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ON THE DILATATION IN FINSLER SPACES

To Prof. K. Shoda in celebration of his 60th birthday

By

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It is well known that a dilatation in the euclidean space, defined as a parallel translation of a plane element to the orthogonal direction by constant length, is a contact transformation. In the present paper we consider a contact structure in Finsler space and prove that a dilatation defined on it is also a contact transformation. Moreover it is proved here that a dilatation on the Riemannian manifold of constant curvature preserves a Riemannian metric constructed appropriately on the dual tangent bundle of the manifold. A greater part of this paper is not essentially new, but is a reproduction of classical results, mainly due to E. Cartan, from a modern geometrical point of view.

§ 1. Contact structure and e-curves

1. We take an m-dimensional differentiable manifold M with local coordinates $x^1, \cdots, x^m$ and a 2-form

$$\alpha = \tfrac{1}{2} a_{ij} dx^i \wedge dx^j \qquad (a_{ij} = -a_{ji}) \tag{1}$$

on it. Throughout the paper we assume differentiability C^∞. Then we have ([6] p. 138)

Theorem. *If there exists an affine connection without torsion for which a tensor field* (a_{ij}) *is parallel, then we have* $d\alpha = 0$.

Conversely, if $d\alpha = 0$, *there exists locally an affine connection without torsion for which* (a_{ij}) *is parallel. Moreover such a connection exists globally when the dimension of the manifold is even and the rank of* α *is maximal.*

2. We take a $2n-1$-dimensional differentiable manifold M with a closed 2-form α of a maximal rank $2n-2$. Especially, if $\alpha = d\omega$ (exact) and $\omega \wedge \alpha^{n-1} \neq 0$, M is called to have a contact structure. We consider a differential equation $\mathrm{i}(X)\alpha = 0$ which holds for all vector fields X. When α is expressed as (1), $\mathrm{i}(X)\alpha = 0$ reduces to

$$a_{ij}dx^j = 0 \qquad (i, j=1, \cdots, 2n-1). \tag{2}$$

As the rank of (a_{ij}) is $2n-2$ these can also be written as

$$\frac{dx^1}{c^1} = \frac{dx^2}{c^2} = \cdots\cdots = \frac{dx^{2n-1}}{c^{2n-1}}$$

with certain functions $c^1, \cdots, c^{2n-1}$. This defines curves on M, which we call *e-curves.* Then we have

Theorem 1. *An e-curve $x=x(t)$ is a path of an affine connection for which (a_{ij}) is parallel. Conversely, if a curve is a path of an affine connection for which (a_{ij}) is parallel and satisfies an initial condition $\left(a_{ij}\dfrac{dx^j}{dt}\right)_{t=0}=0$, it is an e-curve.*

Proof. For an e-curve $x=x(t)$ we have $a_{ij}dx^j/dt=0$, and if we put $dx^i/dt=v^i$, we get $a_{ij}v^j=0$. By covariant differentiation $Da_{ij}/dt\cdot v^j+a_{ij}Dv^j/dt=0$. Hence $a_{ij}Dv^j/dt=0$. As the rank of (a_{ij}) is maximal, we have $Dv^i/dt=kv^i$, and so the curve $x(t)$ is a path.

Conversely, if a curve $x=x(t)$ is a path, we have $Dv^i/dt=kv^i$ for $v^i=dx^i/dt$. As (a_{ij}) is parallel we have $D(a_{ij}v^j)/dt=ka_{ij}v^j$ and by the assumption $(a_{ij}v^j)_{t=0}=0$ we have always $a_{ij}v^j=0$ and so it is an e-curve.

An e-curve is important in our investigation, but affine connections considered above are unnecessary for our later discussion.

3. We consider a two-dimensional submanifold S of M generated by a one-parametric family of e-curves with a parameter ε. We denote by t a parameter on each e-curve. Then 2-form α restricted to S is

$$\alpha = a_{ij}dx^i \wedge dx^j = 2\,a_{ij}\frac{\partial x^j}{\partial t}\frac{\partial x^i}{\partial \varepsilon}d\varepsilon \wedge dt\,.$$

Along e-curves we have $a_{ij}dx^j=0$ and so $a_{ij}\partial x^j/\partial t=0$. Hence $\alpha=0$. Thus we get

Theorem 2. *M is a differentiable manifold with a closed 2-form α of maximal rank. Then the 2-form α vanishes on a two-dimensional submanifold S generated by a one-parametric family of e-curves. If $\alpha=d\omega$ and c bounds a simply connected region on S, we have $\int_c \omega=0$.*

§ 2. Finsler space and contact structure

1. M is an n-dimensional differentiable manifold with local coordinates $x=(x^1, \cdots, x^n)$ for a point on M. Local coordinates on the tangent bundle $\mathrm{T}(M)$ of M are given by (x, y) with $y=(y^1, \cdots, y^n)$ which are vector components in the tangent space at x. Finsler space is a manifold M with a function F on $\mathrm{T}(M)$ such that $F=F(x, y)$ is linear in y and moreover

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial y^i \partial y^j}\right) = n-1 . \qquad (i, j = 1, \cdots, n) \tag{3}$$

By linearity we have

$$y^i \frac{\partial F}{\partial y^i} = F . \tag{4}$$

In the Finsler space a length of a curve $x=x(t)$ $(t_1 \leqq t \leqq t_2)$ is defined by

$$s = \int_{t_1}^{t_2} F\left(x, \frac{dx}{dt}\right) dt .$$

We consider the dual tangent bundle ${}^c\mathrm{T}(M)$ of M and denote the local coordinates by (x, z) with $z=(z_1, \cdots, z_n)$ dual to $y=(y^1, \cdots, y^n)$. Next we put

$$p_i = \frac{\partial F}{\partial y^i} \tag{5}$$

and define a mapping $\varphi: \mathrm{T}(M) \to {}^c\mathrm{T}(M)$ by $(x, y) \to (x, p)$ with $p=(p_1, \cdots, p_n)$. It can be verified that the mapping is globally defined. We put

$$N = \varphi(\mathrm{T}(M)) . \tag{6}$$

N can be obtained explicitly in the following manner. By virtue of (3) we can assume $\det(\partial^2 F/\partial y^a \partial y^b) \neq 0$ $(a, b=1, \cdots, n-1)$ at a point (x, y) without loss of generality. Hence in a neighborhood of a point (x, y) in $\mathrm{T}(M)$ we get from (5)

$$y^a = f^a(x\,;\, p_1, \cdots, p_{n-1}, y^n) \qquad (a = 1, \cdots, n-1) ,$$

and when we put these into $p_n = \partial F/\partial y^n$, we obtain

$$p_n = g(x\,;\, p_1, \cdots, p_{n-1}), \tag{7}$$

because we have $\det(\partial p_i/\partial y^j) = \det(\partial^2 F/\partial y^i \partial y^j)) = 0$ and there exists a functional relation between x, p. Thus $N=\varphi(\mathrm{T}(M))$ is a submanifold of ${}^c\mathrm{T}(M)$. Generally we call *p-manifold* in $\mathrm{T}(M)$ the submanifold which can be locally expressed as

$$G(x, p) = 0 \qquad (\mathrm{grad}_p\, G \neq 0) . \tag{8}$$

Then $N=\varphi(\mathrm{T}(M))$ is a p-manifold by virtue of (7).

EXAMPLE. As to Riemannian metric we have $F(x, y)=\sqrt{g_{ij}(x)y^i y^j}$ and so

$$p_i = \partial F/\partial y^i = g_{ij}y^j/F\,, \quad G(x, p) = g^{ij}p_i p_j - 1 = 0 \tag{9}$$

with (g^{ij}) inversive to (g_{ij}).

We prepare matters necessary for our later discussion. When we put $p_i=\partial F/\partial y^i=\psi^i(x, y)$, we have naturally $G(x, p)=0$, and by differentiating with respect to y^i we get $G_{p_j}\partial p_i/\partial y^j=0$, since $\partial p_i/\partial y_j=\partial p_j/\partial y_i$. On the other hand $F(x, y)$ is linear in y and so $p_i(x, y)$ is homogeneous of degree 0 in y. Hence we have $y^j\partial p_i/\partial y^j=0$. As the rank of a matrix $(\partial p_i/\partial y^j)=(\partial^2 F/\partial y^i\partial y^j)$ is $n-1$, we get

$$y^i = \lambda G_{p_i} \qquad (i = 1, \cdots, n) \tag{10}$$

and hence

$$\lambda p_i G_{p_i} = y^i p_i = y^i\frac{\partial F}{\partial y^i} = F\,. \tag{11}$$

Next we take a curve $c: x=x(t)$ on M. Then a curve c' is defined in $\mathrm{T}(M)$ by $(x, \dot{x})$ and by the mapping $\varphi: (x, \dot{x})\to(x, p)$ a curve $c''=\varphi(c')$ is defined in N. We call the curve c'' a *lift* of a curve c on M. Here we have by virtue of (4)

$$p_i dx^i = \frac{\partial F}{\partial y^i}(x, \dot{x})\dot{x}^i dt = F(x, \dot{x})dt\,.$$

Hence $p_i dx^i$ for a lift c'' is an arc-element of a curve c on M.

2. On the dual tangent bundle ${}^c\mathrm{T}(M)$ with local coordinates (x, z) 1-form $z_i dx^i$ can be defined globally. We restrict this to the p-manifold N and we get

$$\omega = p_i dx^i\,. \tag{12}$$

Hence

$$\alpha = d\omega = dp_i\wedge dx^i\,. \tag{13}$$

ω defines a contact structure on N with exception of certain points. In fact, by (12) and (13)

$$\begin{aligned}\omega\wedge(d\omega)^{n-1} &= (-1)^{n(n-1)/2}(n-1)!\, dx^1\wedge\cdots\wedge dx^n\\ &\quad\wedge\Big(\sum_i(-1)^{i-1}p_i dp_1\wedge\cdots\wedge\widehat{dp_i}\wedge\cdots\wedge dp_n\Big)\,,\end{aligned}$$

where $\widehat{dp_i}$ means a lack of a term dp_i. In case $G_{p_n}\neq 0$ we have by (8)

$$dp_n = -\frac{1}{G_{p_n}}(G_{x^i}dx^i+G_{p_a}dp_a) \qquad (i=1,\cdots,n\,;\ a=1,\cdots,n-1) \tag{14}$$

and so

$$\omega \wedge (d\omega)^{n-1} = (-1)^{(n-1)(n-2)/2}(n-1)!\frac{p_i G_{p_i}}{G_{p_n}} dx^1 \wedge \cdots \wedge dx^n \wedge dp_1 \wedge \cdots \wedge dp_{n-1}.$$

By (11) this vanishes only for y such that $F(x, y)=0$. Thus we get

Theorem 3. *$\omega = p_i dx^i$ defines a contact structure on N except for points (x, p) corresponding to such (x, y) that $F(x, y)=0$ holds.*

By the discussion in section 1 an e-curve is introduced on N according to the 2-form α. An e-curve is a solution of the equation $\mathrm{i}\left(\frac{\partial}{\partial x^i}\right)\alpha=0$, $\mathrm{i}\left(\frac{\partial}{\partial p_a}\right)\alpha=0$, which we will write explicitly in the case where N is given by

$$G(x, p) = 0. \tag{15}$$

We assume $G_{p_n} \neq 0$ without loss of generality. Then we have by (14)

$$\alpha = dp_i \wedge dx^i = dp_a \wedge dx^a - \frac{1}{G_{p_n}}(G_{x^i} dx^i + G_{p_a} dp_a) \wedge dx^n.$$

Hence we get from $\mathrm{i}\left(\frac{\partial}{\partial x^i}\right)\alpha=0$ and $\mathrm{i}\left(\frac{\partial}{\partial p_a}\right)\alpha=0$

$$\frac{dx^1}{G_{p_1}} = \cdots\cdots = \frac{dx^n}{G_{p_n}} = \frac{dp_1}{-G_{x^1}} = \cdots\cdots = \frac{dp_n}{-G_{x^n}}. \tag{16}$$

This is a differential equation of an e-curve on N. Along the solutions $G(x, p)$ is constant and when an initial condition $x(0)$, $p(0)$ satisfies the relation $G(x(0), p(0))=0$, we have always $G(x, p)=0$, and the solution is an e-curve on N.

We project an e-curve $e: x=x(t),\ p=p(t)$ onto a curve $E: x=x(t)$ on M. Then we have $dx^i/dt=\mu G_{p_i}(x, p)$ by virtue of (16) and if y is such that (x, y) is mapped on (x, p) by φ, we have $y^i=\lambda G_{p_i}(x, p)$ by (10). Hence $dx^i/dt=\mu\lambda^{-1}y^i$, and p of the curve e corresponds to dx/dt of E.

Now we can prove the following theorem due to E. Cartan. (cf. [3] p. 187)

Theorem 4. *M is a Finsler space and N is the p-manifold constructed over M. If we project any e-curve on N onto M, we get an extremal of the Finsler space M. Conversely all the extremals of M can be obtained in this way.*

Proof. We take an e-curve c on N and two points a and b on c, whose projections on M are a curve C and two points A and B. We connect the two points A and B by a one-parametric family of curves

$C_\varepsilon: x=x(t, \varepsilon)$ $(t_1 \leqq t \leqq t_2)$ and we assume $C_\varepsilon = C$ for $\varepsilon=0$. We lift these curves C_ε to c_ε on N, which can be expressed as $x=x(t, \varepsilon)$ and $p=p(t, \varepsilon)$. We denote differential for the variable t by dt and that of ε by $\delta\varepsilon$, which are independent. Then we have for $d\omega = dp_i \wedge dx^i$

$$d(\omega(\delta)) - \delta(\omega(d)) = dp_i \delta x^i - \delta p_i dx^i . \tag{17}$$

This formula, due to E. Cartan, is now justified in modern theory as

$$d(\omega(E))(T) - d(\omega(T))(E) = dp(T)dx(E) - dp(E)dx(T)$$

by taking $T = \partial/\partial t$, $E = \partial/\partial\varepsilon$. We use an old style for the sake of brevity and we get

$$\delta\omega(d) = d\omega(\delta) - (dp_i \delta x^i - \delta p_i dx^i) .$$

Along an e-curve c we have $dp_i = -\lambda G_{x^i} dt$, $dx^i = \lambda G_{p_i} dt$ and hence

$$dp_i \delta x^i - \delta p_i dx^i = -\lambda \delta G dt = 0 ,$$

as G vanishes always. Moreover the points A, B corresponding to t_1 and t_2 are fixed each and so $\omega(\delta)=0$ for $t=t_1, t_2$. Thus we have $\delta \int \omega(d) = 0$ along the curve c. As $\omega(d)$ is an arc-elements along the curves C on M the curve C is an extremal.

As e-curves can be taken in such a way that their projection on M passes through any point x on M and its tangent at x takes any direction when we take an initial condition for an e-curve suitably. Hence any extremal on M is a projection of an e-curve.

As an application of Theorem 4 we can prove Jacobi's enveloping theorem by the use of Stokes's theorem. We take a point x on a curve $x=x(t)$ and a direction represented by (x, y). This direction is called transversal to the curve at the point if $p_i dx^i/dt = 0$ for p corresponding to y by the mapping $\varphi: (x, y) \to (x, p)$. We take a one-parametric family of extremals having contact with a curve C and a curve T transversal to the extremals. For two extremals of the family points of contact with C are A, B and the points of intersection with T are A', B' respectively. Then Jacobi's enveloping theorem asserts

$$\widehat{A'A} - \widehat{B'B} = \widehat{BA} ,$$

where $\widehat{A'A}$, $\widehat{B'B}$ mean the length on extremals and $\widehat{BA}$ that of C. This can be proved as follows under the assumption that the region D bounded by the curves $A'ABB'A$ and generated by the extremals is homeomorphic to a simply connected domain on a plane.

We take tangent vectors $(x, \dot{x})$ at each point x of the extremals of the family in question, and $p=p(x)$ such that $\varphi:(x,\dot{x})\rightarrow(x,p)$. Then $\omega=p_i dx^i$ is a 1-form on our Finsler space. We lift the region D to ${}^c\mathrm{T}(M)$ and apply Theorem 2. Then we have

$$0=\int_D d\omega=\int_{A'A}\omega+\int_{AB}\omega+\int_{BB'}\omega+\int_{B'A'}\omega=\widehat{A'A}-\widehat{BA}-\widehat{B'B},$$

which was to be proved.

Hamiltonian function H in the classical theory can be derived as follows. As $F(x,y)$ is linear in y we can put $F(x,y)=y^n L(x,z)$, where $z=(z^1,\cdots,z^{n-1})$ and $z^a=y^a/y^n$ $(a=1,\cdots,n-1)$. Then we have

$$p_a=\frac{\partial F}{\partial y^a}=\frac{\partial L}{\partial z^a},\quad p_n=\frac{\partial F}{\partial y^n}=L-z^a\frac{\partial L}{\partial z^a}.$$

On account of the relation (3) we have $\det(\partial^2 L/\partial z^a\partial z^b)\neq 0$ without loss of generality and we get $z^a=\psi^a(x,p')$ and hence

$$p_n=L(x,\psi(x,p'))-\psi^a(x,p')p_a,$$

where $p'=(p_1,\cdots,p_{n-1})$. This is the equation (7) in explicit form. The second side of the above equation is $-H(x,p)$ and we get

$$\omega=p_a dx^a+p_n dx^n=p_a dx^a-H dx^n.$$

§ 3. Dilatation in Finsler spaces

1. We take a plane element dual to a tangent of an extremal in Finsler space M and translate it along the extremal by constant length. We call this translation a dilatation in Finsler space. On the other hand a homogeneous contact transformation is defined in a space with a contact structure as a transformation preserving the fundamental 1-form $\omega=p_i dx^i$. Then we have the following theorem.

Theorem 5. *A dilatation in Finsler space M induces a homogeneous contact transformation on the corresponding p-manifold N.*

Proof. A dilatation in M induces on N such a translation T of a point (x,p) to a point $(\bar{x},\bar{p})$ along an e-curve that $\int\omega=\int p_i dx^i=\text{const.}$ We take a segment AB of a curve in N and translate it to $\bar{A}\bar{B}$ by T. Then we get a region generated by e-curves and bounded by $AB\bar{B}\bar{A}A$, and we get by Theorem 2

$$\int_{AB}\omega+\int_{B\bar{B}}\omega+\int_{\bar{B}\bar{A}}\omega+\int_{\bar{A}A}\omega=0.$$

By the definition of dilatation we have $\int_{A\bar{A}} \omega = \int_{B\bar{B}} \omega$, and so

$$\int_{AB} \omega = \int_{\bar{A}\bar{B}} \omega . \tag{18}$$

As AB is arbitrary we get

$$\bar{p}_i dx^i = \bar{p}_i d\bar{x}^i , \tag{19}$$

which was to be proved.

Theorem 5 is not essentially new, but it puts a new light from a geometric point of view upon a classical result, where $\omega = p_i dx^i$ is a relative invariant and $d\omega = dp_i \wedge dx^i$ an absolute one. Here we have proved that ω is itself invariant for dilatation.

Theorem 5 has a following application. We define a measure element in N, namely that of plane elements (x, p) in Finsler space M, by

$$dV = \frac{1}{(n-1)!}(-1)^{n(n-1)/2}\, \omega \wedge (d\omega)^{n-1} .$$

Substituting $\omega = p_i dx_i$ we get

$$dV = dx^1 \wedge \cdots \wedge dx^n \wedge (\sum_i (-1)^{i-1} p_i dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n)$$

By virtue of Theorem 5 we get

Theorem 6. *A measure* $\int dV$ *for plane elements in a Finsler space is invariant for a dilatation.*

In a Riemannian space with a metric $ds^2 = g_{ij} dx^i dx^j$ we have as a volume element of points

$$dv = g dx^1 \wedge \cdots \wedge dx^n \qquad (g = \sqrt{\det (g_{ij})}) .$$

By (9) $p = (p_1, \cdots, p_n)$ are covariant components of a unit vector and we can define a measure of unit vectors by

$$d\sigma = g^{-1} \sum_i (-1)^{i-1} p_i Dp_1 \wedge \cdots \wedge \widehat{Dp_i} \wedge \cdots \wedge Dp_n .$$

where Dp_i means a covariant differential of p. Then we have

$$dV = dv \wedge d\sigma$$

by virtue of the relation $Dp_i \equiv dp_i \pmod{dx^1, \cdots, dx^n}$. In this case we can consider a dilatation as a translation of a tangent unit vector along a geodesic by constant length, which we call a *geodesic flow*. The invariance of $\int dV$ for a geodesic flow is fundamental in the ergodic theory and has been treated by several authors. (cf. for example [4] [5])

2. As to Riemannian manifold of constant curvature, not only a volume element but also a Riemannian metric is invariant for a geodesic flow. We take rectangular frames on the tangent spaces of M and represent the Riemannian metric as

$$ds^2 = \sum_i \omega_i^2 \tag{20}$$

with 1-forms ω_i. Connection forms of the Riemannian connection are given by ω_{ij} in such a way that

$$d\omega_i = \omega_j \wedge \omega_{ji} \qquad (\omega_{ij} = -\omega_{ji})\,, \tag{21}$$

and curvature forms are given by

$$d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = \tfrac{1}{2} R_{ijkh}\omega_k \wedge \omega_h\,. \qquad (R_{ijkh} = -R_{ijhk}) \tag{22}$$

We take geodesics and denote by δ a differential along the geodesics and by s an arc-length along them. We put

$$\omega_i(\delta) = v_i \delta s\,, \quad \omega_{ji}(\delta) = \xi_{ji}\delta s\,.$$

When we take a differential d independent of δ we have by (21)

$$d\omega_i(\delta) - \delta\omega_i(d) = \omega_j(d)\omega_{ji}(\delta) - \omega_j(\delta)\omega_{ji}(d)$$

and putting

$$\omega_i(d) = \omega_i\,, \quad \omega_{ji}(d) = \omega_{ji}\,, \quad Dv_i = dv_i + v_j\omega_{ji}$$

we get

$$\delta\omega_i = (-\omega_j \xi_{ji} + Dv_i)\delta s\,. \tag{23}$$

As (v_i) is a unit tangent vector along a geodesic,

$$\delta v_i = -v_j \xi_{ji} \delta s\,. \tag{24}$$

By virtue of (22)

$$d\omega_{ij}(\delta) - \delta\omega_{ij}(d) - \omega_{ik}(d)\omega_{kj}(\delta) + \omega_{ik}(\delta)\omega_{kj}(d) = R_{ijkh}\omega_k(d)\omega_h(\delta)\,.$$

and so

$$\delta\omega_{ij}/\delta s = d\xi_{ij} - \omega_{ik}\xi_{kj} + \xi_{ik}\omega_{kj} - R_{ijkh}\omega_k v_h\,.$$

Now

$$\delta(Dv_i) = \delta(dv_i) + \delta v_j \omega_{ji} + v_j \delta\omega_{ji}$$

and as $\delta(dv_i) = d(\delta v_i)$ we get

$$\begin{aligned}\delta(Dv_i)/\delta s &= d(-v_j\xi_{ji}) - v_k\xi_{kj}\omega_{ji} + v_j(d\xi_{ji} - \omega_{jk}\xi_{ki} + \xi_{jk}\omega_{ki} - R_{jikh}\omega_k v_h) \\ &= -Dv_j\xi_{ji} - v_j R_{jikh}\omega_k v_h\,.\end{aligned} \tag{25}$$

Thus we have on account of (23)

$$\frac{1}{2}\frac{\delta}{\delta s}(\sum_i \omega_i^2) = \sum_i \omega_i \frac{\delta\omega_i}{\delta s} = Dv_i \cdot \omega_i \tag{26}$$

and by (25)

$$\frac{1}{2}\frac{\delta}{\delta s}(\sum_i (Dv_i)^2) = -v_j R_{jihk}\omega_k v_h Dv_i .$$

Here we assume that M is of constant curvature K and then we get for a unit vector (v_i)

$$R_{jikh}v_j v_h = -K(\delta_{jk}\delta_{ih} - \delta_{jh}\delta_{ik})v_j v_h = K(\delta_{ik} - v_i v_k)$$

and so

$$\frac{1}{2}\frac{\delta}{\delta s}(\sum_i (Dv_i)^2) = -K\,Dv_i\omega_i + K(v_i Dv_i)(v_k\omega_k) = -K\,Dv_i\omega_i . \tag{27}$$

From (26) and (27) we get

$$\delta(K\sum_i \omega_i^2 + \sum_i (Dv_i)^2) = 0 .$$

This can be stated as follows.

Theorem 7. *On a Riemannian manifold M of constant curvature K we denote a square of an arc-element by ds^2 and $\sum_i (Dv_i)^2$ by $d\sigma^2$, where Dv_i means a covariant differential of a unit vector v on M. Then $K\,ds^2 + d\sigma^2$ is an invariant of a geodesic flow.*

This theorem has elementary applications in the non-euclidean geometry, but the author is not aware how it effects on the ergodic theory.

§ 4. Certain contact transformations

1. A homogeneous contact transformation f on ${}^c\mathrm{T}(M)$ is a mapping $(x, z) \to (\bar{x}, \bar{z})$ such that $z_i dx^i = \bar{z}_i d\bar{x}^i$. If it maps p-manifold N into itself and (x, p) is mapped on $(\bar{x}, \bar{p})$, we have

$$p_i dx^i = \bar{p}_i d\bar{x}^i , \quad \text{hence} \quad dp_i \wedge dx^i = d\bar{p}_i \wedge d\bar{x}^i .$$

If we take coordinates $\xi^1, \cdots, \xi^{2n-1}$ on N, this can be written as

$$a_{\alpha\beta}(\xi)\,d\xi^\alpha \wedge d\xi^\beta = a_{\alpha\beta}(\bar{\xi})\,d\bar{\xi}^\alpha \wedge d\bar{\xi}^\beta . \qquad (\alpha, \beta = 1, \cdots, 2n-1)$$

If the induced mapping $\xi \to \bar{\xi}$ is *regular*, namely $\det(\partial\bar{\xi}^\alpha/\partial\xi^\beta) \neq 0$, equations $a_{\alpha\beta}(\xi)\,d\xi^\beta = 0$ and $a_{\alpha\beta}(\bar{\xi})\,d\bar{\xi}^\beta = 0$ are equivalent. In fact

$$a_{\alpha\beta}(\bar{\xi})\frac{\partial\bar{\xi}^\alpha}{\partial\xi^\gamma}\frac{\partial\bar{\xi}^\beta}{\partial\xi^\delta} = a_{\gamma\delta}(\xi) , \quad \text{hence} \quad a_{\alpha\beta}(\bar{\xi})\,d\bar{\xi}^\beta \frac{\partial\bar{\xi}^\alpha}{\partial\xi^\gamma} = a_{\gamma\delta}(\xi)\,d\xi^\delta ,$$

This shows that e-curves are mapped on e-curves and we get

Theorem 8. *If a homogeneous contact transformation on ${}^c\mathrm{T}(M)$ maps p-manifold N on itself and the induced mapping is regular, it maps extremals on extremals on M.*

Dilatation maps extremals on themselves, but it maps each extremals on itself. We give here a more general example. A one-parametric family of contact transformations can be given by solving an ordinary differential equation

$$\delta x^i = \frac{\partial U}{\partial p_i}\delta t\,, \qquad \delta p_i = -\frac{\partial U}{\partial x^i}\delta t \tag{28}$$

where t is a parameter. If U satisfies an equation

$$G_{x^i}U_{p_i} - G_{p_i}U_{x^i} = 0 \tag{29}$$

we have $\delta G=0$. If $G(x, p)=0$ is satisfied for an initial condition, it is always satisfied and we get a one parametric family of homogeneous transformations preserving extremals. In the euclidean case we have $F=\sqrt{\sum_i (y^i)^2}$ and we get by (9) $G(x, p)=\sum_i p_i^2-1=0$. Then (29) reduces to $p_i\partial U/\partial x^i=0$, whose general solution is given by

$$U = \varphi(p_1x^2-p_2x^1,\ p_1x^3-p_3x^1,\ \cdots,\ p_1x^n-p_nx^1,\ p_1,\ \cdots,\ p_n)$$

with an arbitrary function φ.

2. When a homogeneous contact transformation $(x, p)\to(\bar{x}, \bar{p})$ in the euclidean space is such that

$$\bar{p} = f(p)\,,$$

it preverves hyperplanes. In fact for a plane element (x, p) on a hyperplane p is constant and also $p_i dx^i=0$. Hence $\bar{p}_i d\bar{x}^i=0$, and as $\bar{p}=$const, we get $\bar{p}_i\bar{x}^i=$const. Thus the plane element (x, p) is also on a fixed hyperplane. Laguerre transformation affords an example of transformations here considered.

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