Nagumo, M.
Osaka Math. J.
15 (1963), 1-10.

PERTURBATION AND DEGENERATION OF EVOLUTIONAL EQUATIONS IN BANACH SPACES

Dedicated to Professor K. Shoda on his sixtieth birthday

By

Mitio NAGUMO

§ 1. Completely well posed evolutional equations

Let E be a Banach space and E_1 be another Banach space such that $E \subset E_1$ and the embeding of E into E_1 is continuous. Let $A(t)$ be a continuous linear mapping of E into E_1 for every fixed t in the real interval $[a, b]$ such that $A(t)u$ is a continuous function on $[a, b]$ into E_1 for every fixed $u \in E$. Then we can easily see, $A(t)u(t)$ is continuous on $[a, b]$ into E_1, if $u(t)$ is continuous on $[a, b]$ into E.

As an E-solution in $[a, b]$ of the evolutional equation

$$(0) \qquad \partial_t u = A(t)u + f(t) \qquad \left(\partial_t = \frac{d}{dt}\right),$$

where $f(t)$ is an E-continuous function on $[a, b]$[1], we understand an E-continuous function $u = u(t)$ on $[a, b]$ such that the strong derivative $\partial_t u = \lim_{h \to 0} h^{-1}\{u(t+h) - u(t)\}$ exists in E_1 for $t \in [a, b]$ and the equation (0) is fulfilled in E_1 for $t \in [a, b]$.

The equation (0) is said to be *E-well posed* (or simply well posed) in $[a, b]$ when for any $\varphi \in E$ there exists one and only one E-solution $u = u(t)$ of (0) with the initial value $u(a) = \varphi$. We say that the equation (0) is *completely E-well posed* in $[a, b]$ when (0) is E-well posed for any closed subinterval of $[a, b]$ and the solution $u = u(t, s, \varphi)$ of (0) with the initial value $u(s) = \varphi$ $(a \leq s \leq b)$ is a continuous function of (t, s, φ) for $a \leq s \leq t \leq b$, $\varphi \in E$. If (0) is (completely) E-well posed in $[a, b]$ then the associated homogeneous equation

$$(1) \qquad \partial_t u = A(t)u$$

1) $f(t)$ is said to be E-continuous on $[a, b]$ when $f(t)$ is continuous on $[a, b]$ into E.

is also (completely) E-well posed in $[a, b]$. When (1) is completely well posed in $[a, b]$ then the solution of (1) with the initial condition $u(s)=\varphi$ ($s\in[a, b]$) can be written in the form

$$u = U(t, s)\varphi, \tag{2}$$

where $U(t, s)$ is a continuous linear operator on E into E for $a\leq s\leq t\leq b$ with the following properties:

1) $U(t, s)\varphi$ is continuous on $a\leq s\leq t\leq b$, $\varphi\in E$ into E,
2) $U(s, s)=1$ (identity) for $s\in[a, b]$
3) $U(t, \sigma)U(\sigma, s)=U(t, s)$ for $a\leq s\leq\sigma\leq t\leq b$,
4) $\partial_t U(t, s)\varphi=A(t)U(t, s)\varphi$ in E_1 for $a\leq s\leq t\leq b$, $\varphi\in E$.

Such an operator $U(t, s)$ is called the *fundamental solution* of (1).

Especially when $A(t)$ does not depend on t: $A(t)=A$, (1) is completely E-well posed in any finite interval $[a, b]$, if and only if (1) is simply E-well posed in some finite interval. For, the fundamental solution of (1) has the form $U=U(t-s)$. In this case, restricting the domain of A to such a set of u that $Au\in E$, A is the infinitesimal generator of the one-parameter semi-group $\{U(t)\}_{t\geq 0}$, since $U(t+s)=U(s)U(t)$ for $s, t\geq 0$. Conversely, if A is the infinitesimal generator of a one-parameter semi-group $\{U(t)\}_{t\geq 0}$, then extending the domain of A on E in such a way that the range of A will be contained in E_1 as given in Remark 1, we obtain a completely E-well posed equation (1) with $A(t)=A$, for any finite interval, with the fundamental solution $U(t-s)=\exp((t-s)A)$.

We can easily obtain the following:

Theorem 1. *If the homogeneous equation* (1) *is completely E-well posed in* $[a, b]$, *and* $f(t)$ *is E-continuous on* $[a, b]$, *then the inhomogenous equation* (0) *is also completely E-well posed in* $[a, b]$ *and any solution of* (0) *satisfies*

$$u(t) = U(t, s)u(s)+\int_s^t U(t, \sigma)f(\sigma)d\sigma \quad \textit{for} \quad a\leq s\leq t\leq b$$

with the fundamental solution $U(t, s)$ *of* (1).

REMARK 1. Let $A(t)$ be a pre-closed linear operator in E for every t, and let there exists a closed operator A_0 with a domain in E such that for the adjoint operators $A^*(t)$ and A_0^* of $A(t)$ and A_0 resp. we have

$$\|A^*(t)u'\|\leq\|A_0^*u'\|+\|u'\| \quad \text{for} \quad u'\in\mathcal{D}^* = \mathcal{D}(A_0^*).$$

Then, defining a new norm of $u\in E$ by

$$|||u||| = \sup_{u' \in \mathscr{D}^*} |\langle u, u' \rangle| (||A_0^* u'|| + ||u'||)^{-1},$$

we get $|||u||| \leq ||u||$. Hence, denoting by E_1 the completion of E with respect to the new norm, we obtain that the injection of E into E_1 is continuous and the extension of $A(t)$ on E is continuous on E into E_1. Cf [1].

On the other hand, if there exists a closed operator A_0 with a domain $\mathscr{D}$ dense in E such that

$$||A(t)u|| \leq ||A_0 u|| + ||u|| \quad \text{for} \quad u \in \mathscr{D},$$

then difining a new norm of $u \in \mathscr{D}$ by

$$|||u||| = ||A_0 u|| + ||u||$$

the vector space $\mathscr{D}$ becomes a Banach space E_0 with the new norm, such that the injection of E_0 into E is continuous and $A(t)$ is continuous on E_0 into E for every $t \in [a, b]$.

§ 2. Stability of solutions of evolutional equations containing a parameter

Now we consider an evolutional equation containing a parameter $\varepsilon \geq 0$:

$$(1)_\varepsilon \qquad \partial_t u = A_\varepsilon(t)u + f_\varepsilon(t).$$

Let $u = u_0(t)$ be an E-solution of $(1)_0$ in $[a, b]$ for $\varepsilon = 0$. $u = u_0(t)$ is said to be *completely E-stable* in $[a, b]$ with respect to the equation $(1)_\varepsilon$ for $\varepsilon \to 0$, when the following condition is fulfilled: For any $\delta > 0$ there evists $\eta(\delta) > 0$ such that, if $0 < \varepsilon < \eta(\delta)$, any E-solution $u = u_\varepsilon(t)$ of $(1)_\varepsilon$ on $[s, b]$ for any $s \in [a, b]$ with $||u_\varepsilon(s) - u_0(s)|| < \eta(\delta)$ satisfies the inequality

$$||u_\varepsilon(t) - u_0(t)|| < \delta \quad \text{for} \quad s \leq t \leq b.$$

Lemma 1. *Let the equation* $(1)_\varepsilon$ *be completely E-well posed in* $[a, b]$ *for* $\varepsilon > 0$. *If an E-solution* $u = u_0(t)$ *of* $(1)_0$ *is completely E-stable in* $[a, b]$ *for* $\varepsilon \to 0$ *with respect to* $(1)_\varepsilon$, *then the fundamental solution* $U_\varepsilon(t, s)$ *of the associated homogeneous equation of* $(1)_\varepsilon$, *for sufficiently small* $\varepsilon > 0$, *with some constant C satisfies the inequality*

$$(2) \qquad ||U_\varepsilon(t, s)|| \leq C \quad \textit{for} \quad a \leq s \leq t \leq b.$$

Proof. Let $u = u_0(t)$ be completely stable in $[a, b]$ for $\varepsilon \to 0$ with respect to $(1)_\varepsilon$, and $u = u_\varepsilon(t)$ and $u = v_\varepsilon(t)$ be solutions of $(1)_\varepsilon$ such that

$u_\varepsilon(s)=v_0(s)$ and $||v_\varepsilon(s)-u_\varepsilon(s)||<\eta(\delta)$ resp. Then, if $0<\varepsilon<\eta(\delta)$ we must have

$$||u_\varepsilon(t)-v_\varepsilon(t)|| \leqq ||u_\varepsilon(t)-u_0(t)||+||v_\varepsilon(t)-u_0(t)||<2\delta \quad \text{for} \quad s\leq t\leq b\,.$$

Hence for any $w\in E$ with $||w||<\eta(\delta)$ holds the inequality $||U_\varepsilon(t, s)w|| <2\delta$ for $a\leq s\leq t\leq b$. This asserts Lemma 1.

In order to give our sufficient conditions for the complete stability of a solution, we shall prepare a definition of quasi-regularity of solutions. An E-solution $u=u_0(t)$ of

$$(0) \qquad \partial_t u = A_0(t)u+f(t)$$

is said to be *quasi-regular* in $[a, b]$ with respect to an operator $A_1(t)$, when for any $\delta>0$ there exists an E-continuous $v_\delta(t)$ on $[a, b]$ such that $\partial_t v_\delta(t)-A_0(t)v_\delta(t)-f(t)$ and $A_1(t)v_\delta(t)$ are bounded and E-continuous on $[a, b]$ and the inequalities

$$||v_\delta(t)-u_0(t)||<\delta \quad \text{and} \quad ||\partial_t v_\delta(t)-A_0 v_\delta(t)-f(t)||<\delta$$

hold for $a\leq t\leq b$.

Especially if $A_0(t)=A_1(t)=A$ and A is the infinitesimal generator of a 1-parameter semi-group and $f(t)$ is E-continuous on $[a, b]$, then an E-solution of (0) is quasi-regular in $[a, b]$ with respect to A. Indeed in this case we have to set $v_\delta(t)=(1-\lambda_\delta^{-1}A)^{-1}u_0(t)$ with sufficiently large $\lambda_\delta>0$.

Now we assume that $(1)_\varepsilon$ is completely E-well posed in $[a, b]$ for $\varepsilon\geq 0$ and the operator $A_\varepsilon(t)$ have the form:

$$(3) \qquad A_\varepsilon(t) = A_0(t)+\varepsilon A_1(t) \qquad (\varepsilon\geq 0)\,.$$

Further let $f_\varepsilon(t)$ be E-continuous on $[a, b]$ and converge to $f_0(t)$ uniformly on $[a, b]$ as $\varepsilon\to 0$. Then we have:

Theorem 2. *Let $u=u_0(t)$ be an E-solution of $(1)_0$ for $\varepsilon=0$ in a finite closed interval $[a, b]$ and be quasi-regular with respect to $A_1(t)$ in $[a, b]$. In order that $u=u_0(t)$ be completely E-stable in $[a, b]$ with respect to $(1)_\varepsilon$, it is necessary and sufficient that for sufficiently small $\varepsilon>0$, the fundamental solution $U_\varepsilon(t, s)$ of the associated homogeneous equation of $(1)_\varepsilon$ is uniformly bounded for $a\leq s\leq t\leq b$.*

Proof. As the nessecity of the condition is already given by Lemma 1, we have only to prove the sufficiency.

For any $\delta>0$ there exists an E-continuous $v_\delta(t)$ on $[a, b]$ such that

$h_\delta(t)=\partial_t v_\delta(t)-A_0(t)v_\delta(t)-f_0(t)$ and $A_1(t)v_\delta(t)$ are E-continuous on $[a, b]$ with the conditions

(4) $\qquad ||v_\delta(t)-u_0(t)||<\delta$ and $||h_\delta(t)||<\delta$ for $a\leq t\leq b$.

Then we get

$$\partial_t(u_\varepsilon - v_\delta) = A_\varepsilon(t)(u_\varepsilon - v_\delta)+\varepsilon A_1(t)v_\delta + g_{\varepsilon,\delta}(t),$$

where $g_{\varepsilon,\delta}(t)=f_\varepsilon(t)-f_0(t)+h_\delta(t)$. Hence, by Theorem 1,

$$\begin{aligned} u_\varepsilon(t)-v_\delta(t) &= U_\varepsilon(t, s)\{u_\varepsilon(s)-v_\delta(s)\} \\ &+\int_s^t U_\varepsilon(t, \sigma)\{\varepsilon A_1(\sigma)v_\delta(\sigma)+g_{\varepsilon,\delta}(\sigma)\}\, d\sigma \quad \text{for} \quad a\leq s\leq t\leq b. \end{aligned}$$

Thus by (2) we have

$$||u_\varepsilon(t)-v_\delta(t)||\leq C\{||u_\varepsilon(s)-v_\delta(s)||+\int_s^t(\varepsilon||A_1(\sigma)v_\delta(\sigma)||+||g_{\varepsilon,\delta}(\sigma)||)\,d\sigma\}.$$

There exist positive constants $\zeta(\varepsilon)$ and B_δ such that $\zeta(\varepsilon)\to 0$ as $\varepsilon\to 0$, $||f_\varepsilon(t)-f_0(t)||<\zeta(\varepsilon)$ for $a\leq t\leq b$ and $||A_1(t)v_\delta(t)||\leq B_\delta$ for $a\leq t\leq b$. Thus, by (4), we get

$$||u_\varepsilon(t)-u_0(t)||\leq C\{||u_\varepsilon(s)-u_0(s)||+\delta+(b-a)(\varepsilon B_\delta+\zeta(\varepsilon)+\delta)\}, \quad \text{for} \quad a\leq s\leq t\leq b.$$

First taking $\delta>0$ sufficiently small and then letting $\varepsilon\to 0$, we complete the proof.

REMARK 2. The sufficiency of the condition in Theorem 2 remains valid even for the case of infinite interval (a, ∞) $(b=\infty)$, if we add to it the condition

$$\int_a^t ||U_\varepsilon(t, s)||\,ds\leq C \text{ for } a\leq t<\infty \text{ with some constant } C.$$

§ 3. Degeneration of evolutional equations

Let us consider the evolutional equation of the singular form in the parameter ε:

$(1)_\varepsilon \qquad \varepsilon\partial_t u = A_\varepsilon(t)u+f_\varepsilon(t)$ with $\varepsilon>0$

and the degenerated equation

$(1)_0 \qquad A_0(t)u+f_0(t) = 0$

We assume that $A_\varepsilon(t)$ and $f_\varepsilon(t)$ have the forms

$$(2)\qquad \begin{aligned} A_\varepsilon(t) &= A_0(t)+\varepsilon A_1(t)\,, \\ f_\varepsilon(t) &= f_0(t)+\varepsilon f_1(t)+\varepsilon h_\varepsilon(t)\,, \end{aligned}$$

where $f_0(t)$, $f_1(t)$ and $h_\varepsilon(t)$ are E-continuous on $[a, b]$ and $h_\varepsilon(t)\to 0$ uniformly on $[a, b]$ as $\varepsilon\to 0$.

A solution $u=u_0(t)$ of the degenerated equation $(1)_0$ is said to be *completely E-stable* in $[a, b]$ with respect to $(1)_\varepsilon$ for $\varepsilon\to 0$, when the following condition is fulfilled: For any $\varepsilon>0$ there exists some $\eta(\delta)>0$ such that, if $0<\varepsilon<\eta(\delta)$, any E-solution $u_\varepsilon(t)$ of $(1)_\varepsilon$ in $[s, b]$ for any $s\in[a, b]$ with

$$\|u_\varepsilon(s)-u_0(s)\|<\eta(\delta)$$

satisfies the inequality

$$\|u_\varepsilon(t)-u_0(t)\|<\delta \quad \text{for} \quad s\leq t\leq b\,.$$

Theorem 3. *Assume that* $(1)_\varepsilon$ *is completely E-well posed in a finite closed interval* $[a, b]$ *for* $\varepsilon>0$ *and* $A_\varepsilon(t)$ *and* $f_\varepsilon(t)$ *have the forms* (2). *Let* $u=u_0(t)$ *be a E-solution of* $(1)_0$ *on* $[a, b]$ *such that* $u_0(t)$, $\partial_t u_0(t)$ *and* $A_1(t)u_0(t)$ *are E-continuous on* $[a, b]$. *In order that* $u=u_0(t)$ *is completely stable in* $[a, b]$ *for* $\varepsilon\to 0$ *with respect to* $(1)_\varepsilon$ *with any E-continuous* $f_1(t)$ *on* $[a, b]$, *it is necessary and sufficient that the fundamental solution* $U_\varepsilon(t, s)$ *of* $\partial_t u=\varepsilon^{-1}A_\varepsilon(t)u$ *satisfies the following conditions:*

1) *There exists a constant C such that*

$$\|U_\varepsilon(t, s)\|\leq C \text{ for } a\leq s\leq t\leq b \text{ and sufficiently small } \varepsilon>0.$$

2) *For any* α, β, t, *and* $v\in E$ *such that* $a\leq\alpha<\beta\leq t\leq b$,

$$\int_\alpha^\beta U_\varepsilon(t, s)v\,ds\to 0 \text{ uniformly on } t\in[\beta, b] \text{ as } \varepsilon\to 0\,.$$

Proof. The necessity of 1) is obtained in the same way as in the proof of Lemma 1.

To prove the necessity of 2), setting $f_1(t)=\varphi(t)-A_1(t)u_0(t)$ with an arbitrary E-continuous $\varphi(t)$ on $[a, b]$, we get from $(1)_\varepsilon$, $(1)_0$ and (2)

$$\partial_t(u_\varepsilon-u_0) = \varepsilon^{-1}A_\varepsilon(t)(u_\varepsilon-u_0)+\varphi(t)+h_\varepsilon(t)\,.$$

Hence for $a\leq s\leq t\leq b$

$$\begin{aligned} u_\varepsilon(t)-u_0(t) &= U_\varepsilon(t,s)\{u_\varepsilon(s)-u_0(s)\} \\ &\quad +\int_s^t U_\varepsilon(t, \sigma)\{\varphi(\sigma)+h_\varepsilon(\sigma)\}\,d\sigma\,. \end{aligned}$$

By 1) we have if $0<\varepsilon<\eta(\delta)$, as $\eta(\delta)\leq\delta$,

$$\| U_\varepsilon(t, s)\{u_\varepsilon(s)-u_0(s)\} \| \leq C\delta$$

and

$$\left\| \int_s^t U_\varepsilon(t, \sigma) h_\varepsilon(\sigma) d\sigma \right\| \leq (b-a) C\zeta(\varepsilon)$$

for $a \leq s \leq t \leq b$, where $\zeta(\delta) \to 0$ as $\varepsilon \to 0$. Hence we have only to show that

$$\text{“} \int_s^t U_\varepsilon(t, \sigma)\varphi(\sigma)d\sigma \to 0 \text{ uniformly on } a \leq s \leq t \leq b$$

as $\varepsilon \to 0$ for any E-continuous $\varphi(t)$ on $[a, b]$” implies 2). Put $\varphi(\sigma)=\psi(\sigma)v$ with any $v \in E$ and a continuous real valued function $\psi(\sigma)$ on $[a, b]$ such that $0 \leq \psi(\sigma) \leq 1$, $\psi(\sigma)=1$ for $\alpha+\delta \leq \sigma \leq \beta$, $\psi(\sigma)=0$ for $a \leq \sigma \leq \alpha$ and for $\beta+\delta \leq \sigma \leq b$. Then by 1)

$$\left\| \int_s^t U_\varepsilon(t, \sigma)\varphi(\sigma) d\sigma - \int_\alpha^\beta U_\varepsilon(t, \sigma) v d\sigma \right\| \leq 2\delta C \|v\| .$$

Therefore we obtain 2), as δ can be taken arbitrarily small.

To prove the sufficiency of 2) with 1), we have only to show that by these conditions

$$\int_s^t U_\varepsilon(t, \sigma)\varphi(\sigma) d\sigma \to 0 \text{ uniformly for } a \leq s \leq t \leq b \text{ as } \varepsilon \to 0 .$$

Divide the interval $[a, b]$ into a finite number of consecutive intervals $[\tau_{\nu-1}, \tau_\nu]$ ($\nu=1, \cdots, N$) in such a way that $\tau_\nu - \tau_{\nu-1} < \delta$ and $\|\varphi(\sigma)-\varphi(\tau_\nu)\| < \delta$ for $\sigma \in [\tau_{\nu-1}, \tau_\nu]$. Then

$$\left\| \int_s^t U_\varepsilon(t, \sigma)\varphi(\sigma) d\sigma - \sum_{s<\tau_\nu \leq t} \int_{\tau_{\nu-1}}^{\tau_\nu} U_\varepsilon(t, \sigma)\varphi(\tau_\nu) d\sigma \right\|$$
$$< C(b-a)\delta + 2C\delta \operatorname{Max} \|\varphi(\sigma)\| .$$

Thus by 2), taking δ sufficiently small and letting $\varepsilon \to 0$, we attain to the desired conclusion. Q. E. D.

When $f_\varepsilon(t)$ has the form, instead of (2),

$$f_\varepsilon(t) = f_0(t) + h_\varepsilon(t) , \tag{3}$$

where $f_0(t)$ and $h_\varepsilon(t)$ have the same meanings as before, we cannot easily have necessary and sufficient conditions for the complete stability of $u_0(t)$, but only sufficient conditions, while we can relax the conditions on $u_0(t)$ somewhat.

Theorem 4. *Let $u=u_0(t)$ be an E-continuous solution of* $(1)_0$ *on* $[a, b]$ *such that for any $\delta>0$ there exists an E-continuous $v_\delta(t)$ on $[a, b]$ with* *bounded and E-continuous* $\partial_t v_\delta(t)$, $A_0(t)v_\delta(t)$ *and* $A_1(t)v_\delta(t)$ *on* $[a, b]$ *satisfying the conditions*

$$\|v_\delta(t)-u_0(t)\|<\delta \quad \text{and} \quad \|A_0 v_\delta(t)+f_0(t)\|<\delta \quad \text{for} \quad t\in[a, b].$$

Assume that the fundamental solution $U_\varepsilon(t, s)$ *of* $\partial_t u=\varepsilon^{-1}A_\varepsilon(t)u$, *satisfies for sufficiently small* $\varepsilon>0$, *the conditions with some constant* C:

1) $\|U_\varepsilon(t, s)\|\leq C$ *for* $a\leq s\leq t\leq b$,

2)′ $\int_a^t \|U_\varepsilon(t, s)\| ds\leq \varepsilon C$ *for* $a\leq t\leq b$.

Then $u=u_0(t)$ *is completely E-stable in* $[a, b]$ *with respect to* $(1)_\varepsilon$ *for* $\varepsilon\to 0$.

Proof will be left to the reader.

REMARK 3. The sufficiency of the conditions 1) and 2)′ in Theorem 4 remains valid even for the case of infinite interval (a, ∞) $(b=\infty)$.

§ 4. Degeneration of evolutional equation when $A_\varepsilon(t)=A$

Consider the evolutional equation of singular form in ε:

$$(1)_\varepsilon \qquad \varepsilon\partial_t u = Au+f_\varepsilon(t) \qquad (\varepsilon>0),$$

where A is the infinitesimal generator of a one parameter semi-group in a *reflexive* Banach space E. As it has been stated in § 1, the operator A can be extended to a continuous linear operator on E into E_1, and the equation $(1)_\varepsilon$ becomes completely E-well posed in any finite interval. The fundamental solution of the associated homogeneous equation has the form

$$U_\varepsilon(t, s) = \exp(\varepsilon^{-1}(t-s)A),$$

as $\exp(tA)$ $(t\geq 0)$ is the transformation generated by A.

Theorem 5. *Let* $u=u_0(t)$ *be an E-solution in a finite closed interval* $[a, b]$ *of the degenerated equation*

$$Au+f_0(t) = 0$$

with E-continuous $\partial_t u_0(t)$ *on* $[a, b]$.

In order that $u_0(t)$ *be completely E-stable in* $[a, b]$ *with respect to* $(1)_\varepsilon$ *for* $\varepsilon\to 0$, *where* f_ε *has the form* (2) *with any E-continuous* f_1, *it is necessary and sufficient that the following conditions are fulfilled*:

1) *With some constant* C, $\|\exp(tA)\|\leq C$ *for* $0\leq t<\infty$.

2) $Av=0$ *with* $v\in\mathcal{D}(A)$ *implies* $v=0$,

where $\mathscr{D}(A)$ denotes the proper domain of A before the extension of A on E.

Proof. From Theorem 3 we get easily the necessity of 1), as $U_\varepsilon(t, s) = \exp(\varepsilon^{-1}(t-s)A)$.

By the mean ergodic theorem in a reflexive Banach space, we obtain from 1) a projective operator P on E into E such that:

$$\lim_{\tau\to\infty} \tau^{-1}\int_0^\tau \exp(tA)v\,dt = Pv \quad \text{for any} \quad v\in E,^{2)} \tag{3}$$

$$P\exp(tA) = \exp(tA)P = P^2 = P, \tag{4}$$

and

$$P(E) = \{u\in\mathscr{D}(A);\ Au = 0\} \tag{5}$$

Thus, for $a\leq\alpha<\beta\leq t\leq b$ and any $v\in E$, setting

$$\tau(\varepsilon) = \varepsilon^{-1}(\beta-\alpha) \text{ we have by (4)}$$

$$\begin{aligned}&\int_\alpha^\beta U_\varepsilon(t, s)v\,ds-(\beta-\alpha)Pv\\ &\quad=\int_\alpha^\beta \exp(\varepsilon^{-1}(t-s)A)v\,ds-(\beta-\alpha)Pv\\ &\quad=(\beta-\alpha)\exp(\varepsilon^{-1}(t-\beta)A)\{\tau(\varepsilon)^{-1}\int_0^\tau \exp(\sigma A)v\,d\sigma-Pv\},\end{aligned}$$

hence by (3) and 1), we have

$$\left\|\int_\alpha^\beta U_\varepsilon(t, s)v\,ds-(\beta-\alpha)Pv\right\|\to 0 \text{ uniformly for } \beta\leq t\leq b,$$

as $\tau(\varepsilon)\to\infty$ for $\varepsilon\to 0$.

Hence the condition 2) in Theorem 3 with 1) is equivalent to:

$$Pv = 0 \quad \text{for every } v\in E.$$

Therefore, by (5) the condition 2) with 1) is equivalent to the condition 2) with 1) in Theorem 3. Q. E. D.

REMARK 4. The sufficiency of the conditions in Theorem 5 remains valid even for the case $b=\infty$, if we replace 2) by

$$\int_0^\infty \|\exp(tA)\|\,dt<\infty.$$

---

2) Here the lim means the strong limit in E.

OSAKA UNIVERSITY

(Received March 5, 1963)

References

[1] M. Nagumo: *Re-topologization of functional space in order that a set of operators will be continuous,* Proc. Japan Acad. **37** (1961), 550–552.

[2] M. Nagumo: *Singular perturbation of Cauchy problem of partial differential equations with constant coefficients,* Proc. Japan Acad. **35** (1959), 449–454.

[3] H. Kumano-go: *Singular perturbation of Cauchy problem of partial differential equations with contant coefficients II,* Proc. Japan Acad. **35** (1959), 541–546.

[4] K. Yosida: *Mean ergodic theorem in Banach spaces,* Proc. Imp. Acad. Japan (1938), 292–294.

[5] E. Hille and R. S. Phillips: Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ. **31**, Providence, 1957.