Kumano-go, H.
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ON THE UNIQUENESS FOR THE SOLUTION OF THE CAUCHY PROBLEM

By

Hitoshi KUMANO-GO

§0. Introduction. We shall consider a linear partial differential operator L with complex valued coefficients in a neighborhood of the origin in $(\nu+1)$-space $(t, x)=(t, x_1, \cdots, x_\nu)$.

In the recent note [4] we have proved the uniqueness of the solution of the Cauchy problem for the differential equation

$$Lu \equiv \sum_{j+|\alpha|\leqq m} a_{j,\alpha}(t, x)\frac{\partial^{j+|\alpha|}}{\partial t^j \partial x^\alpha} u(t, x) = f(t, x) \tag{0.1}$$
$$(\alpha = (\alpha_1, \cdots, \alpha_\nu), \quad |\alpha| = \alpha_1 + \cdots + \alpha_\nu)$$

under some conditions for the characteristic roots. On the other hand S. Mizohata [7] proved the uniqueness of the Cauchy problem for a parabolic equation

$$\begin{aligned} Lu &\equiv \Big(\sum_{i,j=1}^{\nu} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{\nu} b_i(t, x)\frac{\partial}{\partial x_i} + c(t, x) - \frac{\partial}{\partial t}\Big)u(t, x) \\ &= f(t, x) \qquad \Big(\sum_{i,j=1}^{\nu} a_{ij}(t, x)\xi_i\xi_j \geqq \delta\big(\sum_{i=1}^{\nu}\xi_i^2\big) \quad \text{for} \quad \delta > 0\Big) \end{aligned} \tag{0.2}$$

when the data are prescribed on a piece of a time-like surface, and T. Shirota [10] and M. H. Protter [9] gave other proofs for this problem under weaker conditions.

In this note we shall prove a more general uniqueness theorem which can be applied to the parabolic equation (0.2).

The differential equation which we shall study is of the form

$$\begin{aligned} Lu &\equiv L_0 u(t, x) + \sum_{j+m|\alpha:\mathrm{m}|\leqq m-1} b_{j,\alpha}(t, x)\frac{\partial^{j+|\alpha|}}{\partial t^j \partial x^\alpha} u(t, x) \\ &= f(t, x) \quad \Big(|\alpha:\mathrm{m}| = \frac{\alpha_1}{m_1} + \cdots + \frac{\alpha_\nu}{m_\nu};\ m \geqq m_j \quad (j = 1, \cdots, \nu)\Big) \end{aligned} \tag{0.3}$$

where L_0 has the form

$$L_0 u = \sum_{j+m|\alpha:\mathfrak{m}|=m} a_{j,\alpha}(t, x) \frac{\partial^{j+|\alpha|}}{\partial t^j \partial x^\alpha} u(t, x) \tag{0.4}$$

and is called the principal part of L. We prove the uniqueness theorem when the initial data are given on a surface which meets the plane $(t=0)$ only at the origin. If the initial data are given on a plane portion, we must set the condition: $m=m_j$ or $m \geqq 2m_j$ $(j=1, \cdots, \nu)$ instead of $m \geqq m_j$ $(j=1, \cdots, \nu)$, which is caused by Holmgren's transformation.

If we set $m=m_1=\cdots=m_\nu$, then (0.3) takes the same form with (0.1), and for the parabolic equation (0.2) we get the form (0.3) by setting $m=m_1=\cdots=m_{\nu-1}=2$ and $m_\nu=1$.

The tool used in this note is the singular integral operator of A. P. Calderón and A. Zygmund [1]. But we have some difficulties to use this since the homogeneity of the characteristic roots does not hold. We define $r=r(\xi)$ for real vector $\xi=(\xi_1, \cdots, \xi_\nu) \neq 0$ as the positive root of the equation

$$\sum_{j=1}^{\nu} \xi_j^2 r^{-2/m_j} = 1$$

and represent the characteristic roots λ as $\lambda = r^{1/m}\lambda_0$ where λ_0 are homogeneous of order 0 with respect to ξ in some sense, and we define singular integral operators of type $C_{\mathfrak{m}}^m$ (Definition 1 in § 1) with the symbols λ_0.

Although some results are evident from the note [4], we shall mention them for the sake of completeness. The author wishes to express his sincere gratitude to Prof. M. Nagumo for his advices and encouragement.

§ 1. **Notations and definitions.** We denote a point in $(\nu+1)$-dimensional Euclidean space $R^1 \times R^\nu$ by $(t, x)=(t, x_1, \cdots, x_\nu)$ or $(s, y)=(s, y_1, \cdots, y_\nu)$ and denote a point in the dual space of R^ν by $\xi=(\xi_1, \cdots, \xi_\nu)$ or $\eta=(\eta_1, \cdots, \eta_\nu)$.

$(m, \mathfrak{m})=(m, m_1, \cdots, m_\nu)$ expresses a real vector whose elements are positive integers $(m \geqq m_j;\ j=1, \cdots, \nu)$ and $\alpha=(\alpha_1, \cdots, \alpha_\nu)$ expresses a real vector whose elements are non-negative integers.

We use notations:

$$|\alpha| = \alpha_1 + \cdots + \alpha_\nu, \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_\nu^{\alpha_\nu},$$

$$|\alpha:\mathfrak{m}| = \frac{\alpha_1}{m_1} + \cdots + \frac{\alpha_\nu}{m_\nu}, \qquad \partial x^\alpha = \partial x_1^{\alpha_1} \cdots \partial x_\nu^{\alpha_\nu},$$

$$|x|^2 = \sum_{j=1}^{\nu} x_j^2, \quad |\xi|^2 = \sum_{j=1}^{\nu} \xi_j^2, \quad x \cdot \xi = \sum_{j=1}^{\nu} x_j \xi_j,$$

$$\Xi_h = \{(t, x);\ t^2 + |x|^2 < h^2\}.$$

We shall consider a differential polynomial

$$L(t, x, \lambda, \xi) = L_0(t, x, \lambda, \xi) + \sum_{j+m|\alpha:\mathfrak{m}| \leqq m-1} b_{j,\alpha}(t, x) \tag{1.1}$$

in a open domain in $(\nu+1)$-space, where L_0 has the form

$$L_0(t, x, \lambda, \xi) = \sum_{j+m|\alpha:\mathfrak{m}|=m} a_{j,\alpha}(t, x)\lambda^j\xi^\alpha \quad (a_{m,0} = 1) \tag{1.2}$$

and is called the characteristic polynomial of L.

We define for $u \in L^2$ the Fourier transform $F[u]$ by

$$F[u] = \tilde{u}(\xi) = \frac{1}{\sqrt{2\pi}^\nu}\int e^{-ix\cdot\xi\,[1]}u(x)dx$$

and set

$$K(\xi) = (\sum_{j=1}^{\nu} \xi_j^{2m/m_j})^{1/2m}. \tag{1.3}$$

DEFINITION 1. We call $H=\sum_{r=1}^{\infty} a_r h_r$ a singular integral operator of type $C_{\mathfrak{m}}^m$ with the symbol $\sigma(H)=\sum_{r=1}^{\infty} a_r(x)\tilde{h}_r(\xi)$ if the following conditions are satisfied:

$$a_r(x) \in C^\infty_{(x)}, \quad \tilde{h}_r(\xi) \in C^\infty_{(\xi\neq 0)} \qquad (r = 1, 2, \cdots),$$

and for every α and l there exists a positive constant $A_{\alpha,l}$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} a_r(x)\right| \leqq A_{\alpha,l}r^{-l} \qquad (r = 1, 2, \cdots) \tag{1.4}$$

and for every α there exist positive constants B_α and l_α such that

$$\left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha} \tilde{h}_r(\xi)\right| \leqq B_\alpha r^{l_\alpha} K(\xi)^{-m|\alpha:\mathfrak{m}|} \qquad (r = 1, 2, \cdots). \tag{1.5}$$

Then, Hu is defined dy

$$Hu = \frac{1}{\sqrt{2\pi}^\nu}\int e^{ix\cdot\xi}\sigma(H)\tilde{u}(\xi)d\xi$$

or equivalently $Hu = \sum_{r=1}^{\infty} a_r(x)(h_r u)(x)$ where $h_r u$ are defined by $\widetilde{h_r u} = \tilde{h}_r(\xi)\tilde{u}(\xi)$.

DEFINITION 2. Let $\Lambda(\xi)$ be infinitely differentiable in $\xi(\neq 0)$, and for

1) $i=\sqrt{-1}$, without description we use i in two meanings: a square root of -1 and a suffix, these distinction will be easily seen case by case.

every α there exists a positive constant γ_α such that

$$\left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\Lambda(\xi)\right| \leqq \gamma_\alpha K(\xi)^{1-m|\alpha:\mathfrak{m}|}. \tag{1.6}$$

Then, we define a convolution operator Λ by $\widetilde{\Lambda u(\xi)}=\Lambda(\xi)\tilde{u}(\xi)$.

Let $r=r(\xi)$ be a positive root of the equation

$$F(r, \xi) \equiv \sum_{j=1}^{\nu} \xi_j^2 r^{-2/m_j} = 1 \quad \text{for real} \quad \xi(\neq 0). \tag{1.7}$$

As $\lim_{r\to 0} F(r, \xi)=\infty$, $\lim_{r\to\infty} F(r, \xi)=0$ and $\frac{\partial}{\partial r}F(r, \xi)=-2r^{-1}\sum_{j=1}^{\nu}\frac{1}{m_j}\xi_j^2 r^{-2/m_j}<0$ for $\xi\neq 0$ and $r>0$, it follows that its positive root $r=r(\xi)$ is uniquely determined and infinitely differentiable.

We write for $\eta=(\eta_1, \cdots, \eta_\nu)\neq 0$

$$L_0(t, x, \lambda, i\eta|\eta|^{-1}) = \prod_{l=1}^{m}(\lambda+\lambda_{0,l}(t, x, \eta)),$$

then $\lambda_{0,l}(t, x, \eta)$ $(l=1, \cdots, m)$ are homogeneous of order 0 with respect to η.

Now we define a mapping $\xi\to\eta$ by $\eta_j=\xi_j r^{-1/m_j}$ $(j=1, \cdots, \nu)$ with $r=r(\xi)$ determined by (1.7), and define a matrix R by

$$R = \begin{pmatrix} r^{1/m_1} & & 0 \\ & \ddots & \\ 0 & & r^{1/m_\nu} \end{pmatrix}, \tag{1.8}$$

then $\eta=\xi R^{-1}$.

Set

$$L_0(t, x, \lambda, i\xi) = \prod_{l=1}^{\nu}(\lambda+\lambda_l(t, x, \xi)). \tag{1.9}$$

Remarking $|\xi R^{-1}|=1$ by (1.7) it follows that

$$\begin{aligned} L_0(t, x, \lambda, i\xi) &= \sum_{j+m|\alpha:\mathfrak{m}|=m} a_{j,\alpha}(t, x)\lambda^j(i\xi)^\alpha \\ &= \sum_{j+m|\alpha:\mathfrak{m}|=m} a_{j,\alpha}(t, x)\lambda^j(i\xi R^{-1})^\alpha r^{|\alpha:\mathfrak{m}|} \\ &= r\sum_{j+m|\alpha:\mathfrak{m}|=m} a_{j,\alpha}(t, x)(r^{-1/m}\lambda)^j(i\xi R^{-1})^\alpha \\ &= r\prod_{l=1}^{m}(r^{-1/m}\lambda+\lambda_{0,l}(t, x, \xi R^{-1})) \\ &= \prod_{l=1}^{m}(\lambda+r^{1/m}\lambda_{0,l}(t, x, \xi R^{-1})), \end{aligned}$$

hence by (1. 9) we have

$$\lambda_l(t, x, \xi) = r^{1/m}\lambda_{0,l}(t, x, \xi R^{-1}) \qquad (l = 1, \cdots, m). \tag{1. 10}$$

REMARK. $\lambda_0(t, x, \eta)$ is infinitely differentiable and homogeneous of order 0 with respect to η, then, after A. P. Calderón and A. Zygmund [1] we may represent it as

$$\lambda_0(t, x, \eta) = \sum_{r=1}^{\infty} a_r(t, x)\tilde{h}_{0,r}(\eta)$$

where $a_r(t, x)$ satisfy (1. 4) and $\tilde{h}_{0,r}(\eta)$ satisfy

$$\left|\frac{\partial^{|\alpha|}}{\partial\eta^{\alpha}}\tilde{h}_{0,r}(\eta)\right| \leqq B'_{\alpha}r^{l'_{\alpha}}|\eta|^{-|\alpha|} \qquad (r=1, 2, \cdots).$$

Setting $\tilde{h}_r(\xi)=\tilde{h}_{0,r}(\xi R^{-1})$ we have $\lambda_0(t, x, \xi R^{-1})=\sum_{r=1}^{\infty} a_r(t, x)\tilde{h}_r(\xi)$.

On the other hand by (2. 3) of Lemma 1, we have $\left|\frac{\partial^{|\alpha|}}{\partial\xi^{\alpha}}(\xi_j r^{-1/m_j})\right| \leqq C_{\alpha}K(\xi)^{-m|\alpha;\mathfrak{m}|}$ $(j=1, \cdots, \nu)$.

Hence it follows that $\lambda_0(t, x, \xi R^{-1})$ becomes the symbol of some operator of type $C^m_{\mathfrak{m}}$. Similarly we verify that $r^{1/m}$ satisfies (1. 6), so we can define an operator Λ by $\widetilde{\Lambda u}=r^{1/m}\tilde{u}(\xi)$.

§ 2. **Preliminary lemmas.** Our main tool is the inequality (3. 6) of Theorem 1. In this section we shall mention some lemmas for the proof of Theorem 1. All the lemmas except Lemma 1 is essentially the same with the previous note [4], but we shall give brief proofs to some of them so that we may be convinced of them.

Lemma 1. *Let $r=r(\xi)$ be a positive root of*

$$F(r, \xi) \equiv \sum_{j=1}^{\nu} \xi_j^2 r^{-2/m_j} = 1 \quad \textit{for real} \quad \xi \neq 0. \tag{2. 1}$$

Then, we have for some constants C_0[2] *and C_α*

$$C_0^{-1}K(\xi)^m \leqq r(\xi) \leqq C_0K(\xi)^m \tag{2. 2}$$

and

$$\left|\frac{\partial^{|\alpha|}}{\partial\xi^{\alpha}}r(\xi)\right| \leqq C_{\alpha}K(\xi)^{m(1-|\alpha:\mathfrak{m}|)}. \tag{2. 3}$$

Proof. In the previous section we have already studied that r is uniquely determined and infinitely differentiable.

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2) We denote by C (with or without subscript) positive constants.

From (2.1) we have $\xi_i^2 r^{-2/m_i} \geqq \frac{1}{\nu}$ for some i and $\xi_j^2 r^{-2/m_j} \leqq 1$ for every j, hence we get

$$K(\xi)^{2m} \geqq \xi_i^{2m_i} \geqq \left(\frac{1}{\nu}\right)^{2m_i} r^2 \quad \text{and} \quad \nu r^2 \geqq \sum_{j=1}^{\nu} \xi^{2m_j} = K(\xi)^{2m}.$$

This shows the inequality (2.2) holds.

Differentiating the both sides of (2.1) with respect to ξ_i we have

$$\left(\sum_{j=1}^{\nu} \frac{1}{m_j} \xi_j^2 r^{-2/m_j}\right) \frac{\partial}{\partial \xi_i} r - \xi_i r^{1-2/m_i} = 0 \qquad (i = 1, \cdots, \nu). \tag{2.4}$$

More generally we have

$$\begin{aligned} &\left(\sum_{j=1}^{\nu} \frac{1}{m_j} \xi_j^2 r^{-2/m_j}\right) \frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}} r \\ &+ \sum_{\substack{\alpha_0 \leqq \mu_0,\, \mu_0+\cdots+\mu_l = \alpha_0+\alpha \\ \mu_i < \alpha (i=1,\cdots,l)}} a_{\mu,\alpha_0} \xi^{\alpha_0} r^{1-l-|\mu_0:\mathfrak{m}|} \frac{\partial^{|\mu_1|}}{\partial \xi^{\mu_1}} r \cdot \cdots \cdot \frac{\partial^{|\mu_l|}}{\partial \xi^{\mu_l}} r \\ &= 0 \qquad (|\alpha| > 0). \end{aligned} \tag{2.5}$$

From (2.1) we have $\frac{1}{m} \leqq \sum_{j=1}^{\nu} \frac{1}{m_j} \xi_j^2 r^{-2/m_j} \leqq 1$ and by the definition (1.3) of $K(\xi)$

$$|\xi^{\alpha_0}| \leqq |\xi_1|^{\alpha_{0,1}} \cdot \cdots \cdot |\xi_\nu|^{\alpha_{0,\nu}} \leqq K(\xi)^{m|\alpha_0:\mathfrak{m}|}.$$

Hence applying (2.3) for $\mu_i (< \alpha)$, instead of α, as the assumption of the induction we have

$$\begin{aligned} \left|\frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}} r\right| &\leqq CK(\xi)^{m|\alpha_0;\mathfrak{m}| + m(1-l-|\mu_0:\mathfrak{m}|) + m(1-|\mu_1;\mathfrak{m}|) + \cdots + m(1-|\mu_l;\mathfrak{m}|)} \\ &= CK(\xi)^{m - m(|\mu_0+\cdots+\mu_l:\mathfrak{m}| - |\alpha_0:\mathfrak{m}|)} \\ &= CK(\xi)^{m(1-|\alpha:\mathfrak{m}|)}. \end{aligned}$$

Q.E.D.

Lemma 2. i) *Let P and Q be singular integral operator of type $C_{\mathfrak{m}}^m$ with real valued symbols, then the operator norms*

$$\begin{aligned} &\|P\Lambda - \Lambda P^*\|, \qquad \|Q\Lambda - \Lambda Q^*\|, \\ &\|(P^*Q - Q^*P)\Lambda\|, \quad \|\Lambda(P^*Q - Q^*P)\| \end{aligned} \tag{2.6}$$

are all bounded, where P^* and Q^* show the adjoint operators of P and Q respectively.

ii) *Let H, H_1 and H_2 be singular integral operators of type $C_{\mathfrak{m}}^m$, then we have for any positive integer p and q the representations*

$$(2.7)\qquad \begin{aligned} H\Lambda^p-\Lambda^p H &= H_{p,q}\Lambda^{p-1}+H'_{p,q}\,, \\ (H_1H_2-H_1\circ H_2)\Lambda &= H_q+H'_q \end{aligned}$$

where $H_{p,q}$ *and* H_q *are bounded operators together with* $\Lambda^i H_{p,q}\Lambda^j$ *and* $\Lambda^i H'_q\Lambda^j$ $(0\leqq i+j\leqq q)$ *respectively, and* $H_1\circ H_2$ *is an operator of type* $C^m_{\mathfrak{m}}$ *with the symbol* $\sigma(H_1)\cdot\sigma(H_2)$.

iii) *Let* H *be a singular integral operator of type* $C^m_{\mathfrak{m}}$ *such that* $|\sigma(H)|\geqq\delta>0$, *then there exists positive constant* C_ε *such that*

$$(2.8)\qquad \|H\Lambda u\|^2\geqq(1-\varepsilon)\delta^2\|\Lambda u\|^2-C_\varepsilon\|u\|^2 \qquad (u\in C_0^\infty(R^\nu),\ 1>\varepsilon>0)$$

(*In what follows we apply* (2.8) *setting* $\varepsilon=\frac{1}{2}$).

By (1.6) it follows that $\left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\Lambda(\xi)\right|\leqq C_\alpha K(\xi)^{1-|\alpha|}$ if $|\xi|\geqq1$ and by $K(\xi)\geqq C|\xi|^{1/m}$ $(|\xi|\geqq1)$, it follows for every k that $\left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\Lambda(\xi)\right|\leqq C_\alpha|\xi|^{-k}$ $(|\xi|\geqq1)$ for sufficiently large α, so that the operator of type $C^m_{\mathfrak{m}}$ in Definition 1 and operator Λ in Definition 2 are essentially the same with those of M. Yamaguti's [11]. Hence we can prove the lemma by the parallel process, but we omit the proof since it is very troublesome. The reader may consult [11] for i) and ii), and [8] for iii).

Lemma 3. *Let* P *and* Q *belong* $C^m_{\mathfrak{m}}$ *and have real valued symbols. Then we haue the following representation*

$$(2.9)\qquad i(\Lambda Q^*P-P\Lambda Q)\Lambda = H\Lambda+H'P\Lambda+H''\,,$$

where H *belongs to* $C^m_{\mathfrak{m}}$ *with the symbol*

$$(2.10)\qquad \sigma(H)=\sum_{i=1}^{\nu}\left\{\frac{\partial}{\partial x_i}\sigma(P)\frac{\partial}{\partial\xi_i}(\sigma(Q)\Lambda(\xi))-\frac{\partial}{\partial x_i}(\sigma(Q))\frac{\partial}{\partial\xi_i}(\sigma(P)\Lambda(\xi))\right\},$$

and H' *and* H'' *are bounded operators.*

Proof. As a simple case we consider $P=ah$ and $Q=bk$ with $\sigma(P)=a(x)\tilde{h}(\xi)$ and $\sigma(Q)=b(x)\tilde{k}(\xi)$ respectively. Then we can write

$$\begin{aligned}(\Lambda Q^*P-P\Lambda Q)\Lambda &= (\Lambda kbah-ah\Lambda bk)\Lambda \\ &= \{((\Lambda k)b-b(\Lambda k))ah\Lambda+b((\Lambda k)a-a(\Lambda k))h\Lambda+abhk\Lambda^2\} \\ &\quad-\{a((h\Lambda)b-b(h\Lambda))(k\Lambda)+abhk\Lambda^2\}\,.\end{aligned}$$

By (2.6) we have $((\Lambda k)b-b(\Lambda k))ah\Lambda=(\Lambda Q^*-Q\Lambda)P\Lambda=H_1P\Lambda$ with a bounded operator H_1.

For $\alpha(\xi)\in C^\infty(\xi)$ such that

$$\alpha(\xi) = 0 \quad \text{on} \quad \{\xi;\ |\xi| \leqq 1\} \quad \text{and} \quad \alpha(\xi) = 1 \quad \text{on} \quad \{\xi;\ |\xi| \geqq 2\}$$

we consider an operator Λ' defined by $\widetilde{\Lambda' u} = \alpha_0 \Lambda \tilde{u}$, then we have

$$\begin{aligned} b((\Lambda k)a - a(\Lambda k))h\Lambda &= b((\Lambda' k)a - a(\Lambda' k))h\Lambda \\ &\quad + b\{(\Lambda - \Lambda')k((ah)\Lambda - \Lambda(ah)) + (\Lambda - \Lambda')k\Lambda ah - a(\Lambda - \Lambda')kh\Lambda\}\,. \end{aligned}$$

As it is easy to see that the second term is bounded, we may consider only the first term.

For $u \in C_0^\infty(R^\nu)$ we have

$$\begin{aligned} ((\Lambda' k)a - a(\Lambda' k))u &= \int ((\Lambda' k)(x-y)a(y) - a(x)(\Lambda' k)(x-y))u(y)dy \\ &\qquad\qquad \text{(in the distribution sense)} \\ &= -\sum_{i=1}^{\nu} \frac{\partial}{\partial x_i} a(x) \int (x_i - y_i)(\Lambda' k)(x-y)u(y)dy \\ &\quad + \sum_{2 \leqq |\alpha| \leqq l} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} a(x) \int \frac{(x-y)^\alpha}{\alpha!} (\Lambda' k)(x-y)u(y)dy \\ &\quad + \sum_{|\alpha| = l+1} \int (x-y)^\alpha (\Lambda' k)(x-y) a_\alpha(x, y) u(y) dy\,. \end{aligned}$$

The first term is equal to an operator of type C_{m}^m with the symbol $-i\sum_{i=1}^{\nu} \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial \xi_i}(\tilde{k}\alpha_0\Lambda)$. If we estimate the remaining term for sufficiently large fixed l according to M. Yamaguti [11], we see that it is equal to a bounded operator H_2, together with $H_2\Lambda$, applied to u. Hence, we write $ib((\Lambda k)a - a(\Lambda k))h\Lambda = H_3\Lambda + H_4$ where H_3 is an operator of type C_{m}^m with $\sigma(H_3) = \sum_{i=1}^{\nu} \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial \xi_i}(\tilde{k}\Lambda)$ and H_4 is a bounded operator, and so we have the similar result for $ia((h\Lambda)b - b(h\Lambda))k\Lambda$. This shows that (2.9) holds for the simple case.

For the general case if we estimate the operator norms in detail using constants $A_{\alpha,l}$ of (1.4) and B_α of (1.5) we can (2.9). Q.E.D.

Lemma 4. *Let $P(t)$ and $Q(t)$ be singular integral operators of type C_{m}^m with real valued symbols defined in (x)-space with t as a parameter.*

Suppose we can write

$$\begin{aligned} I &\equiv \frac{\partial}{\partial t}\sigma(P) + \sum_{i=1}^{\nu} \left\{ \frac{\partial}{\partial x_i}\sigma(P) \frac{\partial}{\partial \xi_i}(\sigma(Q)\Lambda) - \frac{\partial}{\partial x_i}\sigma(Q) \frac{\partial}{\partial \xi_i}(\sigma(P)\Lambda) \right\} \\ &= \sigma(H) \cdot \sigma(P) \qquad (|\xi| \geqq 1) \end{aligned} \tag{2.11}$$

with some $H' \in C_{\mathrm{m}}^m$ (the condition of M. Matsumura [6]).

Then, for the operator $J = \frac{\partial}{\partial t} + (P + iQ)\Lambda$ *there exists an positive* *constant C depending only on P and Q such that for sufficiently small h, every n*[3] *and $\varphi=1+t/2h$*

$$\int \varphi^{-2n}\|Ju\|^2 dt \geqq C\left\{nh^{-2}\int \varphi^{-2n}\|u\|^2 dt+\frac{1}{n}\int \varphi^{-2n}\|P\Lambda u\|^2 dt\right\} \tag{2.12}$$
for every $u\in C_0^\infty(\Xi_h)$.

Especially, if the condition $|\sigma(P)|\geqq\delta>0$ is added, then we have for a positive constant C'

$$\begin{aligned}&\int \varphi^{-2n}\|Ju\|^2 dt \\ &\geqq C'\left\{nh^{-2}\int \varphi^{-2n}\|u\|^2 dt+\frac{1}{n}\int \varphi^{-2n}\left(\left\|\frac{\partial}{\partial t}u\right\|^2+\|\Lambda u\|^2\right)dt\right\}\end{aligned} \tag{2.13}$$
for every $u\in C_0^\infty(\Xi_h)$.

REMARK 1. i) If $\sigma(P)\equiv 0$, then it is easy to see that (2.11) is satisfied with any operator $H\in C_{\mathfrak{m}}^m$.

ii) In this paper we treat only the operator P with $\sigma(P)=\lambda_0(t,x,\xi R^{-1})$ where $\lambda_0(t,x,\eta)$ is homogeneous of order 0 with respect to η. Hence, if $|\sigma(P)|\geqq\delta>0$, then, $|\lambda_0(t,x,\eta)^{-1}|\leqq\delta^{-1}$ and $\lambda_0(t,x,\eta)^{-1}$ is homogenous of order 0, so that we can expand $\lambda_0(t,x,\eta)^{-1}=\sum_{r=1}^{\infty}a_r(t,x)\tilde{h}_{0,r}(\eta)$ by [1].

This shows that if we consider an operator H with $\sigma(H)=I\cdot\sum_{r=1}^{\infty}a_r(t,x)\tilde{h}_{0,r}(\xi R^{-1})$, then H is of type $C_{\mathfrak{m}}^m$ and (2.11) is satisfied with this H.

iii) If $\sigma(P)$ is independent of t and $\sigma(Q)=\sigma(H_0)\cdot\sigma(P)$ with $H_0\in C_{\mathfrak{m}}^m$ then (2.11) holds for $H\in C_{\mathfrak{m}}^m$ with $\sigma(H)=\sum_{i=1}^{\nu}\left\{\frac{\partial}{\partial x_i}\sigma(P)\frac{\partial}{\partial\xi_i}(\sigma(H_0))\Lambda-\frac{\partial}{\partial x_i}\sigma(H_0)\frac{\partial}{\partial\xi_i}(\sigma(P)\Lambda)\right\}$.

REMARK 2. The condition (2.11) has local property in the following sense:

If there exists a partition of the unity such that $\Theta_i(\eta|\eta|^{-1})\in C^\infty$ $(\eta\neq 0)$ $(i=1,\cdots,p)$, $\sum_{i=1}^{p}\Theta_i^2(\eta|\eta|^{-1})=1$ and (2.11) holds only in *supp*[4] $\Theta_i(\xi R^{-1})$ for each i where H may depend on i, then we can get (2.12) by dividing u according to that partition.

This fact is verified by the same method as in the appendix of the

3) In what follows we shall use n as real number $\geqq 1$.

4) For $u=u(x)$, *supp* u=*closure of* $\{x;u(x)\neq 0\}$.

note [4], so that we assume that (2.11) holds for every $\xi(|\xi|\geqq 1)$.

Proof of Lemma 4. Set $u=\varphi^n v$ for $u\in C_0^\infty(\Xi_h)$, then $v\in C_0^\infty(\Xi_h)$ and $\varphi^{-n}Ju=(v'+iQ\Lambda v)+(n(2h\varphi)^{-1}v+P\Lambda v)$.

We have

$$\begin{aligned}
&\int \varphi^{-2n}\|Ju\|^{2n}dt = \int \|(v'+iQ\Lambda v)+(n(2h\varphi)^{-1}v+P\Lambda v)\|^2 dt \\
&\quad = \int \|v'+iQ\Lambda v\|^2 dt + \int \|n(2h\varphi)^{-1}v+P\Lambda v\|^2 dt \\
\text{(2.14)}\quad &\quad + n(2h)^{-1}\int \varphi^{-1}\{(v',v)+(v,v')\}\,dt + in(2h)^{-1}\int \varphi^{-1}\{(Q\Lambda v,v)-(v,Q\Lambda v)\}dt \\
&\quad + \left\{\int (v'+iQ\Lambda v,\ P\Lambda v)\,dt + \int (P\Lambda v,\ v'+iQ\Lambda v)\,dt\right\} \\
&\quad \equiv \sum_{i=1}^{5} I_i\,.
\end{aligned}$$

By Schwarz's inequality

$$I_2 \geqq \int (n^2(2h\varphi)^{-2}\|v\|^2 - 2n(2h\varphi)^{-1}\|v\|\|P\Lambda v\| + \|P\Lambda v\|^2)\,dt$$

and integrating by parts

$$I_3 = n(2h)^{-1}\int \varphi^{-1}\frac{d}{dt}\|v\|^2 dt = n\int (2h\varphi)^{-2}\|v\|^2 dt\,,$$

hence we have

$$\text{(2.15)}\qquad I_2+I_3 = \frac{2}{3}n\int (2h\varphi)^{-2}\|v\|^2 dt + \frac{1}{4n}\int \|P\Lambda v\|^2 dt \qquad (n\geqq 1)\,.$$

Using (2.16) and Schwarz's inequality

$$\text{(2.16)}\qquad I_4 = in\int (2h\varphi)^{-1}((Q\Lambda-\Lambda Q^*)v,\ v)\,dt \geqq -C_1 n\int (2h\varphi)^{-1}\|v\|^2 dt\,.$$

Estimation for I_5 is fairly complicated. Integrating by part the second term we have

$$\begin{aligned}
I_5 &= \int \{(v'+iQ\Lambda v,\ P\Lambda v)-(P\Lambda v',\ v)-(P'\Lambda v,\ v)-(iPv,\ Q\Lambda v)\}\,dt \\
&= \int \{(v'+iQ\Lambda v,\ P\Lambda v)-((v'+iQ\Lambda v,\ \Lambda P^*v)-(iQ\Lambda v,\ \Lambda P^*v)) \\
&\quad -(P'\Lambda v,\ v)-(iP\Lambda v,\ Q\Lambda v)\}\,dt \\
&= \int (v'+iQ\Lambda v,\ (P\Lambda-\Lambda P^*)v)\,dt - \int ((P'+i(\Lambda Q^*P-P\Lambda Q))\Lambda v,\ v)\,dt \\
&= I_5'+I_5''.
\end{aligned}$$

Using (2. 6) we have

$$I_5' \geqq -\int \|v'+iQ\Lambda v\|^2 dt - \int \|(P\Lambda - \Lambda P^*)v\|^2 dt$$
$$\geqq -I_1 - C_2 \int \|v\|^2 dt .$$

By Lemma 3 and the condition (2. 11) we can write

$$(P'+i(\Lambda Q^*P - P\Lambda Q))\Lambda = H_1 P\Lambda + H_2$$

where H_1 and H_2 are bounded operators, so that we have

$$I_5'' \geqq -C_3 \left\{ \int \|P\Lambda v\| \, \|v\| dt + \int \|v\|^2 dt \right\}$$
$$\geqq -\frac{1}{8n} \int \|P\Lambda v\|^2 dt - C_4 n \int \|v\|^2 dt \qquad (n \geqq 1) .$$

Hence

$$I_1 + I_5 \geqq -\frac{1}{8n} \int \|P\Lambda v\|^2 dt - C_5 n \int \|v\|^2 dt . \tag{2.17}$$

From (2. 14)-(2. 17) it follows

$$\int \varphi^{-2n} \|Ju\|^2 dt \geqq n \int \left(\frac{2}{3} (2h\varphi)^{-2} - C_1 (2h\varphi)^{-1} - C_4 \right) \|v\|^2 dt + \frac{1}{8n} \|P\Lambda v\|^2 dt .$$

Since $\|v\|^2 = \varphi^{-2n} \|u\|^2$, $\|P\Lambda v\|^2 = \varphi^{-2n} \|P\Lambda u\|^2$ and $\varphi^{-1} = (1+t/2h)^{-1} \geqq \frac{1}{3}$ for $h > t > -h$, we have (2. 12) for sufficiently small h.

Furthermore if $|\sigma(P)| \geqq \delta > 0$, then $\|P\Lambda u\|^2 \geqq \frac{1}{2} \delta^2 \|\Lambda u\|^2 - C_6 \|u\|^2$ by (2. 8), and since $\frac{\partial}{\partial t} u = Ju - (P+iQ)\Lambda u$ we have

$$\left\| \frac{\partial}{\partial t} u \right\|^2 \leqq 2(\|Ju\|^2 + \|(P+iQ)\Lambda\|^2) \leqq 2\|Ju\|^2 + C_7 \|\Lambda u\|^2 .$$

Hence we have (2. 13) for sufficiently small h. Q.E.D.

Lemma 5. *Let $H_i(t)$ $(i=1, \cdots, k$ for $k \geqq 2)$ belong to $C_{\mathfrak{m}}^{m}$ defined in (x)-space with t as a parameter, and let $|\sigma(H_i - H_j)| \geqq \delta > 0$ $(i \neq j)$.*

Set $J_i = \frac{\partial}{\partial t} + H_i \Lambda$ $(i=1, \cdots, k)$, and let $J_{i_1} \cdot J_{i_2} \cdot \cdots \cdot J_{i_{k-1}}$ $(i_\nu \neq i_\mu$ for $\nu \neq \mu)$ be the product operators for the permutation from $J_1, J_2, \cdots,$ and J_k. Then, we have for positive constants C and C'

$$\sum_{i_1,\cdots,i_{k-1}} \|J_{i_1}\cdot\cdots\cdot J_{i_{k-1}}u\|^2$$

(2.18)
$$\geqq C\sum_{i+j=k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 - C'\sum_{0\leqq i+j\leqq k-2}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 \qquad u\in C_0^\infty(\Xi_h).$$

Proof is omitted since it is quite the same as that of Lemma 4 of the note [4].

Lemma 6. *Let $H_i(t)=P_i(t)+iQ_i(t)$ $(i=1,\cdots,k)$ belong to C_{m}^m defined in (x)-space with t as a parameter and let $|\sigma(H_i-H_j)|\geqq\delta>0$ $(i\neq j)$.*
Suppose each pair P_i and Q_i $(i=1,\cdots,k)$ satisfies the condition (2.11).
Set $J_i=\frac{\partial}{\partial t}+H_i\Lambda$ $(i=1,\cdots,k)$, then we have for the operator $A=J_1\cdot\cdots\cdot J_k$ with a constant C

(2.19)
$$\int\varphi^{-2n}\|Au\|^2dt\geqq C\sum_{0\leqq i+j=\tau\leqq k-1}(nh^{-2})^{(k-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt,$$
$$u\in C_0^\infty(\Xi_h)$$

for sufficiently small h.
Especially, if $|\sigma(P_i)|\geqq\delta>0$, then furthermore we have

(2.20)
$$\int\varphi^{-2n}\|Au\|^2dt\geqq C'\frac{1}{n}\sum_{0\leqq i+j=\tau\leqq k}h^{-2(k-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt$$
$$u\in C_0^\infty(\Xi_h).$$

Proof. (a) The proof of (2.19). For the case $k=1$ the proof is trivial from (2.12) of Lemma 4.

For the general case $k\geqq 2$, the proof is performed by the induction method.

$$\begin{aligned}J_{i_1}J_{i_2}-J_{i_2}J_{i_1} &= \left(\frac{\partial}{\partial t}(H_{i_1}-H_{i_2})\right)\Lambda+(H_{i_1}\Lambda H_{i_2}\Lambda-H_{i_2}\Lambda H_{i_1}\Lambda)\\ &= \left(\frac{\partial}{\partial t}(H_{i_1}-H_{i_2})\right)\Lambda-\{H_{i_1}(\Lambda H_{i_2}-H_{i_2}\Lambda)+(H_{i_1}H_{i_2}-H_{i_1}\circ H_{i_2})\Lambda\\ &\quad+(H_{i_2}\circ H_{i_1}-H_{i_2}H_{i_1})\Lambda-H_{i_2}(H_{i_1}\Lambda-\Lambda H_{i_1})\}\Lambda.\end{aligned}$$

Hence, using (2.7) we can write

$$J_{i_1}\cdot J_{i_2}-J_{i_2}\cdot J_{i_1}=H'\Lambda+H''+H'''$$

where H' and H'' belong to C_{m}^m and H''' is a bounded operator together with $\Lambda^i H'''\Lambda^j$ $(0\leqq i+j\leqq k)$. Using the above equality in succession we have

$$\sum_{i_2,\cdots,i_k}\left(\left\|\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_k}u\right\|^2+\|\Lambda J_{i_2}\cdot\cdots\cdot J_{i_k}u\|^2\right)$$
$$\geqq C_{10}\sum_{i+j=k}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2-C_{11}\sum_{0\leqq i+j\leqq k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2. \tag{2.26}$$

From (2.19), (2.21), (2.25) and (2.26)

$$\int\varphi^{-2n}\|Au\|^2dt\geq C_{12}\left\{\frac{1}{n}\sum_{i+j=k}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt\right.$$
$$\left.+n\sum_{0\leqq i+j=\tau\leqq k-1}h^{-2(k-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt\right\}$$
$$-C_{13}\sum_{0\leqq i+j\leqq k-1}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt.$$

Hence we have (2.20) for sufficiently small h and every $n(\geqq 1)$. Q.E.D.

§3. A priori inequality. We shall consider a differential polynomial $L=L(t, x, \lambda, \xi)$ in a neighborhood of the origin in $(\nu+1)$-space.

Let

$$L_0(t, x, \lambda, \xi)=\sum_{j+m|\alpha:\mathfrak{m}|=m}a_{j,\alpha}(t, x)\lambda^j\xi^\alpha \quad (a_{m,0}(t, x)=1) \tag{3.1}$$

be a characteristic polynomial of L with infinitely differentiable coefficients.

Now we resolve $L_0(t, x, \lambda, i\eta|\eta|^{-1})$ into the form

$$L_0(t, x, \lambda, i\eta|\eta|^{-1})=\prod_{i=1}^{k}(\lambda+\lambda_{0,i}^{(1)}(t, x, \eta))\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t, x, \eta)) \tag{3.2}$$
$$(0\leqq k<m),$$

and $L_0(t, x, \lambda, i\xi)$ into the form

$$L_0(t, x, \lambda, i\xi)=\prod_{i=1}^{k}(\lambda+\lambda_i^{(1)}(t, x, \xi))\prod_{j=1}^{m-k}(\lambda+\lambda_j^{(2)}(t, x, \xi)) \tag{3.3}$$
$$(0\leqq k<m),$$

and we write

$$\begin{aligned}\lambda_i^{(1)}(t, x, \xi)&=p_i^{(1)}(t, x, \xi)+iq_i^{(1)}(t, x, \xi) \qquad (i=1, \cdots, k)\\ \lambda_j^{(2)}(t, x, \xi)&=p_j^{(2)}(t, x, \xi)+iq_j^{(2)}(t, x, \xi) \qquad (j=1, \cdots, m-k).\end{aligned} \tag{3.4}$$

Theorem 1. *Let* $L(t, x, \lambda, \xi)=L_0(t, x, \lambda, \xi)+\sum_{j+m|\alpha:\mathfrak{m}|=\tau\leqq m-1}b_{j,\alpha}(t, x)\lambda^j\xi^\alpha$ *be a differential polynomial of order* m *with bounded measurable* $b_{j,\alpha}(t, x)$.

Suppose $\lambda_{0,i}^{(1)}$ $(i=1, \cdots, k)$ *and* $\lambda_{0,j}^{(2)}$ $(j=1, \cdots, m-k)$ *in* (3.2) *are distinct respectively* ($\lambda_{0,i}^{(1)}$ *and* $\lambda_{0,j}^{(2)}$ *may coincide at some* i *and* j) *and infinitely differentiable, and suppose each* $p_i^{(1)}$ $(i=1, \cdots, k)$ *in* (3.4) *does not vanish*

$$\|(J_1\cdot\cdots\cdot J_k-J_{i_1}\cdot\cdots\cdot J_{i_k})u\|^2\leqq C_1\sum_{0\leqq i+j\leqq k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2$$

($i_\nu \neq i_\mu$ for $\nu\neq\mu$), consequently

$$\|Au\|^2\geqq C_2\sum_{i_1,\cdots,i_k}\|J_{i_1}\cdot\cdots\cdot J_{i_k}u\|^2-C_3\sum_{0\leqq i+j\leqq k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2. \tag{2.21}$$

Applying (2.12) to the operators $J_{i_1}\cdot\cdots\cdot J_{i_k}=J_{i_1}\ (J_{i_2}\cdot\cdots\cdot J_{i_k})$ and using (2.21) it follows that

$$\begin{aligned}&\int\varphi^{-2n}\|Au\|^2dt\\ &\geqq C_4nh^{-2}\sum_{i_2,\cdots,i_k}\int\varphi^{-2n}\|J_{i_2}\cdot\cdots\cdot J_{i_k}u\|^2dt-C_5\sum_{0\leqq i+j\leqq k-1}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt.\end{aligned} \tag{2.22}$$

By the assumption of the induction

$$\int\varphi^{-2n}\|J_1\cdot\cdots\cdot J_{k-1}u\|^2dt\geqq C_6\sum_{0\leqq i+j=\tau\leqq k-2}(nh^{-2})^{(k-1-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt. \tag{2.23}$$

Considering (2.22)$+\varepsilon nh^{-2}\times$(2.23) for sufficiently small $\varepsilon(>0)$ it follows that

$$\begin{aligned}\int\varphi^{-2n}\|Au\|^2dt&\geqq C_7nh^{-2}\sum_{i_2,\cdots,i_k}\int\varphi^{-2n}\|J_{i_2}\cdot\cdots\cdot J_{i_k}u\|^2dt\\ &+C_8n\sum_{0\leqq i+j\leqq\tau\leqq k-2}(nh^{-2})^{(k-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt\\ &-C_5\sum_{0\leqq i+j\leqq k-1}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt.\end{aligned} \tag{2.24}$$

Applying (2.18) of Lemma 5 to the first term of the right hand side in (2.24), we have (2.19) for sufficiently small h and every $n\geqq 1$.

(b) The proof of (2.20). By the assumption we can apply (2.13) of Lemma 4 to $J_{i_1}\cdot\cdots\cdot J_{i_k}=J_{i_1}\ (J_{i_2}\cdot\cdots\cdot J_{i_k})$ and get

$$\begin{aligned}&\int\varphi^{-2n}\|J_i\cdot\cdots\cdot J_{i_k}u\|^2dt\\ &\geqq C_9\frac{1}{n}\int\varphi^{-2n}\left(\left\|\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_k}u\right\|^2+\|\Lambda J_{i_2}\cdot\cdots\cdot J_{i_k}u\|^2\right)dt.\end{aligned} \tag{2.25}$$

Estimating commutators $\left(\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_k}-J_{i_2}\cdot\cdots\cdot J_{i_k}\frac{\partial}{\partial t}\right)u$ and $(\Lambda J_{i_2}\cdot\cdots\cdot J_{i_k}-J_{i_2}\cdot\cdots\cdot J_{i_k}\Lambda)u$ by (2.7), and using (2.18) of Lemma 5 we have *for* $\xi \neq 0$ *and each pair* $p_j^{(2)}$ *and* $q_j^{(2)}$ $(j=1, \cdots, m-k)$ *in* (3.4) *satisfies the condition*

$$\frac{\partial}{\partial t} p_j^{(2)} + \sum_{i=1}^{\nu} \left\{ \frac{\partial}{\partial x_i} p_j^{(2)} \frac{\partial}{\partial \xi_i} q_j^{(2)} - \frac{\partial}{\partial x_i} q_j^{(2)} \frac{\partial}{\partial \xi_i} p_j^{(2)} \right\} = \sigma(H_j) p_j^{(2)} \tag{3.5}$$
$$(|\xi| \geqq 1\,;\ j=1, \cdots, m-k)$$

with some $H_j \in C_{\mathfrak{m}}^m$.

Then, there exists a positive constant C such that

$$\int \varphi^{-2n} \|Lu\|^2 dt \geqq C \sum_{j+m|\alpha|:\,\mathfrak{m}|=\tau \leqq m-1} h^{-2(m-\tau)} \int \varphi^{-2n} \left\| \frac{\partial^{j+|\alpha|}}{\partial t^j \partial x^\alpha} u \right\|^2 dt \tag{3.6}$$
$$(\varphi = 1+t/2h) \quad if \quad u \in C_0^\infty(\Omega_h)^{5)}$$

for sufficiently small h and every n $(\geqq 1)$.

REMARK. In Theorem 1, if we omit the condition "$\lambda_{0,j}^{(1)}$ $(j=1, \cdots, k)$ are distinct", we can derive the inequalities

$$\begin{aligned} & n \sum_{i+j=m-1} \int \varphi_0^{-2n} \left\| \frac{\partial^i}{\partial t^i} \Lambda_0^j u \right\|^2 dt \\ & \leqq C \left\{ \int \varphi_0^{-2n} \|Lu\|^2 dt + \sum_{i+j=\tau \leqq m-2} n^{2(m-\tau)-1} \int \varphi_0^{-2n} \left\| \frac{\partial^i}{\partial t^i} \Lambda_0^j u \right\|^2 dt \right\} \\ & \qquad \text{if} \quad u \in C_0^\infty(\Omega_{h_0}) \end{aligned} \tag{3.7}$$

for sufficiently small fixed $h_0 (>0)$, and

$$n \sum_{i+m|\alpha|:\,\mathfrak{m}|=\tau \leqq m-1} h^{-2(m-1-\tau)} \int \left\| \frac{\partial^{i+|\alpha|}}{\partial t^i \partial x^\alpha} u \right\|^2 dt \leqq C \int \|Lu\|^2 dt$$
$$\text{if} \quad u \in C_0^\infty(\Omega_h)$$

for sufficiently small $h (\leqq h_0)$ depending on n, where $\varphi_0 = 1+t/2h_0$ and Λ_0 is a convolution operator defined by $\widetilde{\Lambda_0 u}(\xi) = K(\xi)\tilde{u}(\xi)$.

These inequalities are applicable to the existence theorem and to the propagation of regularity of the solutions. The proof has been given in [5], but recently L. Hörmander [3] has already derived a similar inequality to (3.7) by another method for the case $m_j = m$ $(j=1, \cdots, \nu)$.

Furthermore we remark the following: Let H_k be a class a temperate distributions in (x)-space such that $\int (1+K(\xi))^{2k} |\tilde{u}(\xi)|^2 d\xi < \infty$ for $u \in H_k$. Setting

$$\tilde{\psi}_{\varepsilon,s}(\xi) = \frac{(1+K(\xi))^s}{\{1+\varepsilon(1+K(\xi))\}^{2s}} \qquad (s \geqq 0)$$

---

5) In what follows Ω_h means the set $\{(t, x)\,;\, t^2 + K(x)^2 < h^2\}$.

we define a convolution operator $\psi_{\varepsilon,s}$ by $\widetilde{\psi_{\varepsilon,s}u}(\xi)=\psi_{\varepsilon,s}(\xi)\tilde{u}(\xi)$. Then, by a similar method to L. Hörmander [2] pp. 142, we can prove for $|k|\leqq 2s-1\leqq R$,

$$C_R\|u\|_k^2\leqq \int_0 \varepsilon^{2(s-k)-1}\|\psi_\varepsilon u\|^2 d\varepsilon \leqq \|u\|_k^2 \quad \text{for} \quad u\in H_k$$

and $\|(\psi_\varepsilon\Lambda_0-\Lambda_0\psi_\varepsilon)u\|^2\leqq C_R'\|\psi_\varepsilon u\|^2$ for $u\in H_{-s+1}$. This shows that the inequality (3.7) holds even for $\psi_{\varepsilon,s}u$ for sufficiently large n.

If we multiply $\varepsilon^{2(s-b)-1}$ to the both sides of (3.7) for $\psi_{\varepsilon,s}u$ instead of u and integrate it with respect to ε setting $n=-a\log\varepsilon+l(a>0$; l, sufficiently large), then we have

$$\sum_{i+j=m-1}\int\left\|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right\|_{-2g(t)+b}^2 dt$$
$$\leqq C\int\|Lu\|_{-2g(t)+b}^2 dt + C_\delta\sum_{i+j\leqq m-2}\int\left\|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right\|_{-2g(t)+b+\delta}^2 dt$$

for arbitrary small δ (>0) with $g(t)=\log(1+t/2h_0)$; see [3] pp. 359.

Proof of Theorem 1. By (1.10) we can write

$$\text{(3.8)}\qquad \begin{aligned}\lambda_i^{(1)}(t,x,\xi) &= r^{1/m}\lambda_{0,i}^{(1)}(t,x,\xi R^{-1}) \quad (i=1,\cdots,k)\\ \lambda_j^{(2)}(t,x,\xi) &= r^{1/m}\lambda_{0,j}^{(2)}(t,x,\xi R^{-1}) \quad (j=1,\cdots,m-k)\end{aligned}$$

where r and R are defined by (1.7) and (1.8) respectively.

Since $\lambda_{0,i}^{(2)}(t,x,\eta)$ and $\lambda_{0,j}^{(2)}(t,x,\eta)$ are infinitely differentiable, by the remark at the end of §1, $\lambda_{0,i}^{(1)}(t,x,\xi R^{-1})$ and $\lambda_{0,j}^{(2)}(t,x,\xi R^{-1})$ become the symbols of some operators of type C_{m}^{m}.

Now we consider singular integral operators $H_i^{(1)}$ $(i=1,\cdots,k)$ and $H_j^{(2)}$ $(j=1,\cdots,m-k)$ with the symbols $\lambda_{0,i}^{(1)}(t,x,\xi R^{-1})$ and $\lambda_{0,j}^{(2)}(t,x,\xi R^{-1})$ respectively, and consider a convolution operator Λ defined by $\widetilde{\Lambda u}=r^{1/m}\tilde{u}(\xi)$.

Set

$$\text{(3.9)}\qquad A_1A_2=\prod_{i=1}^{k}\left(\frac{\partial}{\partial t}+H_i^{(1)}\Lambda\right)\prod_{j=1}^{m-k}\left(\frac{\partial}{\partial t}+H_j^{(2)}\Lambda\right),$$

then, by the assumption of the theorem we can apply (2.20) to A_1 and (2.19) to A_2 respectively. Applying (2.20) to A_1 we have

$$\int\varphi^{-2n}\|A_1(A_2u)\|^2 dt\geqq C\frac{1}{n}\sum_{0\leqq i+j=\tau\leqq k}h^{-2(k-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j A_2u\right\|^2 dt.$$

Estimating the commutators $\left(\frac{\partial^i}{\partial t^i}\Lambda^j A_2-A_2\frac{\partial^i}{\partial t^i}\Lambda^j\right)u$ by (2.7) and applying (2.19) to A_2 we have

$$\int \varphi^{-2n}\|A_1A_2u\|^2dt \geqq C_1\frac{1}{n}\sum_{0\leqq i+j=\tau\leqq k} h^{-2(k-\tau)}\Big\{C_2\sum_{0\leqq i'+j'=\tau'\leqq m-k-1}(nh^{-2})^{(m-k-\tau')}$$
$$\times\int \varphi^{-2n}\left\|\frac{\partial^{i+i'}}{\partial t^{i+i'}}\Lambda^{j+j'}u\right\|^2 dt - C_3\sum_{0\leqq i'+j'=\tau'\leqq \tau+m-k-1}\int\varphi^{-2n}\left\|\frac{\partial^{i'}}{\partial t^{i'}}\Lambda^{j'}u\right\|^2 dt\Big\}.$$

Remarking $m-k-\tau'\geqq 1$ for $\tau'\leqq m-k-1$ and $\tau+m-k-1\leqq m-1$ for $\tau\leqq k$ we have

$$(3.10)\qquad \int \varphi^{-2n}\|A_1A_2u\|^2dt \geqq C_4\sum_{0\leqq i+j=\tau\leqq m-1} h^{-2(m-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt$$

for sufficiently small h.

On the other hand estimating the commutators by (2. 7) we have

$$(3.11)\qquad A_1A_2u=\sum_{j=0}^m H_j\frac{\partial^j}{\partial t^j}\Lambda^{m-j}u+\sum_{0\leqq i+j\leqq m-1}H_{i,j}\frac{\partial^i}{\partial t^i}\Lambda^j u$$

where H_j belong $C^m_{\mathfrak{m}}$ and $H_{i,j}$ are bounded operators. From (3. 1), (3. 3) and (3. 8) we have $\sigma(H_j)r^{1-j/m}=\sum_{m|\alpha:\mathfrak{m}|=m-j}a_{j,\alpha}(t,x)(i\xi)^\alpha$, hence we have

$$\sum_{j=0}^m H_j\frac{\partial^j}{\partial t^j}\Lambda^{m-j}u=\frac{1}{\sqrt{2\pi}^\nu}\int e^{ix\cdot\xi}L_0\Big(t,x,\frac{\partial}{\partial t},i\xi\Big)\tilde{u}(t,\xi)d\xi$$
$$=L_0\Big(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x}\Big)u,$$

and consequently we have by (3. 11)

$$(3.12)\qquad \left\|\Big(A_1A_2-L_0\Big(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x}\Big)\Big)u\right\|^2\leqq C_5\sum_{0\leqq i+j\leqq m-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2.$$

From (3. 10) and (3. 12) it follows that

$$(3.13)\qquad \int\varphi^{-2n}\|L_0u\|^2dt\geqq C_6\sum_{0\leqq i+j=\tau\leqq m-1}h^{-2(m-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2dt$$

for sufficiently small h.

As $|\xi_j|\leqq K(\xi)^{m/m_j}\leqq C_7r^{1/m_j}$ by (1. 3) and (2. 2), we have

$$\left\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u\right\|^2=\|(i\xi)^\alpha\tilde{u}\|^2\leqq C_8\|r^{|\alpha:\mathfrak{m}|}\tilde{u}\|^2\leqq C_9\|K(\xi)^{m|\alpha:\mathfrak{m}|}\tilde{u}\|^2.$$

On the other hand, using Fourier transform we have for $u\in C_0^\infty(\Omega_h)$ $h^{-2(a-b)}\|\Lambda_0^b u\|^2\leqq C_a\|\Lambda_0^a u\|^2\leqq C'_a\|r^{a/m}\tilde{u}\|^2$ $(0\leqq b\leqq a)$ where Λ_0^a is defined by $\widetilde{\Lambda_0^a u}(\xi)=K(\xi)^a\tilde{u}(\xi)$.

Hence, we have

$$(3.14)\qquad \sum_{j+m|\alpha|:\,m|=\tau\leq m-1} h^{-2(m-\tau)}\left\|\frac{\partial^{j+|\alpha|}}{\partial t^j\partial x^\alpha}u\right\|^2 \leq C_{10}\sum_{j+i=\tau\leq m-1} h^{-2(m-\tau)}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2$$

for sufficiently small h.

From (3.13) and (3.14) we have (3.6) for sufficiently small h. Q.E.D.

§4. **Uniqueness.** We are concerned with the uniqueness for the solution of the Cauchy problem. Let $S(t, x)$ be a continuous real valued function defined in a neighborhood of the origin such that the set $\{(t, x); S(t, x)\geqq 0\}$ lies in the half space $t\geqq 0$ and meets the plane $t=0$ only at the origin, then we have the following.

Theorem 2. *Let L be a differential polynomial which satisfies the condition of Theorem 1.*

Suppose $u=u(t, x)\in C^m_{(t,x)}$ defined in a neighborhood of the origin satisfies the differential equation $L\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u(t, x)=0$ and vanishes on $\{(t, x); S(t, x)\leqq 0\}$.

Then $u=u(t, x)$ vanishes identically in a neighborhood of the origin.

Proof. For $\psi(t)\in C^\infty_{(t)}$ such that

$$(4.1)\qquad \begin{aligned}\psi(t) &= 1 \quad \text{for} \quad t\leqq 2^{-l}h\,,\\ \psi(t) &= 0 \quad \text{for} \quad t\geqq 2^{-(l-1)}h\,.\end{aligned}$$

We consider $w(t, x)=\psi(t)u(t, x)$, then by the assumption of u $w(t, x)$ belongs to $C^m_0(\Omega_h)$ for a sufficiently large fixed l. By approximating w by $u_n\in C^\infty_0(\Omega_h)$ it is easy to see that the inequality (3.6) holds for $w(t, x)\in C^m_0(\Omega_h)$, so that we have

$$\int \varphi^{-2n}\|Lw(t, x)\|^2dt \geqq C_1\int \varphi^{-2n}\|w\|^2dt\,.$$

By (4.1) it follows that $Lw=Lu=0$ and $w=u$ for $t\leqq 2^{-l}h$, hence we have

$$\int_{t\geqq 2^{-l}h}\varphi^{-2n}\|Lw\|^2dt \geqq C_1\int_{t\leqq 2^{-(l+1)}h}\int \varphi^{-2n}\|u\|^2dt\,.$$

Remarking $\varphi\geqq 1+2^{-(l+1)}$ for $t\geqq 2^{-l}h$ and $\varphi\leqq(1+2^{-(l+2)})$ for $t\leqq 2^{-(l+1)}h$, we have

$$\left(\frac{1+2^{-(l+2)}}{1+2^{-(l+1)}}\right)^{2n}\int_{t\geqq 2^{-l}h}\|Lw\|^2dt \geqq C_1\int_{t\leqq 2^{-(l+1)}h}\|u\|^2dt$$

and letting $n\to\infty$ we get u vanishes identically in $0\leqq t\leqq 2^{-(l+1)}h$. This completes the proof. Q.E.D.

Next we consider the case when the Cauchy data are given on a plane portion. In this case we transform the plane portion to a convex surface by Holmgren's transformation, and apply Theorem 2. Hence, for a while we investigate how a differential operator is transformed by Holmgren's transformation.

Let $(m, \mathfrak{m})=(m, m_1, \cdots, m_\nu)$ satisfy the condition

$$m_j = m \quad \text{or} \quad m_j \leqq \frac{1}{2} m \quad (j = 1, \cdots, \nu). \tag{4.2}$$

We consider a differential operator

$$M\left(s, y, \frac{\partial}{\partial s}, \frac{\partial}{\partial y}\right) = M_0\left(s, y, \frac{\partial}{\partial s}, \frac{\partial}{\partial y}\right) + \sum_{j+m|\alpha:\mathfrak{m}|\leqq m-1} b_{j,\alpha}(s, y) \frac{\partial^{j+|\alpha|}}{\partial s^j \partial y^\alpha} \tag{4.3}$$

in a neighborhood of the origin, where M_0 is the principal part of M and of the form

$$M_0\left(s, y, \frac{\partial}{\partial s}, \frac{\partial}{\partial y}\right) = \sum_{j+m|\alpha:\mathfrak{m}|=m} a_{j,\alpha}(s, y) \frac{\partial^{j+|\alpha|}}{\partial s^j \partial y^\alpha} \quad (a_{m,0}(0, 0) = 1). \tag{4.4}$$

We take Holmgren's transformation

$$\begin{aligned} t &= s + \sum_{j=1}^{\nu} y_j^2, \\ x_i &= y_i \quad (i = 1, \cdots, \nu). \end{aligned} \tag{4.5}$$

Then, as we have

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i} + 2x_i \frac{\partial}{\partial t} \quad (i = 1, \cdots, \nu),$$

the associated operator L defined by $Lu(t-|x|^2, x)=Mu(s, y)$ is of the form

$$L\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = M\left(t-|x|^2, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial t}, \cdots, \frac{\partial}{\partial x_\nu} + 2x_\nu \frac{\partial}{\partial t}\right). \tag{4.6}$$

Remarking (4.2) it is evident that the characteristic polynomial L_0 of L is obtained

$$L_0(t, x, \lambda, \xi) = M_0(t-|x|^2, x, \lambda, \xi_1 + 2x_1\delta_{m,m_1}\lambda, \cdots, \xi_\nu + 2x_\nu\delta_{m,m_\nu}). \tag{4.7}$$[6]

For $\lambda^j\xi^\alpha(j+m|\alpha:\mathfrak{m}|=m)$ if we replace one of ξ_j $(m_j \leqq \frac{1}{2}m)$ by λ, then $\lambda^j\xi^\alpha$ changes to $\lambda^{j+1}\xi_1^{\alpha_1}\cdots\xi_j^{\alpha_j-1}\cdots\xi_\nu^{\alpha_\nu}$ and for this

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6) $\delta_{i,j}=\begin{cases}1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{cases}$

$$(j+1)+m\left(\frac{\alpha_1}{m_1}+\cdots+\frac{\alpha_j-1}{m_j}+\cdots+\frac{\alpha_\nu}{m_\nu}\right)=(j+m|\alpha:\mathfrak{m}|)+1-\frac{m}{m_j}$$
$$\leq m+1-2=m-1,$$

hence L becomes

$$L(t,x,\lambda,\xi)=L_0(t,x,\lambda,\xi)+\sum_{j+m|\alpha:\mathfrak{m}|\leq m-1} b^*_{j,\alpha}(t,x)\lambda^j\xi^\alpha. \tag{4.8}$$

By (4.7), if we write

$$L_0(t,x,\lambda,\xi)=M_0(t-|x|^2,\lambda,\xi)+\sum_{j+m|\alpha:\mathfrak{m}|=m} a^*_{j,\alpha}(t,x)\lambda^j\xi^\alpha,$$

we have $a^*_{j,\alpha}(t,x)=0(|x|)$.

This shows that if the characteristic roots $\lambda_0(t,x,\eta)$ of $M_0(s,y,i\eta|\eta|^{-1})=0$ are distict and infinitely differentiable, then those of L_0 are also distinct and infinitely differentiable for sufficiently small y.

Theorem 3. *Let $M=M\left(s,y,\frac{\partial}{\partial s},\frac{\partial}{\partial y}\right)$ be a differential polynomial of the form* (4.3).

Let $L=a^*\left(\frac{\partial^m}{\partial t^m}+**\right)$ be the associated operator obtained by the transformation (4.6).

Suppose $a^{*^{-1}}L=\frac{\partial^m}{\partial t^m}+**$ satisfies the conditions of Theorem 1, *and $u=u(s,y)\in C^m_{(s,y)}$ defined in a neighborhood of the origin satisfies the differential equation $M\left(s,y,\frac{\partial}{\partial s},\frac{\partial}{\partial y}\right)u(s,y)=0$ and satisfies the initial conditions*

$$\frac{\partial^{j-1}}{\partial t^{j-1}}u(0,y)=0 \quad (j=1,\cdots,m). \tag{4.9}$$

Then, $u(s,y)$ vanishes identically in a neighborhood of the origin.

Proof. If we set $U(s,y)=u(s,y)$ for $s\geq 0$, and $U(s,y)=0$ for $s\leq 0$, then $U(s,y)\in C^m_{(s,y)}$ and $MU=0$ in a neighborhood of the origin.

Now we take Holmgren's transformation (4.5), then $U=U(t-|x|^2,x)=0$ on $\{(t,x);\ t\leq|x|^2\}$ and $a^{*^{-1}}LU(t-|x|^2,x)=0$.

Here we remark that $a^*=M_0(t-|x|^2,x,1,2x_1\delta_{m,m_1},\cdots,2x_\nu\delta_{m,m_\nu})$ by (4.7), and $|a^*|\geq\frac{1}{2}$ for sufficiently small t and x as $a_{m,0}(0,0)=1$. Hence, for the operator $a^{*^{-1}}L$ we can apply Theorem 2 and get that $U(s,y)=U(t-|x|^2,x)$ vanishes identically in a neighborhood of the origin, so that $u(s,y)=U(s,y)=0$ for $s(\geq 0)$, so we get $u(s,y)=0$ for $s\leq 0$. This completes the proof. Q.E.D.

EXAMPLE. i) Consider a parabolic polynomial

$$M_0 = \lambda^2+2(\sum_{i=1}^{\nu-1} a_i(s, y)\xi_i)\lambda+ \sum_{i,j=1}^{\nu-1} a_{ij}(s, y)\xi_i\xi_j-b(s, y)\xi_\nu \qquad (b \neq 0)$$

where

$$(4.10) \qquad \lambda^2+2(\sum_{i=1}^{\nu-1} a_i\xi_i)\lambda+ \sum_{i,j=1}^{\nu-1} a_{ij}\xi_i\xi_j \geqq \delta(\lambda^2+ \sum_{i=1}^{\nu-1}\xi_i^2) \qquad (\delta>0).$$

Setting $A=(\sum_{i=1}^{\nu-1} a_i\eta_i)$ and $B=\sum_{i,j=1}^{\nu-1} a_{ij}\eta_i\eta_j-(\sum_{i=1}^{\nu-1} a_i\eta_i)^2+ib\eta_\nu|\eta|$ we have

$$\begin{aligned} M_0(s, y, \lambda, i\eta|\eta|^{-1}) &= \{\lambda+\lambda_{0,1}(s, y, \eta)\}\{\lambda+\lambda_{0,2}(s, y, \eta)\} \\ &= \{\lambda+|\eta|^{-1}(iA+B)\}\{\lambda+|\eta|^{-1}(iA-B)\}, \end{aligned}$$

and $\sum_{i,j=1}^{\nu-1} a_{ij}\eta_i\eta_j-(\sum_{i=1}^{\nu-1} a_i\eta_i)^2 \geqq \delta_1|\eta|^2$ $(\delta_1>0)$ from (4.10). In this case $(m, \mathfrak{m})=(2, 2, \cdots, 2, 1)$ satisfies (4.2), and $\lambda_{0,1}$ and $\lambda_{0,2}$ are distinct and the real parts of these roots do not vanish. Hence for this operator or the product operator of two such parabolic operators the uniqueness theorem holds when the initial values are prescribed on a plane portion.

More generally for the operator $M_0\left(s, y, \frac{\partial}{\partial s}, \frac{\partial}{\partial x}\right)$ of (4.4), if the equation $M_0(s, y, \lambda, i\eta|\eta|^{-1})=0$ has distinct roots whose real parts do not vanish, then for this operator the same proposition as the above case holds.

ii) Consider $M=M_0+\sum_{j+m|\alpha:\mathfrak{m}|\leqq m-1} b_{j,\alpha}(s, y)\lambda^j\xi^\alpha$ with $(m, \mathfrak{m})$ satisfying (4.2).

If we assume that the coefficients of $M_0(t, x, i\lambda, i\xi)$ are real and the characteristic roots of $M_0(t, x, \lambda, i\eta|\eta|^{-1})=0$ are distinct, then, the associated characteristic polynomial L_0 of (4.7) has distinct roots which are purely imaginary or have non vanishing real parts because $L_0(t, x, i\lambda, i\xi)$ has real coefficients. Hence, for such operator M, or more generaly for the product operator $M_1 M$ the uniqueness theorem holds, where M_1 is an operator whose characteristic polynomial has distinct roots with non vanishing real parts. A non-trivial example is made by the following way. Set

$$F(s, y, \theta, \xi) =$$
$$\sum_{j=1}^{l} C_j(\theta-a_1K(\xi)^{2m})\cdot\cdots\cdot(\theta-a_{j-1}K(\xi)^{2m})(\theta-a_{j+1}K(\xi)^{2m})\cdot\cdots\cdot(\theta-a_lK(\xi)^{2m})$$

$(a_j=a_j(s,y)\ (j=1, \cdots, l);\ 0<a_1<\cdots<a_l;\ l\geqq 2;\ C_j>0\ (j=1,\cdots,l), \sum_{j=1}^{l} C_j=1)$ with $K(\xi)$ of (1.3). Then the equation $F(s, y, \theta, \xi)=0$ has distinct positive roots since $sign\, F(s, y, a_j K(\xi)^{2m}, \xi) = sign\,(-1)^{l-j}$ $(j=1, \cdots, l)$.

Hence, if we set $M_0(s, y, \lambda, \xi) = F(s, y, (i\lambda)^{2m}, i\xi)$, then $M_0(s, y, i\lambda, i\xi)$ is of order $2(l-1)m$ and has real coefficients, and the equation $M_0(s, y, \lambda, i\eta|\eta|^{-1}) = 0$ has distinct roots. This shows that the uniqueness holds for the operator

$$M\left(s, y, \frac{\partial}{\partial s}, \frac{\partial}{\partial y}\right) = M_0\left(s, y, \frac{\partial}{\partial s}, \frac{\partial}{\partial y}\right) + \sum_{j+m|\alpha|:\mathfrak{m}|\leq 2(l-1)m-1} b_{j,\alpha}(s, y) \frac{\partial^{j+|\alpha|}}{\partial s^j \partial y^\alpha}.$$

OSAKA UNIVERSITY

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