Yosida, K.
Osaka Math. J.
15 (1963), 51-57.

HOLOMORPHIC SEMI-GROUPS IN A LOCALLY CONVEX LINEAR TOPOLOGICAL SPACE

Dedicated to Professor Kenjiro Shoda on his sixtieth birthday

BY

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The purpose of the present note is to show that the analytical theory of holomorphic semi-groups in a Banach space, given in a preceding note[1]. can be extended to locally convex linear topological spaces. The result may thus be applied to the "abstract Cauchy problem" in such spaces.

Let X be a *locally convex, sequentially complete linear topological space*, and $L(X, X)$ be the set of all continuous linear operators defined on X into X. Let $T_t \in L(X, X)$, $t \geqq 0$, satisfy the conditions:

(i) $T_t T_s = T_{t+s} (t, s \geqq 0)$, $T_0 = I =$ the identity operator,

(ii) $\lim_{t \to t_0} T_t x = T_{t_0} x$ for all $t_0 \geqq 0$ and $x \in X$,

(iii) $\{T_t\}$ is *equi-continuous* in $t \geqq 0$ in the sense that, for any continuous semi-norm $p(x)$ on X, there exists a continuous semi-norm $q(x)$ on X such that $p(T_t x) \leqq q(x)$ for all $t \geqq 0$ and all $x \in X$.

Such a system $\{T_t\}$ is said to constitute an *equi-contituous semi-group of class* (C_0). *The infinitesimal generator* A of T_t is defined by

(iv) $Ax = (D_t T_t x)_{t=0} = \lim_{t \downarrow 0} t^{-1}(T_t - I)x$, i.e., the domain $D(A)$ of A is the set of those $x \in X$ for which the right hand limit exists, and when $x \in D(A)$ we have $Ax = \lim_{t \downarrow 0} t^{-1}(T_t - I)x$.

As in the case where X is a Banach space and $\sup_{t \geqq 0} \| T_t \| < \infty$, such A is characterized by the following properties:

(v) A is a closed linear operator with dense domain $D(A)$, i.e., $D(A)^a = X$[2],

1) K. Yosida: *On the differentiability of semi-groups of linear operators*, Proc. Japan Acad. **34** (1958), 337-340. Cf. also E. Hille-R. S.Phillips: Functional Analysis and Semi-groups, Providence (1957).

2) M^a denotes the closure of $M \subseteqq X$.

(vi) the resolvent $R(\lambda; A)=(\lambda I-A)^{-1}\in L(X, X)$ exists for $Re(\lambda)>0$ and the system of linear operators $\{(\lambda R(\lambda; A))^n\}$ is equi-continuous in $\lambda\geqq 1$ and in $n=1, 2, \cdots$

Moreover, the resolvent $R(\lambda; A)$ is obtained from the original group by

(vii) $(\lambda R(\lambda; A))^n x=\dfrac{\lambda^n}{(n-1)!}\displaystyle\int_0^\infty e^{-\lambda t}t^{n-1}T_t x dt$ for $Re(\lambda)>0$ and $x\in X$.

After these preliminaries, we are ready to discuss those semi-groups T_t which can, as functions of the parameter t, be continued holomorphically into a sector of the complex plane containing the positive t-axes.

Lemma. *Suppose that, for all* $t>0$, $T_tX\subseteqq D(A)$, *Then, for any* $x\in X$, T_tx *is infinitely differentiable in* $t>0$ *and we have*

$$T_t^{(n)}x = (T'_{t/n})^n x \quad \text{for all} \quad t>0, \tag{1}$$

where $T'_t=D_tT_t$, $T''_t=D_tT'_t$, $\cdots$, $T_t^{(n)}=D_tT_t^{(n-1)}$

Proof. It $t>t_0>0$, then $T'_tx=AT_tx=T_{t-t_0}AT_{t_0}x$ by the commutativity of A and T_s, which is an easy consequence from (i) and (iv). Thus $T'_tX\subseteqq T_{t-t_0}X\subseteqq D(A)$ when $t>0$, and so T''_tx exists for all $t>0$ and $x\in X$. Since A is a closed linear operator, we have $T''_tx=D_t(AT_t)x = A\cdot\lim_{n\uparrow\infty} n(T_{t+(1/n)}-T_t)x = A(AT_t)x=AT_{t/2}AT_{t/2}x = (T'_{t/2})^2x$. Repeating the argument, we obtain (1).

Theorem. *For an equi-continuous semi-group* T_t *of class* (C_0) *in a locally convex, sequentially complete linear topological space* X, *the following three conditions are mutually equivalent.*

(I) *For all* $t>0$, $T_tX\subseteqq D(A)$ *and there exists a positive constant* $C\leqq 1$ *such that the family of operators* $\{CtT'_t)^n\}$ *is equi-continuous in* $n=1, 2, \cdots$ *and* $0<t\leqq 1$.

(II) T_t *admits a holomorphic extension* T_λ *given by*

$$T_\lambda x = \sum_{n=0}^{\infty}(\lambda-t)^n T_t^{(n)}x/n! \quad \text{for} \quad |\arg\lambda|<\mathrm{Tan}^{-1}(Ce^{-1}), \tag{2}$$

in such a way that

(3) *the family of operators* $\{e^{-\lambda}T_\lambda\}$ *is equi-continuous in* λ *for* $|\arg\lambda|<\mathrm{Tan}^{-1}(2^{-1}Ce^{-1})$.

(III) *Let* A *be the infinitesimal generator of* T. *Then there exists a positive constant* C_1 *such that the family of operators* $\{(C_1\lambda R(\lambda; A))^n\}$ *is equi-continuous in* $n=1, 2, \cdots$ *and* λ *with* $Re(\lambda)\geqq 1+\varepsilon$, *where* $\varepsilon>0$.

Proof. *The implication* (I)$\to$(II). Let p be any continuous semi-norm on X. Then, by hypothesis, there exists a continuous semi-norm q on X such that $p((tT_t')^n x) \leqq C^{-n} q(x)$ for $1 \geqq t \geqq 0$, $n \geqq 0$ and $x \in X$. Hence, by (1), we obtain, for any $t > 0$,

$$p((\lambda-t)^n T_t^{(n)} x/n!) \leqq \frac{|\lambda-t|^n}{t^n} \frac{n^n}{n!} \frac{1}{C^n} p\left(\left(\frac{t}{n} CT'_{t/n}\right)^n x\right)$$
$$\leqq \left(\frac{|\lambda-t|}{t} C^{-1} e\right)^n q(x), \text{ whenever } 0 < t/n \leqq 1.$$

Thus the right side of (2) surely converges for $||\arg\lambda|| < \mathrm{Tan}^{-1}(Ce^{-1})$, and so, by the sequential completeness of X, $T_\lambda x$ is well defined and is holomorphic in λ for $|\arg\lambda| < \mathrm{Tan}^{-1}(Ce^{-1})$. Next put $S_t = e^{-t}T_t$. Then $S_t' = -e^{-t}T_t + e^{-t}T_t'$ and so, by $0 \leqq te^{-t} \leqq 1$ $(0 \leqq t)$ and (I), we easily see that $\{(2^{-1}CtS_t')^n\}$ is equi-continuous in $t > 0$ and $n \geqq 0$, in virtue of the equi-continuity of $\{T_t\}$. The equi-continuous semi-group S_t of class (C_0) satisfies the condition that $S_t X \subseteqq D(A-I) = D(A)$, where $(A-I)$ is the infinitesimal generator of S_t. Therefore, by the same reasoning as applied to T_t above, we can prove that the holomorphic extension $e^{-\lambda}T_\lambda$ of $S_t = e^{-t}T_t$ satisfies the estimate (3).

By the way, we can prove the following

Corollary (due to E. Hille). *If, in particular, X is a complex B-space and* $\overline{\lim}_{t\downarrow 0} ||tT_t'|| < e^{-1}$,[1] *then* $X = D(A)$.

Proof. For a fixed $t > 0$, we have $\overline{\lim}_{n\to\infty} ||(t/n)T'_{t/n}|| < e^{-1}$, and so the series

$$\sum_{n=0}^{\infty} (\lambda-t)^n T_t x/n! = \sum \frac{(\lambda-t)^n}{t^n} \frac{n^n}{n!} \left(\frac{t}{n} T'_{t/n}\right)^n x$$

converge in some circle

$$\{\lambda ; |\lambda-t|/t < 1+\delta \quad \text{with a} \quad \delta > 0\}$$

of the complex λ-plane. This circle surely contains $\lambda = 0$ in its interior.

The implication (II)$\to$(III). We have, by (vii),

$$(4) \qquad (\lambda R(\lambda; A))^n x = \frac{\lambda^{n+1}}{n!} \int_0^\infty e^{-\lambda t} t^n T_t x dt \quad \text{for} \quad Re(\lambda) > 0, \qquad x \in X.$$

Hence

$$((\sigma+1+i\tau)R(\sigma+1+i\tau; A))^{n+1} x = \frac{(\sigma+1+i\tau)^{n+1}}{n!} \int_0^\infty e^{-(\sigma+i\tau)t} t^n S_t x dt, \qquad \sigma > 0,$$

Let $\tau<0$. Since the integrand is holomorphic, we can deform, by the estimate (3) and Cauchy's integral theorem, the path of integration: $0\leq t<\infty$ to the ray: $re^{i\theta}$ $(0\leq r<\infty)$ contained in the sector $0<\arg\lambda<\operatorname{Tan}^{-1}(2^{-}Ce^{-1})$ of the complex λ-plane. We thus obtain

$$((\sigma+1+i\tau)R(\sigma+1+i\tau\,;A))^{n+1}x=\frac{(\sigma+1+i\tau)^{n+1}}{n!}\times$$
$$\int_0^\infty e^{-(\sigma+i\tau)re^{i\theta}}r^n\,e^{in\theta}S_{re^{i\theta}}x\,e^{i\theta}dr\,,$$

and so, by (3),

$$\begin{aligned}&p(((\sigma+1+i\tau)R(\sigma+1+i\tau\,;A))^{n+1}x)\\ &\leqq\frac{|(\sigma+1+i\tau)|^{n+1}}{n!}\int_0^\infty e^{(-\sigma\cos\theta+\tau\sin\theta)r}r^n p(S_{re^{i\theta}})\,dr\\ &\leqq q'(x)\,\frac{|\sigma+1+i\tau|^{n+1}}{|\tau\sin\theta-\sigma\cos\theta|^{n+1}}\,,\end{aligned}$$

where q' is a continuous semi-norm on X. A similar estimate is obtained for the case $\tau>0$ also. Hence, combined with (vi), we have proved (III).

The implication (III)$\rightarrow$(I). For any continuous semi-norm p on X, there exists a continuous semi-norm q on X such that

$$p((C_1\lambda R(\lambda\,;A))^n x)\leqq q(x)\quad\text{whenever}\quad Re(\lambda)\geqq 1+\varepsilon,\ \varepsilon>0\quad\text{and}\quad n\geqq 0\,.$$

Hence, if $Re(\lambda_0)\geqq 1+\varepsilon$, we have

$$p(((\lambda-\lambda_0)R(\lambda_0\,;A))^n x)\leqq\frac{|\lambda-\lambda_0|^n}{(C_1|\lambda_0|)^n}q(x)\qquad(n=0,1,2,\cdots)\,.$$

Thus, if $|\lambda-\lambda_0|/C_1|\lambda_0|<1$, the resolvent $R(\lambda\,;A)$ exists and is given by $R(\lambda: A)x=\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n R(\lambda_0\,;A)^{n+1}x$ such that

$$p(R(\lambda\,;A)x)\leqq(1-C_1^{-1}|\lambda_0|^{-1}|\lambda-\lambda_0|)^{-1}q(R(\lambda_0\,;A)x)\,.$$

Therefore, by (III) there exists an angle θ_0 with $\pi/2<\theta_0<\pi$ such that $R(\lambda\,;A)$ exists and satisfies the estimate

$$p(R(\lambda\,;A)x)\leqq\frac{1}{|\lambda|}q'(x)\tag{5}$$

with a continuous semi-norm q' on X in the sectors $\pi/2\leqq\arg\lambda\leqq\theta_0$ and $-\theta_0\leqq\arg\lambda\leqq-\pi/2$ and also for $Re(\lambda)\geqq 0$, when $|\lambda|$ is sufficiently large.

Hence the integral

$$(6) \qquad \hat{T}_t x = (2\pi i)^{-1}\int_{C_2} e^{\lambda t}R(\lambda\,;\,A)xd\lambda \qquad (t>0,\ x\in X)$$

converges if we take the path of integration $C_2=\lambda(\sigma)$, $-\infty<\sigma<\infty$, in such a way that $\lim_{|\sigma|\uparrow\infty}|\lambda(\sigma)|=\infty$ and, for some $\varepsilon>0$,

$$\pi/2+\varepsilon\leqq\arg\lambda(\sigma)\leqq\theta_0 \quad\text{and}\quad -\theta_0\leqq\arg\lambda(\sigma)\leqq-\pi/2-\varepsilon$$

when $\sigma\uparrow+\infty$ and $\sigma\downarrow-\infty$, respectively; for not large $|\sigma|$, $\lambda(\sigma)$ lies in the right half plane of the complex λ-plane.

We shall show that $\hat{T}_t$ coincides with the semi-group T_t itself[3]. We first show that $\lim_{t\downarrow 0}\hat{T}_t x=x$ for all $x\in D(A)$. Let x_0 be any element $\in D(A)$, and choose any complex number λ_0 to the right of the contour C_2 of integration, and denote $(\lambda_0 I-A)x_0=y_0$. Then, by the resolvent equation,

$$\begin{aligned}\hat{T}_t x_0 &= \hat{T}_t R(\lambda_0\,;\,A)y_0 = (2\pi i)^{-1}\int_{C_2} e^{\lambda t}R(\lambda\,;\,A)R(\lambda_0\,;\,A)y_0\,d\lambda\\ &= (2\pi i)^{-1}\int_{C_2} e^{\lambda t}(\lambda_0-\lambda)^{-1}R(\lambda\,;\,A)y_0\,d\lambda\\ &\quad -(2\pi i)^{-1}\int_{C_2} e^{\lambda t}(\lambda_0-\lambda)^{-1}R(\lambda_0\,;\,A)y_0\,d\lambda\,.\end{aligned}$$

The second integral on the right is equal to zero, as may be seen by shifting the path of integration to the left. Hence

$$\hat{T}_t x_0 = (2\pi i)^{-1}\int_{C_2} e^{\lambda t}(\lambda_0-\lambda)^{-1}R(\lambda\,;\,A)y_0\,d\lambda, \quad y_0=(\lambda_0 I-A)x_0\,.$$

Because of the estimate (5), the passage to the limit $t\downarrow 0$ under the integral sign is justified, and so

$$\lim_{t\downarrow 0}\hat{T}_t x_0 = (2\pi i)^{-1}\int_{C_2}(\lambda_0-\lambda)^{-1}R(\lambda\,;\,A)y_0\,d\lambda, \quad y_0=(\lambda_0 I-A)x_0\,.$$

To evaluate the right hand integral, we make a closed contour out of the original path of integration C_2 by adjointing the arc of the circle $|\lambda|=r$ which is to the right of the path C_2, and throwing away that portion of the original path C_2 which lies outside the circle $|\lambda|=r$. The value of the integral along the new arc and the discarded arc tends to zero as $r\downarrow\infty$, in virtue of (5). Hence the value of the integral is equal to the residue inside the new closed contour, that is, the value

3) Adapted from P. D. Lax and A. N. Milgram: *Parabolic equations*, Contributions to the Theory of Partial Differential Equations, Princeton (1954).

$R(\lambda_0\,;\,A)y_0=x_0$. We have thus proved $\lim_{t\downarrow 0}\hat{T}_t x_0=x_0$ when $x_0\in D(A)$.

We next show that $\hat{T}'x=A\hat{T}_t x$ for $t>0$ and $x\in X$. We have $R(\lambda\,;\,A)X=D(A)$ and $AR(\lambda\,;\,A)=\lambda R(\lambda\,;\,A)-I$, so that, by the convergence factor $e^{\lambda t}$, the integral $(2\pi i)^{-1}\int_{C_2} e^{\lambda t}AR(\lambda\,;\,A)x\,d\lambda$ has a sense. This integral is equal to $A\hat{T}_t x$, as may be seen by approximating the integral (6) by Riemann sum and using the fact that A is *closed*: $\lim_{n\to\infty} x_n=x$ and $\lim_{n\to\infty} Ax_n=y$ imply $x\in D(A)$ and $Ax=y$. Therefore

$$A\hat{T}_t x = (2\pi i)^{-1}\int_{C_2} e^{\lambda t}AR(\lambda\,;\,A)x\,d\lambda, \quad t>0\,.$$

On the other hand, by differentiating (6) under the integral sign, we obtain

$$\hat{T}'_t x = (2\pi i)^{-1}\int_{C_2} e^{\lambda t}\lambda R(\lambda\,;\,A)x\,d\lambda, \quad t>0\,. \tag{8}$$

In fact, the difference of these two integrals is $(2\pi i)^{-1}\int_{C_2} e^{\lambda t}x\,d\lambda$, and the value of the last integral is zero, as may be seen by shifting the path of integration to the left.

Thus we have proved that $\hat{x}(t)=\hat{T}_t x_0$, $x_0\in D(A)$, satisfies i) $\lim_{t\downarrow 0}\hat{x}(t)=x_0$, ii) $d\hat{x}(t)/dt=A\hat{x}(t)$ for $t>0$, and iii) $\{\hat{x}(t)\}$ is bounded when $t\uparrow\infty$, as may be seen from (6). On the other hand, since $x_0\in D(A)$ and since $\{T_t\}$ is equi-continuous in $t\geqq 0$, we see that $x(t)=T_t x_0$ also satisfies $\lim_{t\downarrow t_0} x(t)=x_0$, $dx(t)/dt=Ax(t)$ for $t\geqq 0$, and $\{x(t)\}$ is bounded when $t\geqq 0$. Let us put $\hat{x}(t)-x(t)=y(t)$. Then $\lim_{t\downarrow 0} y(t)=0$, $dy(t)/dt=Ay(t)$ for $t>0$ and $\{y(t)\}$ is bounded when $t\uparrow\infty$. Hence we may consider the Laplace transform

$$L(\lambda\,;\,y) = \int_0^\infty e^{-\lambda t}y(t)dt, \quad Re(\lambda)>0\,.$$

We have

$$\int_\alpha^\beta e^{-\lambda t}y'(t) = \int_\alpha^\beta e^{-\lambda t}Ay(t)\,dt = A\int_\alpha^\beta e^{-\lambda t}y(t)\,dt\,, \quad 0\leqq\alpha<\beta<\infty\,,$$

by approximating the integral by Riemann sum and using the fact that A is closed. By partial integration, we obtain

$$\int_\alpha^\beta e^{-\lambda t}y'(t)\,dt = e^{-\lambda\beta}y(\beta)-e^{-\lambda\alpha}y(\alpha)+\lambda\int_\alpha^\beta e^{-\lambda t}y(t)\,dt$$

which tends to $\lambda L(\lambda\,;\,y)$ as $\alpha\downarrow 0$, $\beta\uparrow\infty$. For, $y(0)=0$ and $\{y(\beta)\}$ is bounded as $\beta\uparrow\infty$. Thus again, by using the closure property of A, we obtain

$$AL(\lambda\,;\,y) = \lambda L(\lambda\,;y), \quad Re(\lambda) > 0\,.$$

Since the inverse $(\lambda I - A)^{-1}$ exists for $Re(\lambda)>0$, we must have $L(\lambda\,;\,y)=0$ when $Re(\lambda)>0$. Thus, for any continuous linear functional $f\in X'$, the dual space of, we have

$$\int_0^\infty e^{-\lambda t}f(y(t))\,dt = 0 \quad \text{when} \quad Re(\lambda)>0\,.$$

We set $\lambda=\sigma+i\tau$ and put

$$g_\sigma(t) = e^{-\sigma t}f(y(t)) \quad \text{or} \quad =0 \quad \text{according as} \quad t\geqq 0 \quad \text{or} \quad t<0\,.$$

Then, the above equality shows that the Fourier transform

$$(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-i\tau t}g_\sigma(t)\,dt \quad \text{vanishes identically in} \quad \tau,\ -\infty<\tau<\infty\,,$$

so that, by Fourier's integral theorem, $g_\sigma(t)=0$ identically. Thus $f(y(t)) =0$ and so we must have $y(t)=0$ identically, in virtue of Hahn-Banach's theorem.

Therefore $\hat{T}_t x = T_t x$ for all $t>0$ and $x\in D(A)$. $D(A)$ being dense in X and $\hat{T}_t$, T_t both belong to $L(X, X)$, we easily conclude that $\hat{T}_t x = T_t x$ for all $x\in X$ and $t>0$. Hence, by defining $\hat{T}_0=I$, we have $\hat{T}_t=T_t$ for all $t\geqq 0$. Hence, by (7). $T'_t x=(2\pi i)^{-1}\int_{C_2} e^{\lambda t}\lambda R(\lambda\,;\,A)x\,dt$, $t>0$, and so, by (1) and (5), we obtain

$$(T'_{t/n})^n x = T_t^{(n)}x = (2\pi i)^{-1}\int_{C_2} e^{\lambda t}\lambda^n R(\lambda\,;\,A)x\,d\lambda\,.$$

Hence

$$(tT'_t)^n x = (2\pi i)^{-1}\int_{C_2} e^{n\lambda t}(t\lambda)^n R(\lambda\,;\,A)x\,d\lambda\,.$$

Therefore, by (III),

$$p((tT'_t)^n x) \leqq (2\pi)^{-1}q(x)\int_{C_2} |e^{n\lambda t}|\,t^n|\lambda|^{n-1}d|\lambda|\,.$$

The last integral is majoraized by C_3^n with some positive constant C_3, when $1\geqq t>0$. Hence we have proved (I).

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(Received March 6, 1963)