k) & \to & \pi(Y_{k+2}) \\ \downarrow & & & & \\ \pi(S_{k+1}) & & & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

The left vertical sequence and the two horizontal sequences are exact. By induction hypothesis the two right terms are zero. Then the same argument as in the above proof of 3. 1, making use of the lemma shows that $\pi(Y_k)=0$. Especially we obtained that

$$\pi(S \cup e_1 \cup \cdots \cup e_{q-s-1} \cup e_q) = 0\,.$$

The exact sequence

$$\begin{array}{l} \pi(S \cup e_1 \cup \cdots \cup e_{q-s-1} \cup e_q) \to \pi(X) \to \pi(X, S \cup e_1 \cup \cdots \cup e_{q-s-1} \cup e_q) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \approx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \pi(S_{q-s} \cup \cdots \cup e_{q-1}) \end{array}$$

shows that $\pi(X)=\pi(S_{q-s}\cup\cdots\cup e_{q-1})$ and the group is isomorphic to Z_{p^s} because the Adem relation proves that the space $S_{q-s}\cup\cdots\cup e_{q-1}$ satisfies the conditions of 3.1. q.e.d.

Proof of the lemma. At first we summarize some well-known results. By the Adem relation, if $i<p$, we have

$$\mathfrak{P}_p^i\mathfrak{P}_p^j=\binom{i+j}{i}\mathfrak{P}_p^{i+j} \tag{1}$$

$$\mathfrak{P}_p^i\Delta_p^1\mathfrak{P}_p^j=\binom{i+j-1}{i}\Delta_p^1\mathfrak{P}_p^{i+j}+\binom{i+j-1}{j}\mathfrak{P}_p^{i+j}\Delta_p^1 \tag{2}$$

Consider the following exact sequences

$$0\to Z_{p^h}\to Z_{p^{h+1}}\to Z_p\to 0 \tag{3}$$

$$0\to Z_p\to Z_{p^{h+1}}\to Z_{p^h}\to 0\,. \tag{4}$$

The coboundary operators associated with (3), (4) are denoted by δ_h, δ'_h respectively. In [9] (§2.1) the cohomology operations Δ_p^i $(1\leqq i)$ are defined:

$$\Delta_p^h:\Delta_p^{h-1}\text{-kernel}\,(\subset H^{n-1}(X,Z_p))\to H^n(X,Z_p)\bmod\delta'_{h-1}\text{-image},$$

then, the following relations hold:

$$\Delta_p^h\text{-kernel}=\delta_h\text{-kernel},\ \Delta_p^h\text{-image}=\delta'_h\text{-image}/\delta'_{h-1}\text{-image}.$$

Let $F\to E\to B$ be a Serre fiber space with base space B $l(>1)$-connected and fiber F $m(>1)$-connected, and $n<l+m+2$, then we have the following exact sequence

$$0\to H^1(B,Z_p)\xrightarrow{p^*}H^1(E,Z_p)\xrightarrow{i^*}H^1(F,Z_p)\to\cdots$$
$$\to H^n(B,Z_p)\xrightarrow{p^*}H^n(E,Z_p)\xrightarrow{i^*}H^n(F,Z_p)\,.$$

Let α and β be respectively elements of $H^s(E,Z_p)$ and of $H^{s+1}(B,Z_p)$ such that $\delta_{r-1}(\alpha)=0$ and $\Delta_p^r(\alpha)=p^*(\beta)\bmod\delta'_{r-1}$-image. Then by [9] Th. 3.2

$$\tau\cdot\Delta_p^{r+1}i^*(\alpha)=-\Delta_p^1(\beta)\bmod\tau\cdot\delta'_rH^s(F,Z_{p^r}) \tag{5}$$

Let α, β and γ be respectively elements of $H^s(E,Z_p)$, of $H^{s+1}(B,Z_p)$ and of $H^s(B,Z_p)$ such that $\Delta_p^r(\alpha)=p^*(\beta)$ $(r\geqq 2)$ and $\alpha=p^*(\gamma)$, then by [9] Th. 3.8, there exists an element ε of $H^s(F,Z_p)$ with the following properties:

(6) $$\tau(\mathcal{E}) = \Delta_p^1(\gamma),$$
$$\tau\Delta_p^r(\mathcal{E}) = \Delta_p^1(\beta) \bmod \tau\delta'_{r-1}H^s(F, Z_{p^{r-1}}).$$

To prove the lemma we consider the Cartan-Serre fiber space

$$X(N+2(p-1)) \to X \to K(Z, N)$$

for $X=S \cup e_1$, and the associated exact sequence, where $X(r)$ is $(r-1)$-connected and ${}^p\pi_i(X(r))={}^p\pi_i(X)$ $i \geqq r$.

$$0 \to H^N(Z, N, Z_p) \xrightarrow{p^*} H^N(X, Z_p) \xrightarrow{l^*} H^N(X(N+2(p-1))Z_p) = 0 \cdots$$
$$\xrightarrow{\tau} H^{N+2(p-1)}(Z, N, Z_p) \xrightarrow{p^*} H^{N+2(p-1)}(X, Z_p) \xrightarrow{i^*} H^{N+2(p-1)}(X(N+2(p-1))$$
$$\xrightarrow{\tau} H^{N+2(p-1)+1}(Z, N, Z_p) \xrightarrow{p^*} H^{N+2(p-1)+1}(X,Z_p) = \to 0 \cdots$$
$$0 \to H^{N+4(p-1)-1}(X(N+2(p-1), Z_p) \xrightarrow{\tau} H^{N+4(p-1)}(Z, N, Z_p) \to 0$$

Then there exist elements a_1 and b_1 of $H^{N+2(p-1)}(X(N+2(p-1)), Z_p)$ and of $H^{N+4(p-1)-1}(X(N+2(p-1)), Z_p)$ such that $\tau a_1=\Delta_p^1\mathfrak{P}_p^1u_1$ and $\tau b_1=\mathfrak{P}_p^2u_1$, where u_1 is the generator of $H^N(Z, N, Z_p)$. Since $H^i(X, Z_p)=0$ for $i>N+2(p-1)$ we have that the transgression $\tau: H^{N+i}(X(N+2(p-1)), Z_p) \to H^{N+i+1}(Z, N, Z_p)$ are isomorphic onto for $N+2(p-1) \leqq i < 2N-1$. Then we have relations:

(3. 1. 1) $$\Delta_p^1 b_1 = \mathfrak{P}_p^1 a_1$$

(3. 1. 2) $$2\Delta_p^1\mathfrak{P}_p^{i-2}b_1 = i\mathfrak{P}_p^{i-2}\Delta_p^1 b_1 = i(i-1)\mathfrak{P}_p^{i-1}a_1 \quad \text{for} \quad 2 \leqq i \leqq p.$$

Next consider the Cartan-Serre fiber space

$$X(N+4(p-1)-1) \to X(N+2(p-1)) \to K(Z,N+2(p-1))$$

and the associated exact sequence

$$0 \to H^{N+2(p-1)}(Z,N+2(p-1), Z_p) \xrightarrow{p^*} H^{N+2(p-1)}(X(N+2(p-1)),Z_p) \to 0$$
$$\cdots \to H^{N+4(p-1)-1}(X(N+2(p-1)), Z_p) \xrightarrow{i^*} H^{N+4(p-1)-1}(X(N+4(p-1)-1, Z_p)$$
$$\xrightarrow{\tau} H^{N+4(p-1)}(X,N+2(p-1), Z_p) \xrightarrow{p^*} H^{N+4(p-1)}(X(N+2(p-1)), Z_p)$$
$$\xrightarrow{i^*} H^{N+4(p-1)}(X(N+4(p-1)-1), Z_p) \xrightarrow{\tau} H^{N+4(p-1)+1}(Z, N+2(p-1), Z_p) \to \cdots$$

Denote by u_2 the generator of $H^{N+2(p-1)}(Z, N+2(p-1), Z_p)$ and by b_2 the i^*-image of b_1. Since $p^*u_2=a_1$, we have

(3. 2. 1) $$\tau\Delta_p^2 b_2 = -\Delta_p^1\mathfrak{P}_p^1 u_2,$$

by (3. 1. 1) and (5) above, and

$$\Delta_p^2 \mathfrak{P}_p^{i-2} b_2 = \frac{i(i-1)}{2} \mathfrak{P}_p^{i-1} \Delta_p^2 b_2 \quad \text{for} \quad 2 \leqq i < p. \tag{3.2.2}$$

by (3. 1. 2). Thus we have

$$ {}^p\pi_{N+4(p-1)-1}(X) = Z_{p^2}. \tag{3.2.3}$$

When $p=3$ the proof is completed. When $p>3$, we shall prove the following assertions (A_l) and (B_l) for $2 \leqq l \leqq p-1$ by induction on l at the same time:

$$(A_l) \qquad {}^p\pi_{N+2l(p-1)-1}(X) = Z_{p^2},$$

denoting by b_l a generator of $H^{N+2l(p-1)-1}(X(N+2l(p-1)-1), Z_p)$ there holds the following relation

$$(B_l) \qquad \Delta_p^2 \mathfrak{P}_p^{i-l} b_l = \mathcal{E}(l, i) \mathfrak{P}_p^{i-l} \Delta_p^2 b_l \neq 0 \quad \text{for} \quad p > i \geqq l$$
$$\text{with} \quad \mathcal{E}(l, i) \in Z_p.$$

The case for $l=2$ is proved by (3. 2. 2) and (3. 2. 3). Assume (A_l) ane (B_l), and consider the Cartan-Serre fiber space

$$X(N+2(l+1)(p-1)-1) \xrightarrow{i} X(N+2l(p-1)-1) \xrightarrow{p} K(Z_{p^2}, N+2l(p-1)-1).$$

Denote by u_{l+1} and by b_{l+1} generators of $H^{N+2l(p-1)-1}(Z_{p^2}, N+2l(p-1)-1, Z_p)$ and $H^{N+2(l+1)(p-1)-1}(X(N+2(l+1)(p-1)-1, Z_p)$. Since $p^* u_{l+1} = b_l$ and $\Delta_p^1 \mathfrak{P}_p^1 b_l = 0$, we have $\tau b_{l+1} = \Delta_p^1 \mathfrak{P}_p^1 u_{l+1}$. By (B_l), $\Delta_p^2 \mathfrak{P}_p^1 b_l = \mathcal{E}(l, l+1) \mathfrak{P}_p^1 \Delta_p^2 b_l$, hence by (6) the relation

$$(C_{l+1}) \qquad \tau \Delta_p^2 b_{l+1} = \mathcal{E}(l, l+1) \Delta_p^1 \mathfrak{P}_p^1 \Delta_p^2 u_{l+1} \neq 0$$

holds. Further using (6) and the relation above we have the relation

$$\mathcal{E}(l, l+1) \mathfrak{P}_p^{i-(l+1)} \Delta_p^2 b_{l+1} = \mathcal{E}(l, i) \Delta_p^2 \mathfrak{P}_p^{i-(l+1)} b_{l+1} \quad \text{for} \quad p > i \geqq l+1.$$

Since the group Z_p is also a field this relation are reduced to the following

$$(B_{l+1}) \qquad \Delta_p^2 \mathfrak{P}_p^{i-(l+1)} b_{l+1} = \mathcal{E}(l+1, i) \mathfrak{P}_p^{i-(l+1)} \Delta_p^2 b_{l+1} \quad \text{for} \quad p > i \geqq l+1.$$

By (C_{l+1}) we obtain $\Delta_p^2 b_{l+1} \neq 0$ and that

$$(A_{l+1}) \qquad {}^p\pi_{N+2(l+1)(p-1)-1}(X) = Z_{p^2}.$$

Thus we complete the proof of the lemma.

REMARK. This lemma is a part of Proposition 4. 21 in [7] IV which is obtained by the composition method.

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