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ON PARTIALLY HYPOELLIPTIC OPERATORS

By

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§ 1. Introduction

In this note we shall consider the differential operator $P(D)=P(D_{x'}, D_{x''})$ with complex constant coefficients defined in some open set $\Omega\subset R^m\times R^n$ whose points are denoted by $x=(x', x'')=(x'_1, \cdots, x'_m, x''_1, \cdots, x''_n)$, where $D_{x'}=(D_{x'_1}, \cdots, D_{x'_m})=(-\sqrt{-1}\,\partial/\partial x'_1, \cdots, -\sqrt{-1}\,\partial/\partial x'_m)$ and $D_{x''}=(D_{x''_1}, \cdots, D_{x''_n})=(-\sqrt{-1}\,\partial/\partial x''_1, \cdots, -\sqrt{-1}\,\partial/\partial x''_n)$.

L. Gårding and B. Malgrange [3] introduced the notions of partial hypoellipticity, partial ellipticity and conditional ellipticity for the operator $P(D)$, and characterized each of these notions completely by the property of the algebraic variety $V(P)=\{\zeta=(\zeta', \zeta'')\in C^m\times C^n\,;\, P(\zeta)=0\}$. J. Friberg [1] and L. Hörmander [6] proved that, if $P(\zeta)$ is a polynomial of finite type σ in a fixed direction, any solution of $P(D)u=0$ is hypoanalytic of type σ in the same direction. J. Friberg [1] expected that if $P(\zeta)$ is partially hypoelliptic of type σ in x', $P(\zeta)$ will be conditionally hypoelliptic of type σ in x'. In this note we shall prove the above fact. The method of the proof is based on the idea of Gårding and Malgrange [3] and that of Friberg [1]. The theorem 5.1 of [3] follows from our results by setting $\sigma=1$.

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§ 2. Definitions and Algebraic Considerations

Let $\alpha=(\alpha^{1'}, \cdots, \alpha^{m'}, \alpha^{1''}, \cdots, \alpha^{n''})$ be a multi-integer whose elements are non-negative integers.
In what follows we use the following notations:

$$|\alpha| = \alpha^{1'}+\cdots+\alpha^{m'}+\alpha^{1''}+\cdots+\alpha^{n''},$$
$$D^{\alpha} = D^{\alpha'}_{x'}D^{\alpha''}_{x''} = D^{\alpha^{1'}}_{x'_1}\cdots D^{\alpha^{m'}}_{x'_m}D^{\alpha^{1''}}_{x''_1}\cdots D^{\alpha^{n''}}_{x''_n}\,.$$

Definition 2.1. Let Ω be an open set in $R^m\times R^n$ and $f(x', x'')\in\mathcal{D}'(\Omega)$ be a distribution. We say that f is *regular in* x' if, for every pair of open sets $A\subset R^m$, $B\subset R^n$, $A\times B\subset\Omega$ and for any $\varphi\in\mathcal{D}(B)$, the distribution in x'

$$f(x',\varphi)=\int f(x',x'')\varphi(x'')dx'' \tag{2.1}$$

is an infinitely differentiable function.

DEFINITION 2.2. Let Ω be an open set in $R^m\times R^n$ and $f(x',x'')\in\mathcal{D}'(\Omega)$ be a distribution. We say that f is *analytic in* x' if, for every pair of open sets $A\subset R^m$, $B\subset R^n$, $A\times B\subset\Omega$ and for any $\varphi\in\mathcal{D}(B)$, the distribution in x'

$$f(x',\varphi)=\int f(x',x'')\varphi(x'')dx'' \tag{2.2}$$

is an analytic function.

DEFINITION 2.3. $P=P(D_{x'},D_{x''})$ is said to be *partially hypoelliptic in* x' if, for an open set Ω, every distribution solution of $Pf=0$ in Ω is regular in x'.

DEFINITION 2.4. When the operator $P(D)$ is hypoelliptic, we say P is *strictly stronger* than Q (it is written by $Q\ll P$) if $Q(\xi)/P(\xi)\to 0$ as $\xi\to\infty$, $\xi\in R^N$.

The following Gårding-Malgrange's theorem [3] is important.

Theorem 2.1. *An operator P is partially hypoelliptic in x' if and only if one of the following equivalent two conditions are satisfied*

(I) $P(\zeta',\zeta'')=0$, *ζ'' and $\mathcal{I}_m\zeta'$ are bounded then $\mathcal{R}_e\zeta'$ is bounded.*

(II) $P(\zeta',\zeta'')=P_0(\zeta')+\sum P_\gamma(\zeta')\cdot(\zeta'')^\gamma \quad |\gamma|>0$
where $P_0(\zeta')$ is hypoelliptic and $P_\gamma(\zeta')\ll P_0(\zeta')$.

(I) *is equivalent to*

(I′) *There exist $\sigma>0$ and $C>0$ such that*
$P(\zeta',\zeta'')=0$ *implies*
$|\mathcal{R}_e\zeta'|\leqq C(1+|\zeta''|+|\mathcal{I}_m\zeta'|)^\sigma$.

Corollary 2.1. *An operator $P(D)$ is partially hypoelliptic if and only if every solution of $P(D)u=0$ such that $u\in C^\infty_{(x'')}$ belongs to $C^\infty_{(x)}$.*

(See, L. Hörmander [6] or proof of Theorem 3.1. in [3].)

DEFINITION 2.5. A function $u(x)\in C^\infty(\Omega)$ is said to be *hypoanalytic of type σ in Ω* (we denote it $u(x)\in A_{\sigma(x)}$) if for every compact subset K of Ω, there exists a positive constant C depeinding only on K and u such that

$$\operatorname*{Max.}_{x\in K}|D^pu(x)|\leqq C^{p+1}p^{\sigma\cdot p} \qquad p=0,1,2,\cdots \tag{2.3}$$

where $|D^p u(x)|^2 = \sum_{|\alpha|=p} \frac{p!}{\alpha'!\,\alpha''!} |D_{x'}^{\alpha'} D_{x''}^{\alpha''} u|^2$.

DEFINITION 2.6. $P(\zeta)$ is said to be *hypoelliptic polynomial of type* σ if

$$(2.4) \qquad |\mathcal{R}_e \zeta| \leq C(1+|\mathcal{I}_m \zeta|)^\sigma \qquad \text{for all} \quad \zeta \in V(p).$$

Here we take σ as small as possible.

Lemma 2.1. *P is hypoelliptic of type* σ, *if and only if*

$$(2.5) \qquad \sum_{|\alpha|>0} |P^{(\alpha)}(\xi')|^2 |\xi'|^{2|\alpha|/\sigma} \leq C' \sum_{|\alpha|\geq 0} |P^{(\alpha)}(\xi')|^2 \qquad (\xi' \in R^m)$$

or equivalently

$$(2.5)' \qquad \sum_{|\alpha|>0} |P^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha|/\sigma} \leq C''|P(\xi')| \qquad (|\xi'|>A_1)$$

with some constant A_1. (For a proof, see p. 25–28 of Friberg [1].)

Since $P(\zeta)=P(\zeta', \zeta'')$ is a polynomial in $C^m \times C^n$, $P(\zeta)$ can be written as a finite sum;

$$P(\zeta', \zeta'') = P_0(\zeta') + \sum_{N \geq |\gamma|>0} P_\gamma(\zeta')\cdot(\zeta'')^\gamma$$

where $\gamma=(\gamma^1, \cdots, \gamma^n)$ with non-negative integer γ^i. Then the following important theorem is established.

Theorem 2.2. *In order that a polynomial* $P(\zeta)$ *satisfies the condition for some constant* $C_0>0$

$$(2.6) \qquad |\mathcal{R}_e \zeta'| \leq C_0(1+|\mathcal{I}_m \zeta'|+|\zeta''|)^\sigma \qquad (\zeta \in V(p)),$$

it is necessary and sufficient that the following estimate holds with some constant C_1.

$$(2.7) \qquad \sum_{|\alpha+\gamma|\geq 0} |P_\gamma^{(\alpha)}(\xi')|^2 |\xi'|^{2|\alpha+\gamma|/\sigma} \leq C_1(|P_0(\xi')|^2+1) \qquad (\xi' \in R^m).$$

REMARK. If $P(\zeta)$ satisfies the inequality (2.6), then $P_0(\zeta')$ $(=P(\zeta', 0))$ is hypoelliptic of type σ as a polynomial in ζ'. Thus the following inequality is valid from Lemma 2.1, with some $C_2>0$,

$$(2.8) \qquad \sum_{|\alpha|>0} |P_0^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha|/\sigma} \leq C_2|P_0(\xi')| \qquad (\xi' \in R^m,\ |\xi'|>A_2)$$

Proof of Theorem 2.2. It is easily verified that (2.7) is equivalent to

$$(2.7)' \qquad \sum_{\substack{|\alpha+\gamma|>0 \\ |\alpha|\geq 0}} |P_\gamma^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha+\gamma|/\sigma} \leq C_1'|P_0(\xi')| \qquad (|\xi'|>A_3)$$

for suitably chosen A_3 and C_1' which depend only on P and C_1. Setting

$W_t=\{\zeta=(\zeta',\zeta'');\ |\mathcal{J}_m\zeta'|+|\zeta''|<t|\mathcal{R}_e\zeta'|^{1/\sigma}\}$ and writing $\zeta'=\xi'+i\eta'$ where $\xi',\eta'\in R^m$, $\zeta\in W_t$ implies

$$(2.9)\quad \begin{aligned} |P(\zeta)| &= |P_0(\zeta')+\sum_{|\gamma|>0}P_\gamma(\zeta')\cdot(\zeta'')^\gamma| \\ &= |P_0(\xi')+\sum_{|\alpha|>0}c_\alpha P_0^{(\alpha)}(\xi')\cdot(i\eta')^\alpha+\sum_{|\gamma|>0}\sum_{|\alpha|\geqq 0}c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot(i\eta')^\alpha(\zeta'')^\gamma| \\ &\geqq |P_0(\xi')|-C\sum_{0<|\alpha|\leqq\rho_0}t^{|\alpha|}|P_0^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha|/\sigma} \\ &\quad -C\sum_{0<|\gamma|\leqq N}\sum_{0\leqq|\alpha|\leqq\rho}t^{|\alpha|+|\gamma|}|P_\gamma^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha+\gamma|/\sigma} \end{aligned}$$

where $\rho_0=$degree of P_0, $\rho=$degree of P. From the inequality (2.7)$'$ we have the following estimate

$$(2.9)'\qquad |P(\zeta)|\geqq|P_0(\xi')|\,\{1-CC_1'\sum_{0<|\alpha+\gamma|\leqq\rho+N}t^{|\alpha+\gamma|}\}\,.$$

It is obvious that there exists a sufficiently small positive number t_0 such that if $0<t\leqq t_0$ then $1-CC_1'\sum_{0<|\alpha+\gamma|\leqq\rho+N}t^{|\alpha+\gamma|}>0$, and from (2.8) follows $|P_0(\xi')|>0$ for $|\xi'|>A_2$.

These facts show that if $|\eta'|+|\zeta''|<t_0|\xi|^{1/\sigma}$ and $1<A_4^{-1/\sigma}|\xi'|^{1/\sigma}$ then $P(\zeta)\neq 0$ where $A_4=\max.(A_2,A_3)$. Now let $C'=\min.(t_0,A_2^{-1/\sigma})$ then

$$1+|\eta'|+|\zeta''|<C'|\xi'|^{1/\sigma}\qquad\text{implies}\quad P(\zeta)\neq 0\,.$$

Thus the sufficiency of (2.7) is proved.

(Necessity). Writing $\zeta'=\xi'+i\eta'$ ($\xi',\eta'\in R^m$) as above, (2.6) is equivalent to

$$(2.6)'\qquad |\xi'|^{1/\sigma}\leqq C_0'(|\eta'|+|\zeta''|)\qquad(\zeta\in V(P),\ |\xi'|>A_5)$$

for some positive C_0' and A_5. From Taylor's formula $P(\zeta)$ can be written as follows,

$$(2.11)\qquad \begin{aligned} P(\zeta) &= P_0(\xi')+\sum_{0<|\alpha|}c_\alpha P_0^{(\alpha)}(\xi')\cdot(i\eta')^\alpha \\ &\quad +\sum_{|\gamma|>0}\sum_{|\alpha|\geqq 0}c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot(i\eta')^\alpha(\zeta'')^\gamma\,. \end{aligned}$$

Now let $\eta'=|\xi'|^{1/\sigma}\tilde{\eta}'$, $\zeta''=|\xi'|^{1/\sigma}t\cdot\tilde{\zeta}''$ where $\tilde{\eta}'\in R^m$, $\tilde{\zeta}''\in C^n$ ($|\tilde{\zeta}''|=1$), $t\in C^1$ and $t\cdot\tilde{\zeta}''=(t\tilde{\zeta}_1'',\cdots,t\tilde{\zeta}_n'')$, then (2.11) is transformed into

$$(2.12)\qquad \begin{aligned} P(\zeta) &= P_0(\xi')+\sum_{0\leqq|\alpha|}c_\alpha P_0^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha \\ &\quad +\sum_{|\gamma|>0}\sum_{|\alpha|\geqq 0}c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha(\tilde{\zeta}'')^\gamma t^{|\gamma|}\,. \end{aligned}$$

First of all fix the length of $\tilde{\eta}'$ ($=\mathcal{E}$) suitably, then according to (2.8) there exist constants C_3, C_3' such that

$$(2.13)\qquad \begin{aligned} C_3|P_0(\xi')| &\leqq |P_0(\xi')+\sum_{|\alpha|>0}c_\alpha P_0^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha| \\ &\leqq C_3'|P_0(\xi')|\qquad(|\xi'|>A_3)\,. \end{aligned}$$

Thus from the condition (2.6)$'$, if $t(\in C^1)$ is a root of the algebraic equation:

$$P_0(\xi')+\sum_{|\alpha|>0} c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha + \sum_{|\gamma|>0}\sum_{|\alpha|\geqq 0} c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha(\tilde{\zeta}'')^\gamma t^{|\gamma|}=0, \tag{2.14}$$

then $\zeta=(\xi'+i|\xi'|^{1/\sigma}\tilde{\eta}', |\xi'|^{1/\sigma}t\cdot\tilde{\zeta}'')\in V(P)$ and $|t|>C_4$ for some $C_4>0$ (For example $C_4=(1-C_0'\mathcal{E})C_0'^{-1}$), for arbitraly $\tilde{\eta}'\in R^m$ ($|\tilde{\eta}'|=\mathcal{E}$), $\tilde{\zeta}\in C^n$ ($|\tilde{\zeta}''|=1$) and uniformly in ξ' for $|\xi'|>A_6$ where $A_6=\max.(A_3, A_5)$. This shows that every solution τ of the algebraic equation:

$$\tau^\rho+\sum_{k=1}^{\rho}\sum_{|\gamma|=k}\frac{\sum_{|\alpha|\geqq 0} c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha}{\sum_{|\alpha|\geqq 0} c_\alpha P_0^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha}(\tilde{\zeta}'')^\gamma\tau^{\rho-k}=0 \tag{2.14$'$}$$

satisfies $|\tau|<1/C_4$ uniformly in $\tilde{\eta}'$, $\tilde{\zeta}''$, $|\xi'|$ determined above.

This implies that every coefficient of τ^k ($k=0, \cdots, \rho-1$) is uniformly bounded, i.e.

$$\sum_{|\gamma|=k}\frac{\sum_{|\alpha|\geqq 0} c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha}{\sum_{|\alpha|\geqq 0} c_\alpha P_0^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha}(\tilde{\zeta}'')^\gamma = Q(\xi', \tilde{\eta}', \tilde{\zeta}'') \tag{2.15}$$

is uniformly bounded in $\tilde{\eta}'\in R^m$ ($|\tilde{\eta}'|=\mathcal{E}$), $\tilde{\zeta}''\in C^n$ ($|\tilde{\zeta}''|=1$) and $|\xi'|>A_6$. Therefore by virtue of uniformity in $\tilde{\zeta}''$ we can choose suitably finite number of vectors $\tilde{\zeta}''_{(i)}$ ($i=1, \cdots, M$) such that the coefficients of $(\tilde{\zeta}'')^\gamma$ are solvable in $Q(\xi', \tilde{\eta}', \tilde{\zeta}''_{(i)})$ ($i=1, \cdots, M$). Thus

$$\frac{\sum_{|\alpha|\geqq 0} c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha}{\sum_{|\alpha|\geqq 0} c_\alpha P_0^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha} = R(\xi', \tilde{\eta}') \tag{2.16}$$

is bounded for arbitraly $\tilde{\eta}'\in R^m$ ($|\tilde{\eta}'|=\mathcal{E}$) and $|\xi'|>A_6$. From the inequality (2.13), the absolute value of $R(\xi', \tilde{\eta}')$ is not smaller than the absolute value of

$$\frac{\sum_{|\alpha|\geqq 0} c_\alpha P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha}{C_3'|P_0(\xi')|} = \tilde{R}(\xi', \tilde{\eta}') \tag{2.17}$$

for arbitrary $\tilde{\eta}'\in R^m$ ($|\tilde{\eta}'|=\mathcal{E}$) and $|\xi'|>A_6$. The same argument as above is applicable to $\tilde{\eta}'$ in place of $\tilde{\zeta}''$. Hence $\{P_\gamma^{(\alpha)}(\xi')\cdot|\xi'|^{|\alpha+\gamma|/\sigma}\}|P_0(\xi')|^{-1}$ is bounded for $|\xi'|>A_6$. This completes the proof.

REMARK. As in the proof of Lemma 3.9 in Hörmander [6] the best possible choice of σ, such that for some C

$$|\mathcal{R}_e\zeta'|\leqq C(1+|\mathcal{I}_m\zeta'|+|\zeta''|)^\sigma \qquad (\zeta\in V(P))$$

is always a rational number, therefore we may assume in this note $\sigma=r/s$ ($\geqq 1$) with mutually prime positive integers r and s. Then the inequality (2.7) is equivalent to

$$(2.7)'' \quad \sum_{\substack{|\alpha+\gamma|>0 \\ |\alpha|\geqq 0}} |P_\gamma^{(\alpha)}(\xi')|^{2r} |\xi'|^{2s|\alpha+\gamma|} \leqq C_1''(|P_0(\xi')|^{2r}+1) \qquad (\xi' \in R^m).$$

§ 3. A priori estimates and the main theorem

In this section we introduce a new norm (similar to the norm introduced in [1]) which depend on the operator $P(D)$ and δ with $0<\delta\leqq 1$. Let K be any given relatively compact subset in $\Omega\subset R^m\times R^n$ such that $\bar{K}\subset\Omega$. We then define the norm of u ($\in C^\infty(\Omega)$) as follows

$$(3.1) \quad |u, K|_\delta^2 = \sum_{0\leqq|\gamma|} \sum_{\substack{0<|\alpha_1|\leqq\cdots\leqq|\alpha_r| \\ 0\leqq k<s|\alpha_1|}} \|Q_\gamma^{(\alpha_1)}(D)\cdots Q_\gamma^{(\alpha_r)}(D)\cdot D^k u, K\|^2 \delta^{2(\sigma k-\sum|\alpha_i|)}$$

where $Q_\gamma(D)=P_\gamma(D_{x'})D_{x''}^\gamma$ and $\|f, K\|$ denotes the usual L^2 norm of f on K.

Now by the above definition $\sigma k-\sum|\alpha_i|<0$, because

$$\sigma k-\sum|\alpha_i|<\sigma s|\alpha_1|-\sum|\alpha_i| = r|\alpha_1|-\sum|\alpha_i| = \sum(|\alpha_1|-|\alpha_i|)\leqq 0.$$

Since the degree of $P(\zeta)$ is ρ, there exists at least one α_0 ($|\alpha_0|=\rho$) such that $P^{(\alpha_0)}(D)=c\neq 0$. Thus $|u, K|_1$ contains terms of type $\|c^r D^k u, K\|^2$ for all k with $0\leqq k<s\rho$, and since $\sigma k-\sum|\alpha_i|\geqq -\rho\cdot r$ and $0<\delta\leqq 1$, the following estimates are established:

$$(3.2) \qquad |u, K|_1 \leqq |u, K|_\delta \leqq |u, K|_1\cdot\delta^{-\rho r}$$

$$(3.3) \qquad c\sum_{0\leqq k<\rho\cdot s} \|D^k u, K\|^2 \leqq |u, K|_1^2 \leqq |u, K|_\delta^2.$$

On the other hand the total degree of the polynomial $Q_\gamma^{(\alpha_1)}(\xi)^2\cdots Q_\gamma^{(\alpha_r)}(\xi)^2\cdot|\xi|^{2k}$ is smaller than $2\max_{|\alpha_i|>0}.\{\rho\cdot r-\sum|\alpha_i|+s\min.|\alpha_i|\}=2\{\rho\cdot r-(r-s)\}$. Hence the following important lemma is established.

Lemma. 3.1. There exist constants C_5 and C_6 (independent of u) such that the ineqalities

$$(3.4) \qquad C_5\sum_{0\leqq k<s\cdot\rho} \|D^k u, K\|^2 \leqq |u, K|_1^2 \leqq C_6 \sum_{|\alpha|\leqq\rho\cdot r-(r-s)} \|D^\alpha u, K\|^2$$

is valid for all $u\in C^\infty(\Omega)$.

REMARK. $\rho\cdot r-(r-s)>s\cdot\rho$ except in the trivial case $\rho=1$ or $r=s=\sigma=1$. The case $r=s=\sigma=1$ is treated in [3]. When $r=s=\sigma=1$, our norm is equivalent to that of §5 in [3].

Lemma 3.2. *Let K_0, K_1 be relatively compact subdomains of Ω such that $K_0 \subset K_1 \subset \bar{K}_1 \subset \Omega$ and* $\operatorname{dist}(\partial K_0, \partial K_1) = \delta > 0$. *Then it is possible to find a function $\varphi(x) \in C_0^\infty(K_1)$ which is equal to 1 on K_0 such that*

$$|D^\alpha \varphi(x)| \leqq \tilde{c}_\alpha \delta^{-|\alpha|} \qquad (x \in K_1) \tag{3.5}$$

where $\tilde{c}_\alpha$ is a constant depending only on α and $m+n$.

Proof is easy. cf. p. 205 in Hörmander [7].

Hereafter we only consider the case such that $|\alpha| \leqq \rho \cdot r$, thus we may suppose

$$|D^\alpha \varphi(x)| \leqq \tilde{c} \delta^{-|\alpha|} \qquad (x \in K_1,\ |\alpha| \leqq \rho \cdot r) \tag{3.5$'$}$$

where $\tilde{c} = \max\limits_{|\alpha| \leqq \rho \cdot r} \tilde{c}_\alpha$.

Parseval's formula shows the

Lemma 3.3. *If $R_i(\xi)$ $(i = 1, 2, \cdots, r)$ are polynomials with constant coefficients then the following inequality is valid for all $v(x) \in C_0^\infty$.*

$$||R_1(D) \cdots R_r(D) v(x)||^2 \leqq r^{-1} \sum_{i=1}^{r} ||R_i(D)^r v(x)||^2, \tag{3.6}$$

where integration is taken over the full space.

Now we state the most important estimate.

Theorem 3.1. *Let $P(\zeta)$ be a polynomial which satisfies the condition:*

$$|\mathcal{R}_e \zeta'| \leqq C_0(1 + |\mathcal{I}_m \zeta'| + |\zeta''|)^\sigma \qquad (\zeta \in V(P)), \tag{2.6}$$

and K_0, K_1 be two relatively compact subdomains of Ω such that $K_0 \subset K_1 \subset \bar{K}_1 \subset \Omega$ and $\operatorname{dist}(\partial K_0, \partial K_1) = \delta$ $(0 < \delta \leqq 1)$.

Then there exists a constant C_7 (independent of u, δ, K_0 and K_1) such that

$$\begin{aligned} &\delta^\sigma |Du, K_0|_\delta \\ &\leqq C_7 \Big\{ \sum_{k=0}^{r-s+1} |D_{x''}^k u, K_1|_\delta \delta^k + \sum_{0 \leqq |\alpha| \leqq \rho(r-1)} ||D^\alpha P(D) u, K_1|| \, \delta^{-\rho(r-1)} \Big\} \end{aligned} \tag{3.7}$$

is valid for all $u \in C^\infty(\Omega)$.

Proof. First we estimate the quantity

$$\begin{aligned} &\delta^{2\sigma} |Du, K_0|_\delta^2 \\ &= \sum_{|\gamma| > 0} \sum_{\substack{0 < |\alpha_1| \leqq \cdots \leqq |\alpha_r| \\ 0 \leqq k < s|\alpha_1|}} ||Q_\gamma^{(\alpha_1)}(D) \cdots Q_\gamma^{(\alpha_r)}(D) \cdot D^{k+1} u, K_0||^2 \delta^{2\sigma(k+1) - 2\Sigma|\alpha_i|}. \end{aligned} \tag{3.8}$$

We can split the above sum into two parts so that in the first part $k+1 < s|\alpha_1|$ and in the second part $k+1 = s|\alpha_1|$. In the first part each term is contained in the members of $|u, K_0|_\delta^2$, hence there exists a positive constant C_8 such that

$$\text{The 1st part} \leqq C_8|u, K_0|_\delta^2 \leqq C_8|u, K_1|_\delta^2 . \tag{3.9}$$

In the second part each term is estimated as follows (we set $v(x)=\varphi(x)\cdot u(x)\in C_0^\infty(K_1)$),

$$\begin{aligned}(3.10)\qquad & \|Q_\gamma^{(\alpha_1)}(D)\cdots Q_\gamma^{(\alpha_r)}(D)\cdot D^{s|\alpha_1|}u, K_0\|^2\delta^{2\sigma s|\alpha_1|-2\Sigma|\alpha_i|} \\ &= \|Q_\gamma^{(\alpha_1)}(D)\cdots Q_\gamma^{(\alpha_r)}(D)\cdot D^{s|\alpha_1|}u, K_0\|^2\delta^{2\Sigma(|\alpha_1|-|\alpha_i|)} \\ &\leqq \|Q_\gamma^{(\alpha_1)}(D)\cdots Q_\gamma^{(\alpha_r)}(D)\cdot D^{s|\alpha_1|}v\|^2\delta^{-2\Sigma(|\alpha_i|-|\alpha_1|)} \\ &\leqq r^{-1}\sum_{i=1}^r \|Q_\gamma^{(\alpha_i)}(D)^r D^{s|\alpha_i|}v\|^2\delta^{-2r(|\alpha_i|-|\alpha_1|)} .\end{aligned}$$

The last inequality is obtained from Lemma 3.3. by setting $R_i(D)=Q_\gamma^{(\alpha_i)}(D)\delta^{-(|\alpha_i|-|\alpha_1|)}$. The last sum in (3.10) is composed of terms of two different types generally.

One type is

$$\|Q_\gamma^{(\alpha)}(D)^r D^{s|\alpha|}v\|^2 \qquad (\text{when } |\alpha_i|=|\alpha_1|) \tag{3.11}$$

and another type is

$$\|Q_\gamma^{(\alpha)}(D)^r D^{sk}v\|^2\delta^{-2r(|\alpha|-k)} \qquad \text{with} \quad k<|\alpha| . \tag{3.12}$$

A term of the second type is estimated as follows. Since $v=\varphi\cdot u$ belongs to $C_0^\infty(K_1)$, applying lemma 3.2 we obtain

$$\begin{aligned}(3.13)\qquad & \|Q_\gamma^{(\alpha)}(D)^r D^{sk}v\|^2\delta^{-2r(|\alpha|-k)} \\ &= \|Q_\gamma^{(\alpha)}(D)^r D^{sk}(\varphi\cdot u), K_1\|^2\delta^{-2r(|\alpha|-k)} . \\ &\leqq C\sum_{\substack{|\beta_i|\geqq 0\\ sk\geqq j\geqq 0}} \|Q_\gamma^{(\alpha+\beta_1)}(D)\cdots Q_\gamma^{(\alpha+\beta_r)}(D)\cdot D^{sk-j}u, K_1\|^2\delta^{-2(\Sigma|\alpha+\beta_i|-rk+j)} \\ &\qquad\qquad (k<|\alpha|) .\end{aligned}$$

Here, $sk-j\leqq sk<s\cdot|\alpha|\leqq s\min_i|\alpha+\beta_i|$, and (the exponent of δ^2)

$$-\sum|\alpha+\beta_i|+rk-j=\sigma sk-j-\sum|\alpha+\beta_i|\geqq\sigma(sk-j)-\sum|\alpha+\beta_i| ,$$

by the assumption on σ: $\sigma\cdot s=r$ and $\sigma\geqq 1$. The fact that $0<\delta\leqq 1$ implies

$$\delta^{-2(\Sigma|\alpha+\beta_i|-rk+j)}\leqq\delta^{2\sigma(sk-j)-2\Sigma|\alpha+\beta_i|} .$$

Therefore the right hand side of (3.13) is majorated by

$$\begin{aligned}(3.13)'\qquad & C\cdot\sum_{\substack{|\alpha+\beta_i|>0\\ 0\leqq k<s\cdot\min|\alpha+\beta_i|}} \|Q_\gamma^{(\alpha+\beta_1)}(D)\cdots Q_\gamma^{(\alpha+\beta_r)}(D)\cdot D^k u, K_1\|^2\delta^{2\sigma k-2\Sigma|\alpha+\beta_i|} \\ &\leqq C_9|u, K_1|_\delta^2 \qquad \text{for some} \quad C_9>0 .\end{aligned}$$

i.e.

(3.14) $\quad ||Q_\gamma^{(\alpha)}(D)^r D^{sk}v||^2 \delta^{-2r(|\alpha|-k)} \leqq C_9 |u, K_1|_\delta^2 \quad$ with $\quad k<|\alpha|$.

Finally we consider the terms of the first type, (3.11).

$$(3.15) \qquad \begin{aligned} &||Q_\gamma^{(\alpha)}(D)^r D^{s|\alpha|}v, K_1||^2 \\ &= \int |Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2 d\xi \\ &+ \sum_{k=1}^{s|\alpha|}\binom{s|\alpha|}{k}\int |Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi'|^{2(s|\alpha|-k)}|\xi''|^{2k}|\hat{v}(\xi)|^2 d\xi \end{aligned}$$

where $\hat{v}(\xi)$ denotes the Fourier transform of $v(x)$.

The last sum of the above equality is estimated as follows:

$$(3.16) \qquad \begin{aligned} &\sum_{k=1}^{s|\alpha|}\binom{s|\alpha|}{k}\int |Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi'|^{2(s|\alpha|-k)}|\xi''|^{2k}|\hat{v}(\xi)|^2 d\xi \\ &\leqq C\int |Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi|^{2(s|\alpha|-1)}|\xi''|^2|\hat{v}(\xi)|^2 d\xi \\ &= C||Q_\gamma^{(\alpha)}(D)^r D^{s|\alpha|-1}D_{x''}(\varphi\cdot u), K_1||^2 \\ &\leqq C' \sum_{\substack{0\leqq|\beta_i| \\ 0\leqq k\leqq s|\alpha|-1}} ||Q_\gamma^{(\alpha+\beta_1)}(D)\cdots Q_\gamma^{(\alpha+\beta_r)}(D)\cdot D^{s|\alpha|-1-k}u, K_1||^2 \delta^{-2\Sigma|\beta_i|-2(k+1)} \\ &+ C'' \sum_{0\leqq|\beta_i|} ||Q_\gamma^{(\alpha+\beta_1)}(D)\cdots Q_\gamma^{(\alpha+\beta_r)}(D)\cdot D^{s|\alpha|-1}(D_{x''}u), K_1|| \delta^{-2\Sigma|\beta_i|}. \end{aligned}$$

We denote by I_1 and I_2, the first and the second sum of the right hand side of the above inequality respectively. Then we must calculate the exponent of δ^2 and the orders of operators.

In I_1, the exponent of δ^2, $-\sum|\beta_i|-(k+1)$, is

$$\geqq \sigma(s|\alpha|-k-1)-\sum|\alpha+\beta_i|,$$

and $s\cdot|\alpha|-1-k\leqq s|\alpha|-1<s|\alpha|\leqq s\cdot\min.|\alpha+\beta_i|$, thus I_1 is majorated by $C|u, K_1|_\delta^2$ with some positive constant C.

In I_2 the exponent of δ^2, $-\sum|\beta_i|$, is

$$= \sigma(s|\alpha|-1)-\sum|\alpha+\beta_i|+\sigma$$

and $s\cdot|\alpha|-1<s\cdot|\alpha|\leqq s\cdot\min.|\alpha+\beta_i|$, hence $I_2\leqq C|D_{x''}u, K_1|_\delta^2\delta^{2\sigma}$ with another constant C.

Therefore we obtain the following inequality,

$$(3.17) \qquad \begin{aligned} &\sum_{k=1}^{s|\alpha|}\binom{s|\alpha|}{k}\int |Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi'|^{2(s|\alpha|-k)}|\xi''|^{2k}|\hat{v}(\xi)|^2 d\xi \\ &\leqq C_{10}\{|u, K_1|_\delta^2+|D_{x''}u, K_1|_\delta^2\cdot\delta^{2\sigma}\} \end{aligned}$$

with some constant $C_{10}>0$.

Next if we define $\xi^{\gamma-\alpha''}$ as follows,

1) $\gamma-\alpha''=(\gamma'-\alpha^{1''},\cdots,\gamma^n-\alpha^{n''})$ when $\gamma^i-\alpha^{i''}\geqq 0$ for all i.
2) $\xi^{\gamma-\alpha''}\equiv 0$ when $\gamma^i-\alpha^{i''}<0$ for some i.

(we shall write $\gamma\supset\alpha''$ for the case 1), and $\gamma\not\supset\alpha''$ for the case 2)), then $(\partial/\partial\xi'')^{\alpha''}\{(\xi'')^\gamma\}=C_{\gamma,\alpha''}(\xi'')^{\gamma-\alpha''}$ with suitable constant $C_{\gamma,\alpha''}$.

We shall estimate the quantity

$$(3.18)\qquad I=\int|Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2d\xi$$
$$=\int|P_\gamma^{(\alpha')}(\xi')\cdot(\partial/\partial\xi'')^{\alpha''}\{(\xi'')^\gamma\}|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2d\xi$$

in two cases.

In the first case; $\gamma\supset\alpha''$ and $|\gamma-\alpha''|>0$, we obtain the following inequality with suitable β'' ($|\beta''|=1$):

$$I\leqq C\int|P_\gamma^{(\alpha')}(\xi')\cdot(\xi'')^{\gamma-\alpha''}|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2d\xi$$
$$\leqq C\int|P_\gamma^{(\alpha')}(\xi')\cdot(\xi'')^{\gamma-\alpha''-\beta''}|^{2r}|\xi''|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2d\xi$$
$$\leqq C\int|P_\gamma^{(\alpha')}(\xi')\cdot(\xi'')^{\gamma-\alpha''-\beta''}|^{2r}|\xi|^{2(s|\alpha|+s-1)}|\xi''|^{2(r-s+1)}|\hat{v}(\xi)|^2d\xi$$
$$\leqq C'\|Q_\gamma^{(\alpha',\alpha''+\beta'')}(D)^r\cdot D^{s|\alpha|+s-1}(D_{x''}^{r-s+1}v),K_1\|^2$$
$$\times\delta^{2(r-s+1)+2\sigma(s|\alpha|+s-1)-2\Sigma|\alpha+\beta''|}$$
$$\leqq C'\sum_{\substack{0\leqq|\tilde{\alpha}_i|\\0\leqq k,k'}}\|Q_\gamma^{(\alpha+\tilde{\alpha}_1+\beta'')}(D)\cdots Q_\gamma^{(\alpha+\tilde{\alpha}_r+\beta'')}(D)\cdot D^{s|\alpha|+s-1-k}D_{x''}^{r-s+1-k'}u,K_1\|^2$$
$$\times\delta^{2(r-s+1-k')}\delta^{2\sigma(s|\alpha|+s-1-k)-2\Sigma|\alpha+\tilde{\alpha}_i+\beta''|}.$$

Therefore we obtain

$$(3.19)\qquad I\leqq C_{11}\sum_{0\leqq k\leqq r-s+1}|D_{x''}^k u,K_1|_\delta^2\delta^{2k}\qquad\text{with some}\quad C_{11}>0.$$

In the second case; $\gamma=\alpha''$, from theorem 2.2 we obtain

$$(3.20)\qquad I=\int|P_\gamma^{(\alpha')}(\xi')|^{2r}|\xi'|^{2s|\alpha'+\gamma|}|\hat{v}(\xi)|^2d\xi$$
$$\leqq C_1''\int(|P_0(\xi')|^{2r}+1)|\hat{v}(\xi)|^2d\xi$$
$$=C_1''\int|\hat{v}(\xi)|^2d\xi$$
$$+C_1''\int|P(\xi)-\sum_{|\gamma|>0}P_\gamma(\xi')\cdot(\xi'')^\gamma|^{2r}|\hat{v}(\xi)|^2d\xi.$$

The first term of the right hand side of the above inequality is of course less than $C|u, K_1|_\delta^2$ with some C. The second term in (3.20) is (for suitable $C>0$)

$$(3.21)\qquad \leqq C\int |P(\xi)|^{2r}|\hat{v}(\xi)|^2 d\xi$$
$$+ C\sum_{|\gamma|>0}\int |P_\gamma(\xi')\cdot(\xi'')^\gamma|^{2r}|\hat{v}(\xi)|^2 d\xi$$
$$= C\,||P(D)^r(\varphi\cdot u), K_1||^2$$
$$+ C\sum_{|\gamma|>0}\int |P_\gamma(\xi')\cdot(\xi'')^{\gamma-\beta''}|^{2r}|\xi''|^{2r}|\hat{v}(\xi)|^2 d\xi \qquad \text{with } |\beta''|=1.$$

The first term in (3.21) is

$$(3.22)\qquad \leqq C\sum_{|\alpha_i|\geqq 0}||P^{(\alpha_1)}(D)\cdots P^{(\alpha_r)}(D)u, K_1||^2\delta^{-2\Sigma|\alpha_i|}$$
$$= C\sum_{|\alpha_i|>0}||P^{(\alpha_1)}(D)\cdots P^{(\alpha_r)}(D)u, K_1||^2\delta^{-2\Sigma|\alpha_i|}$$
$$+ C\sum_{|\alpha_i|>0}\sum_{1\leqq k\leqq r}||P^{(\alpha_1)}(D)\cdots P^{(\alpha_{r-k})}(D)\cdot P(D)^k u, K_1||^2\delta^{-2\sum_{i=1}^{r-k}|\alpha_i|}$$
$$\leqq C|u, K_1|_\delta^2 + C\sum_{|\alpha|\leqq\rho(r-1)}||D^\alpha P(D)u, K_1||^2\delta^{-2\rho(r-1)}$$

by the definition of P and lemma 3.3.

The second term in (3.21) is

$$(3.23)\qquad \leqq C'\sum_{|\gamma|>0}\sum_{\substack{0\leqq|\alpha_i|\\ 0\leqq k,k'}}||Q_\gamma^{(\alpha_1+\beta'')}(D)\cdots Q_\gamma^{(\alpha_r+\beta'')}(D)\cdot D^{s-1-k'}(D_{x''}^{r-s+1-k}u), K_1||^2$$
$$\times\delta^{-2(k+k')-2\Sigma|\alpha_i|}$$
$$\leqq C'\sum_{|\gamma|>0}\sum_{\alpha_i,k,k'}||Q_\gamma^{(\alpha_1+\beta'')}(D)\cdots Q_\gamma^{(\alpha_r+\beta'')}(D)\cdot D^{s-1-k'}(D_{x''}^{r-s+1-k}u), K_1||^2$$
$$\times\delta^{2\{\sigma(s-1-k')-\Sigma|\alpha_i+\beta''|\}}\delta^{2(r-s+1-k)}$$
$$\leqq C''\sum_{0\leqq k\leqq r-s+1}|D_{x''}^k u, K_1|_\delta^2\cdot\delta^{2k}$$

Therefore we obtain the estimate:

$$(3.20)'\quad I\leqq C_{12}\Big\{\sum_{k=0}^{r-s+1}|D_{x''}^k u, K_1|_\delta^2\cdot\delta^{2k} + \sum_{|\alpha|\leqq\rho(r-1)}||D^\alpha P(D)u, K_1||^2\delta^{-2\rho(r-1)}\Big\}.$$

The inequality (3.9), (3.14), (3.17), (3.19) and (3.20)′ show that the estimate (3.7) is established. This complete the proof of theorem 3.1. (In the above proof constants C's are independent of u, δ, K_0 and K_1).

Theorem 3.2. *Let $P(\zeta)$ be a polynomial of the type σ considered in theorem* 3.1, *ρ be the degree of $P(\zeta)$, and K and L be arbitrary relatively compact subdomains of Ω such that $K\subset L\subset\bar{L}\subset\Omega$ and* dist$(\partial K, \partial L)=\delta$ $(0<\delta\leqq 1)$.

Then there exists a constant C_{13} such that the inequality

$$(3.24)\quad (\delta/p)^{\sigma p}|D^p u, K|_{\delta/p} \leq C_{13}^p\{\sum_{k=0}^{p(r-s+1)}(\delta/p)^k|D_{x''}^k u, L|_{\delta/p} + \sum_{k=0}^{p(r-s+1)}\sum_{|\alpha|\leqq\rho(r-1)}||D^\alpha P(D)\cdot D^k u, L||(\delta/p)^{k-\rho(r-1)}\}\qquad p=0,1,2,\cdots.$$

is valid for all $u\in C^\infty(\Omega)$. *The constant* C_{13} *does not depend on* p, u, K *and* L.

Proof. By the assumptions on K and L there exists an increasing sequence of relatively compact domains: $K_0, K_1, \cdots, K_p$ such that $K=K_0\subset K_1\subset\cdots\subset K_p=L$ and $\operatorname{dist}(\partial K_i, \partial K_{i+1})=\delta/p=h<1$. Thus every pair K_i, K_{i+1} satisfies the conditions imposed on K_0 and K_1 with h in place of δ in theorem 3.1. If $u\in C^\infty(\Omega)$ then for every i ($i=0,1,\cdots,p$), $D^i u\in C^\infty(\Omega)$. Successive applications of theorem 3.1 to K_i, K_{i+1} show that the conclusion is obtained as follows,

$$\begin{aligned} h^{\sigma p}|D^p u, K_0|_h &= h^{(p-1)\sigma}h^\sigma|D(D^{p-1}u), K_0|_h \\ &\leq h^{(p-1)\sigma}C_7\{\sum_{k=0}^{r-s+1}|D_{x''}^k(D^{p-1}u), K_1|_h\cdot h^k \\ &+\sum_{|\alpha|\leqq\rho(r-1)}||D^\alpha P(D)(D^{p-1}u), K_1||h^{-\rho(r-1)}\} \\ &\cdots\cdots \\ &\leq\{C_7(r-s+1)\}^p\{\sum_{k=0}^{p(r-s+1)}|D_{x''}^k u, K_p|_h h^k \\ &+\sum_{k=0}^{p(r-s+1)}\sum_{|\alpha|\leqq\rho(r-1)}||D^\alpha P(D)D^k u, K_p||h^{k-\rho(r-1)}. \end{aligned}$$

This completes the proof of theorem 3.2 with $C_{13}=C_7(r-s+1)$.

Lemma 3.4. *Let* $P(\zeta)$ *be a polynomial which satisifes the condition,*

$$(2.6)\qquad |\mathcal{R}_e\zeta'|\leqq C_0(1+|\mathcal{I}_m\zeta'|+|\zeta''|)^\sigma\qquad(\zeta\in V(P))$$

(*hereafter we call such* P *partially hypoelliptic of type* σ *in* x') *and* u *be an infinitely differentiable solution of* $P(D)u=f$ (*where* f *belongs to* $A_{1(x')}$ *in* Ω) *such that* u *and* $D^k u\in A_{1(x'')}$ ($k=1,2,\cdots,r\rho-(r-s)$).

Then the following estimate is valid.

$$(3.25)\qquad |D^p u, K|_1\leqq C_{14}^{p+1}p^{\sigma p}\qquad\text{for every } p\geqq 0,$$

where C_{14} *does not depend on* p *and* u.

Proof. Considering $D^k u\in A_{1(x'')}$ ($k=0,1,\cdots,r\rho-(r-s)$) aud $f\in A_{1(x')}$, we may suppose that there exists a constant C_{15} such that the following inequalities are valid.

(3.26) $$C_6 \sum_{|\alpha| \leq \rho r-(r-s)} \| D_{x''}^k D^\alpha u, L\|^2 \leq (C_{15}^{k+1} k^k)^2$$

(3.27) $$\sum_{|\alpha| \leq \rho(r-1)} \| D^k D^\alpha f, L\| \leq C_{15}^{k+1} k^k .$$

(3.26) and (3.4) imply that

(3.26)′ $$| D_{x''}^k u, L|_h \leq | D_{x''}^k u, L|_1 (\delta/p)^{-\rho r} \leq C_{15}^{k+1} k^k \delta^{-\rho r} p^{\rho \cdot r}$$

Hence theorem 3.2 implies

$$\begin{aligned}
| D^p u, K|_1 &\leq | D^p u, K|_{\delta/p} \\
&\leq C_{13}^p (\delta/p)^{-p\sigma} \{ \sum_{k=0}^{p(r-s+1)} (\delta/p)^k | D_{x''}^k u, L|_{\delta/p} \\
&+ \sum_{k=0}^{p(r-s+1)} \sum_{|\alpha| \leq \rho(r-1)} \| D^k D^\alpha f, L\| (\delta/p)^{k-\rho(r-1)} \} \\
&\leq C_{13}^p \delta^{-p\sigma} p^{p\sigma} \{ \sum_{k=0}^{p(r-s+1)} \delta^k p^{-k} C_{15}^{k+1} k^k \delta^{-\rho r} p^{\rho \cdot r} \\
&+ \sum_{k=0}^{p(r-s+1)} C_{15}^{k+1} k^k \delta^{k-\rho(r-1)} p^{-k} p^{\rho(r-1)} \} \\
&\leq 2C_{13}^p \delta^{-p\sigma} \delta^{-\rho r} p^{\sigma p} \sum_{k=0}^{p(r-s+1)} C_{15} \{ C_{15} \delta(r-s+1)\}^k p^{\rho \cdot r} \\
&\leq 2C_{13}^p \delta^{-\rho r} \delta^{-\sigma p} p^{\sigma p} e^p \{(\rho r)!\} C_{15} \sum_{k=0}^{p(r-s+1)} \{ C_{15} \delta(r-s+1)\}^k \\
&\leq C_{14}^{p+1} \cdot p^{\sigma p} .
\end{aligned}$$

This completes the proof.

Lemma 3.4. and ineqality (3.4) show the inequality,

(3.28) $$\sum_{k=0}^{s \cdot \rho - 1} \| D^p D^k u, K\| \leq C_{14}^{p+1} p^{\sigma p} .$$

Sobolev's lemma. *If* $u \in C^{m+n}(\Omega)$ *and* $M \subset \bar{M} \subset K \subset \bar{K} \subset \Omega$, *then there exists a constant* C_{16} *such that*

(3.29) $$\operatorname{Sup}_{x \in M} | u(x)|^2 \leq C_{16} \sum_{|\alpha| \leq m+n} \| D^\alpha u, K\|^2 .$$

Proof is well known and we shall omit it.

Main Theorem. *Let a polynomial* $P(\zeta)$ *satisify the inequality* (2.6) (*i.e.* $P(\zeta)$ *is partially hypoelliptic of type* σ *in* x') *and* $f \in A_{1(x)}$ *in* Ω. *The solution* u ($\in C^\infty(\Omega)$) *of* $P(D)u = f$ *in* Ω *such that* $D^k u \in A_{1(x'')}$ ($k = 0, 1, \cdots, r\rho-(r-s)$), *also belongs to* $A_{\sigma(x)}$ *in* M *with* $M \subset \bar{M} \subset K \subset \bar{K} \subset L \subset \bar{L}$, *compact in* Ω.

Proof. From Sobolev's lemma we get that for every non-negative integer p,

(3.29)′ $$\operatorname{Sup}_{x \in M} | D^p u(x)| \leq C_{16} \||\, D^{m+n}(D^q u), K\||\, ,$$

where $|||D^q(u), K|||^2$ stands for $\sum_{j=0}^q ||D^j(u), K||^2$. Lemma 3.4 and (3.28) imply that a solntion u satisfies

$$(3.28)' \qquad |||D^{s\rho-1}(D^\rho u), K||| \leqq C_{14}^{p+1} p^{\sigma p}.$$

Therefore we have, if $s\cdot\rho-1<m+n$, setting $\mu=m+n-(s\cdot\rho-1)$

$$(3.30) \qquad \begin{aligned} \sup_{x\in M} |D^p u(x)| &\leqq C_{16} |||D^{p+\mu} D^{s\rho-1} u, K||| \\ &\leqq C_{16} C_{14}^{p+\mu+1} (p+\mu+1)^{\sigma(p+\mu+1)} \\ &\leqq C_{16} C_{14}^{p+\mu+1} e^{\sigma(p+\mu+1)} [(p+\mu+1)!]^\sigma \\ &\leqq C_{16} \{(\mu+1)!\}^\sigma \prod_{q=1}^{\mu+1} \left(\frac{\mu+q+1}{2q}\right)^\sigma 2^{p\sigma} C_{14}^{p+\mu+1} e^{\sigma(p+\mu+1)} (p!)^\sigma \end{aligned}$$

because $(p+\mu+1)!=(\mu+1)!\prod_{q=1}^{\mu+1}\left(\frac{\mu+q+1}{2}\right)2^p p!$.

Thus we obtain

$$(3.31) \qquad \sup_{x\in M} |D^p u(x)| \leqq C_{17}^{p+1} (p!)^\sigma.$$

Next if $s\cdot\rho-1\geqq m+n$, then obviously we obtain

$$(3.31)' \qquad \sup_{x\in M} |D^p u(x)| \leqq C_{17}^{p+1} (p!)^\sigma.$$

This completes the proof.

REMARK. Corollary 2.1 shows that if $u\in C^\infty_{(x'')}$ and u is a distribution solution of $P(D)u=f$ with $f\in C^\infty(\Omega)$, then in virtue of the partial hypoellipticity of P, the solution u also belongs to $C^\infty(\Omega)$.

Therefore let u be a distribution solution of $P(D)u=f$ such that $D^k u\in A_{1(x'')}$ $(k=0, 1, \cdots, r\cdot\rho-(r-s))$ and if $f\in A_{1(x)}$, and $P(\zeta)$ satisfy the condiction (2.6), then using theorem 3.3, u belongs to $A_{(\sigma)x}$.

The above fact may be said, "if $P(\zeta)$ is partially hypoelliptic of type σ in x', then $P(D)$ is conditionally hypoelliptic of type σ iu x'". Moreover if a distribution solution u of $P(D)u=0$ possess the property described in the above remark, then it will be proved that $P(\zeta)$ satisfies the condition (2.6) analogically to the proof of Gårding and Malgrange [3] for the conditionally elliptic case.

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