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NOTE ON ORBIT SPACES

To Professor K. Shoda on his 60-th birthday

BY

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Let V be an affine or projective variety with universal domain K and let G be an algebraic linear group acting on V as a group of automorphisms of V. Let d be the maximum of the dimension of G-orbits on V and let U be the set of points of V whose G-orbits are of dimension d.

Then one can ask whether or not the set of G-orbits on U forms naturally an algebraic variety. Though the answer is not affirmative in general, it is an important question to ask the nature of the set of G-orbits on U. As one approach to this kind of problem, we observe the following objects:

Let L be the function field of V (over K) and let L_G be the field of G-invariants in L. For each point P of V, we consider the locality of P, which we shall denote by the same P, and we consider the ring $P \cap L_G$, which we shall denote by P_G. P_G is nothing but the ring of G-invariants in P. Now we can ask the following questions:

QUESTION 1. *Is P_G a locality?*

QUESTION 2. *Does there exist an algebraic variety W such that the set of localities of points of W coincides with the set $\{P_G \mid P \in U\}$?*

The answers to these questions are not affirmative in general.

The main purpose of the present paper is to give some results concerning the above questions in rather special cases.

In § 1, we give some preliminaries. In § 2, we give some results in the case where G is a torus group and V is affine. Though Question 1 is affirmative in this case, Question 2 is not affirmative in the case where G is the multiplicative group of K. In § 3, we show that if V is a non-singular affine variety and if every rational representation of G is completely reducible, then Question 1 is affirmative. In § 4, we show that if V is an affine variety whose coordinate ring R is a unique factorization domain, if invertible elements of R are G-invariants and if the radical of G is unipotent, then these 2 questions are affirmative, provided that the ring R_G of G-invariants in R is finitely generated. Then we give an important example to the theory of orbit spaces and then we give an application of the result to the case of projective varieties. In § 5, we show that Question 1 is not affirmative even if G is simple, K is of characteristic zero and V is normal.

The notation stated at the beginning is maintained throughout this paper. When V is an affine variety, R denotes the coordinate ring of V over K, and R_G denotes the ring of G-invariants in R. $\mathfrak{m}_P$ $(P \in V)$ denotes the maximal ideal of P.

1. Preliminaries.

We begin with a remark that Question 1 is not affirmative in general. A counter example is readily obtained by our counter example to the 14-th problem of Hilbert (see, for instance, [3]) by virtue of Theorem 4.1 below. Note that in that example, V is non-singular (cf. Theorems 3.4 and 5.1).

Consider the case where V is an affine variety. Then G becomes a group of automorphisms of R such that for each element a of R, the module $\sum_{g \in G} a^g K$ is a finite K-module.

If either G is a torus group (K being arbitrary) or K is of characteristic zero and the radical of G is a torus group, then we know that every rational representation of G is completely reducible. Therefore we have the following result, whose proof can be found in our lecture note [3].

Lemma 1.1. *With the assumption made above, and denoting by $F(P)$ the closure of the G-orbit of $P \in V$,* (1) *R_G is finitely generated over K,* (2) *the relation $\sim$, defined by that $P \sim Q$ if and only if $F(P) \cap F(Q) \neq$ empty, is an equivalence relation, and* (3) $(R_G)_{\mathfrak{m}_P \cap R_G} = \bigcap_{Q \in F(P)} Q_G$. *In particular,* (4) *if $Q \in F(P)$ is such that G-orbit of Q is closed, then $Q_G = (R_G)_{\mathfrak{m}_Q \cap R_G}$. Furthermore,* (5) *$P \sim Q$ if and only if $(R_G)_{\mathfrak{m}_P \cap R_G} = (R_G)_{\mathfrak{m}_Q \cap R_G}$.*

2. Torus groups.

Theorem 2.1. *Assume that V is an affine variety and that G is a torus group. Then P_G is a locality for any $P \in V$. Let W be a G-admissible* (*irreducible*) *subvariety of V which carries P and let P' be the locality of P on W. Then the ring P'_G of G-invariants in P' is the natural homomorphic image of P_G.*

Proof. Let ϕ be the natural homomorphism from P onto P'. Assume that $\phi(f'/f)$ is in P'_G $(f, f' \in R, f(P) \neq 0)$. Consider the module

$M=\sum_{g\in M} f^g K$. This is generated by G-semi-invariants, say $f_1, \cdots, f_t$. Since $f(P)\neq 0$, there is at least one f_i, say f_1 such that $f_1(P)\neq 0$. $f_1=\sum_{i=1}^{N} a_i f^{g_i}$ with $a_i\in K$ and $g_i\in G$. Set $f_1'=\sum a_i f'^{g_i}$. Then we have $\phi(f'/f)=\phi(f_1'/f_1)$. Thus we may assume that f is G-semi-invariant: $f^g=a(g)f$ $(a(g)\in K)$. Then $\phi(a(g)f'-f'^g)=0$. Consider the module $M'=\sum_{g\in G} f'^g K$ and its submodule $M'\cap\phi^{-1}(0)$. By the complete reducibility of rational representations, we see that there is a representative f'' of f' modulo $M'\cap\phi^{-1}(0)$ such that $f''^g=a(g)f''$ for any $g\in G$. Thus f''/f is G-invariant and $\phi(f''/f)=\phi(f'/f)$. Since it is obvious that the homomorphic image of a G-invariant by ϕ is a G-invariant, we complete the proof of the last half. Consider now the closed set $F(P)$ given in Lemma 1.1. Let Q be a point of $F(P)$ such that G-orbit of Q is closed. Note that Lemma 1.1, (2) implies that $F(P)$ contains only one closed G-orbit. If $Q_G\neq P_G$, then $Q_G\subset P_G$ by Lemma 1.1, hence there is an element f'/f in P_G which is not in Q_G. By the proof above, we may assume that $f(P)\neq 0$ and that f, f' are semi-invariants. Then, considering the affine variety $V-$(closed set defined by f), we can omit Q. If this process is repeated, then the dimension of G-orbit of new Q is greater than that of previous Q, by virtue of the uniqueness of closed orbit in $F(P)$. Therefore, after a finite number of steps, we have the case where $Q_G=P_G$. Q_G is a locality by Lemma 1.1, and therefore P_G is a locality. Thus we complete the proof of Theorem 2.1.

With the same V as before, assume now that G is a torus group of dimension 1, i.e., there is an isomorphism a from G onto the multiplicative group of K. Then R is generated by G-semi-invariants, say $f_1, \cdots, f_n$. Each f_i defines a character a_i of G in such a way that $f_i^g=a_i(g)f_i$. These a_i are powers of a.

Theorem 2.2. *If all the a_i are powers of a with non-negative exponent, then the set $\{P_G \mid P\in U\}$ is the set of localities of a quasiprojective variety. Furthermore, for $P\in U$, the correspondence $\{P^g \mid g\in G\}\rightarrow P_G$ is one to one.*

Proof. Let $\mathfrak{a}$ be the ideal of R generated by all G-semi-invariants which are not G-invariants. Then every element of $R/\mathfrak{a}$ is G-invariant. This shows that if Q is a point of the closed set F defined by $\mathfrak{a}$, then Q is G-invariant. Let $h_1, \cdots, h_r$ be a basis for $\mathfrak{a}$ such that $h_i^g=a^{n_i}(g)h_i$ with positive n_i. Let $\mathfrak{b}$ be the ideal for the closed set $V-U$. Then $\mathfrak{b}$ is generated by G-semi-invariants. Since $F\subseteq V-U$, we see that $\mathfrak{ab}$ defines $V-U$. Thus there are a finite number of G-semi-invariants $k_1, \cdots, k_s$ in R such that (1) $k_i^g=a^{t_i}(g)k_i$ for any $g\in G$ with $t_i>0$ and (2) the ideal $\sum k_i R$ defines $V-U$. Then, each k_i may be replaced by its power (of positive exponent) without losing these two properties. Therefore we may assume that all t_i are the same, which we shall denote by t. Now, if $k_i(P)\neq 0$, then the G-orbit of P is a closed set in the affine veriety defined by $R[k_i^{-1}]$, whence P_G is a ring of quotients of the ring R_i of G-invariants in $R[k_i^{-1}]$ by Lemma 1.1. Thus the set of P_G $(P\in U)$ is the set of localities of dimension zero which are rings of quotients of some R_i. Let k_{ij} $(j=1,\cdots,u_i)$ be elements which generate R_i over K, and we take a natural number v such that $k'_{ij}=k_{ij}k_i^v\in R$ for all i, j. We now consider the projective variety W defined by homogeneous coordinates $(k_1^v,\cdots,k_s^v,k'_{11},\cdots,k'_{1u_1},k'_{21},\cdots,k'_{su_s})$. Then the affine ring of $W-$(the closed set defined by $k_i^v=0$) is obviously R_i. Thus there is an open subset W' of W such that $\{P_G|P\in U\}$ is the set of localities of points of W'. Now we have only to prove that the correspondence $\{P^g|g\in G\}\to P_G$ is one to one. Assume that $U\ni Q\notin\{P^g|g\in G\}$. If $F(P)\cap F(Q)$ is empty, then Lemma 1.1 shows that $P_G\neq Q_G$. So we assume that $F(P)\cap F(Q)$ is not empty. The ideal $\mathfrak{i}_P$ for $F(P)$ is generated by G-semi-invariants. Then considering $\mathfrak{a}\mathfrak{i}_P$, we see that there are a finite number of G-semi-invariants $m_1,\cdots,m_w$ such that (1) $m_i^g=a^{t'_i}(g)m_i$ with $t'_i>0$ and (2) every point of the closed set defined by $\sum m_iR$ is either a G-invariant point or a point in $F(P)$. Since Q is in U and is not in $F(P)$, we have $m_i(Q)\neq 0$ for some i, say 1. Take a linear combination k' of $k_i^{t'_1}$ so that $k'(P)$, $k'(Q)$ are different from zero. Then $f=m_1^t/k'$ is G-invariant, and is regular at P and Q. Furthermore $f(P)=0$ and $f(Q)\neq 0$. Therefore $P_G\neq Q_G$. Thus we complete the proof of Theorem 2.2.

Corollary 2.3. *Assume that $V_1,\cdots,V_n$ are affine varieties and assume that G_i are torus groups of dimension* 1 *acting on V_i. If the operation of G_i on V_i satisfies the condition in Theorem* 2.2, *then, for $V=V_1\times\cdots\times V_n$ and $G=G_1\times\cdots\times G_n$, we have the same conclusion as in Theorem* 2.2.

Here we give a remark that the assumption in Theorem 2.2 is important. Namely, (i) if we do not assume the non-negativity of exponents of characters, then such a quasi-projective variety (or an abstract variety) as W' above may not exist (see Example 2.4 below), and even if such W' exists, the correspondence $\{P^g|g\in G\}\to P_G$ may not be one to one (see Example 2.5 below) and (ii) if G is a torus group of dimension greater than 1, then the non-negativity of exponents is not sufficient (see Examples 2.6 and 2.7 below).

EXAMPLE 2.4. Consider the affine 3-space defined by $R=K[x_1,x_2,x_3]$. Let G be the set of matrices

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}$$

with $t \in K$, $t \neq 0$. Consider $P=(a, b, 0)$ with $a \neq 0$ and $Q=(0, 0, c)$ with $c \neq 0$. Then P, Q are in U. P_G is a ring of quotients of the ring R_1 of G-invariants in $R[x_1^{-1}]$ by Lemma 1.1. R_1 is obviously $K[x_1x_3, x_2x_3, x_2/x_1]$. Similary, Q_G is a ring of quotients of $K[x_1x_3, x_2x_3]$. Therefore we see easily that Q_G is strictly contained in P_G.

EXAMPLE 2.5. Consider the affine plane V defined by $R=K[x, y]$ and let G be the set of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ $(t \in K, t \neq 0)$. Then obviously $L_G=K(xy)$. Each curve $xy=a$ $(a \in K)$ is a G-orbit for $a \neq 0$. Then curve $xy=0$ consists of three orbits, which are $\{(0, b) \mid b \neq 0\}$, $\{(a, 0) \mid a \neq 0\}$ and $\{(0, 0)\}$. If P is on one of these orbits, then P_G dominates $K[xy]_{(xy)}$ which is a valuation ring, hence $P_G=K[xy]_{(xy)}$.

EXAMPLE 2.6. Consider the affine 4-space defined by $R=K[x_1, x_2, x_3, x_4]$ and let G be the set of matrices

$$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & tu & 0 \\ 0 & 0 & 0 & t^2u \end{pmatrix}$$

with t, $u \in K$, $tu \neq 0$. $V-U$ is the set of points such that 3 of the coordinates are zero, hence is defined by $\sum_{i \neq j} x_ix_jR$. If x_ix_j is different from zero at P, then P_G is a ring of quotients of the ring R_{ij} of G-invariants in $R[x_i^{-1}, x_j^{-1}]$. But $R_{14}=K[x_1^2x_2/x_4, x_1x_3/x_4]$, $R_{23}=K[x_1x_2/x_3, x_2x_4/x_3^2]$ and we see that the set of all P_G $(P \in U)$ is not the set of localities of points of any abstract variety.

EXAMPLE 2.7. If we consider the restriction of above G on the three space V defined by $R=K[x_1, x_2, x_3]$, then we see easily the set of P_G $(P \in U)$ is the set of localities of points of the projective variety defined by (x_1x_2, x_3). But, $P=(0, 1, 1)$ and $Q=(1, 0, 1)$ belongs distinct orbits of dimension 2 and $P_G=Q_G$.

We give some remarks.

REMARK 2.8. If G is a torus group, then $\{P_G \mid P \in U\}$ is the set of localities of points of a finite number of affine varieties of L_G.

The proof is immediate.

REMARK 2.9. Let G be again a torus group and let V be an affine variety. If, for a G-admissible open set U' contained in U, there are semi-invariants $f_0, \cdots, f_n$ in R such that (1) characters of G given by f_i are all the same, and (2) the closed set defined by $\sum f_i R$ is $V-U'$, then there is an open set W' of the projective variety defined by the homogeneous coordinates $(f_0, \cdots, f_n)$ such that (i) the set of localities of points of W' is the set of $\{P_G | P \in U'\}$ and (ii) the correspondence $\{P^g | g \in G\} \to P_G$ is one to one (for $P \in U'$).

For the proof, that of Theorem 2.2 is adapted easily.

3. Non-singular case.

Lemma 3.1. *Let P and Q be points on V which is assumed to be normal. If $P_G \not\subseteq Q_G$, then there exists a G-admissible divisorial closed set W of V which carries Q but not P.*

Proof. Let f be an element of P_G which is not in Q_G. Then the pole of f is the required set.

REMARK 3.2. The converse of Lemma 3.1 is not true in general. For instance, in Example 2.5 in § 2, the line $y=0$ is G-admissible and carries Q but does not carry P, though $P_G = Q_G$.

It is known that

Lemma 3.3. *If V is a non-singular affine variety and if W is a divisorial closed set of V, then $V-W$ is an affine variety.*

For the proof, we refer to [1] and [2].

Now we have

Theorem 3.4. *If V is a non-singular affine variety and if every rational representation of G is completely reducible, then P_G is a locality for any $P \in V$.*

Proof. Let Q be a point of the closed set $F(P)$ (defined in Lemma 1.1) such that its G-orbit is closed. Then $Q_G \subseteq P_G$. If $Q_G \neq P_G$, then there is a G-admissible divisorial closed set W of V which carries Q but not P by Lemma 3.1. $V-W$ is affine by Lemma 3.3, whence we may omit such Q by the same reason as we gave in the proof of Theorem 2.1. Thus we have the case $P_G = Q_G$, which is a locality by Lemma 1.1.

4. Semi-simple groups.

Let V be an affine variety as before.

Theorem 4.1. *If R is a unique factorization domain such that every* *invertible element is G-invariant and if the radical of G is unipotent, then* (1) L_G *is the field of quotients of* R_G, (2) $P_G=(R_G)_{(\mathfrak{m}_P\cap R_G)}$ *and* (3) *if furthermore G is connected then* R_G *is a unique factorization domain.*

Proof. The general case follows easily from the case where G is connected. Therefore we assume that G is connected. Then we see that

(*) every rational representation of G into the multiplicative group of a field containing K is trivial.

Let f'/f be an element of L_G, where f, f' are elements of R which have no common factor. Since $(f'/f)^g=f'/f=f'^g/f^g$ $(g\in G)$ and since the number of prime factors of f is equal to that of f^g, we see that $f^g=a_g f$ with $a_g\in R_G$. Therefore (*) above shows that f is invariant[1]. Thus f, f' are in R_G, and L_G is the field of quotients of R_G. If $f'/f\in P$, then $f'/f=h'/h$ with $h(P)\neq 0$ $(h, h'\in R)$. Since f'/f is the reduced expression of h'/h, we have $f(P)\neq 0$, whence $f'/f\in (R_G)_{(\mathfrak{m}_P\cap R_G)}$. Thus $(R_G)_{(\mathfrak{m}_P\cap R_G)}=P_G$. If $f\in R_G$, then each prime factor of f in R is invariant because G is connected (and by virtue of (*) above), whence R_G is a unique factorization domain.

Corollary 4.2. *If furthermore* R_G *is finitely generated, hence in particular if G is semi-simple and K is of characteristic zero, then the set of* P_G $(P\in V)$ *is the set of localities of the affine variety defined by* R_G.

One important remark to be added here is that:

Consider the case where G is semi-simple and K is characteristic zero. Then each P_G corresponds to the equivalence class of P given by Lemma 1.1, hence it happens sometimes that infinitely many G-orbits in U corresponds to one P_G. Namely, there are many examples of an affine variety V which carries a closed subset F of U such that (1) F is the union of infinitely many G-orbits and (2) if a rational function f on V is G-invariant and if f is regular at one point of F, then F is regular at every point of F and the value of f on F is constant all over F.

Existence of such an example is easily seen. But we shall give such an example under more restriction, namely, we shall construct an example as follows:

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1) Since we are using elements of L_G in this representation of G, we have to show that this representation can be extended to a rational representation of an algebraic group over the algebraic closure $\bar{L}_G$ of L_G containing G. This can be shown as follows:

G acts on $\bar{R}=\bar{L}_G\otimes_{L_G}L_G[R]$ as a subgroup of $GL(n, \bar{L}_G)$ with a suitable n, hence the closure $\bar{G}$ of G in $GL(n, \bar{L}_G)$ acts on $\bar{R}$ (cf. [3]). $H=\{g\mid f^g\bar{L}_G=f\bar{L}_G\}$ is a closed subgroup of $\bar{G}$ (cf. [3]). Since H contains G, we see that $H=\bar{G}$. Thus $f\bar{L}_G$ is a representation module of $\bar{G}$.

The simple group $G=SL(3, K)$ is acting of an affine space V, and V contains G-admissible non-empty open subset U' of U which satisfies the following two conditions. (1) If P is a generic point of U' and if Q is a point of U', then $\{P^g \mid g \in G\}$ is uniquely specialized in U' to $\{Q^g \mid g \in G\}$ over the specialization $P \to Q$, namely the set of (Q, Q^g) $(Q \in U', g \in G)$ is closed in $U' \times U'$. (2) U' contains a closed set F which is the union of infinitely many (mutually distinct) G-orbits such that if a rational function f on V is G-invariant and is regular at one point of F, then f is regular at every point of F and the value of f on F is constant.

EXAMPLE 4. 3. Consider the space V of homogeneous forms of degree 5 in three variables x, y, z. Then V is an affine space of dimension 21. An element of $G=GL(3, K)$ gives a linear transformation of the variables x, y, z, and therefore it gives a linear transformation of the space V.

Let F_1 be the smallest G-admissible set in V containing all of the forms of the type $f_5(x, y)+zf_4(x, y)+az^2x^3$. Here, $f_n(x, y)$ denotes an arbitrary homogeneous form of degree n in x and y. Let F_2 be the smallest G-admissible set in V containing all of the forms of the type $f_5(x, y)+zxf_3(x, y)+z^2x^2f_1(x, y)$. Let F_3 be the set of all forms which have linear factors. Then:

The complement U' of $F_2 \cup F_3$ is the required example, with $F=F_1 \cap U'$.

Let P be a generic point of V and let Q be a point of U. Let g be a generic point of $G=SL(3, K)$. Assume that $(P, P^g) \to (Q, Q')$ is a specialization. Then the specialization is obtained as the specialization given by a zero-dimensional valuation v of the function field $K(P, g)$. From now on for a while, we mean specialization only the one given by v. Assume that $g_1, g_2 \in SL(3, K(g))$ are specialized to non-singular matrices g_1^*, g_2^*, then P^{gg_2}, P^{g_1} are specialized to $Q'^{g_2^*}$ and $Q^{g_1^*}$ respectively. There are such g_1, g_2 with additional condition that $g_1^{-1}gg_2$ is a diagonal matrix. Therefore, considering $P^{g_1}, Q^{g_1^*}$ instead of P, Q, we assume that g is a diagonal matrix: $g=\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}$. Since $g \in SL(3, K(g))$, we have $t_1t_2t_3=1$. We set $u_i=v(t_i)$, whence $u_1+u_2+u_3=0$. If all the u_i are zero, then g is specialized to a non-singular matrix, and therefore Q' is in the orbit of Q. We consider the other case. We may assume that $u_1 \geq u_2 \geq u_3$. Let $P=\sum_{i+j+k=5} a_{ijk}x^iy^jz^k$ and $Q=\sum_{i+j+k=5} b_{ijk}x^iy^jz^k$. Then $P^g=\sum a_{ijk}t_{ijk}x^iy^jz^k$ with $t_{ijk}=t_1^it_2^jt_3^k$ hence $v(t_{ijk})=u_1i+u_2j+u_3k$. Therefore that P^g has a finite specialization implies that if $b_{ijk} \neq 0$, then $v(t_{ijk}) \geq 0$. Set $Q'=\sum c_{ijk}x^iy^jz^k$. Then we see furthermore (i) if $v(t_{ijk})>0$ or if both

b_{ijk} and $v(t_{ijk})$ are zero, then $c_{ijk}=0$ and (ii) if $v(t_{ijk})<0$ (hence $b_{ijk}=0$), then by choice of the manner of approaching zero of a_{ijk}, c_{ijk} can be arbitrarily given. Now we observe the situation in more detail.

(1) If $u_2=0$, then $u_3=-u_1$, hence we see immediately that both Q and Q' must be in F_2. (2) Assume now that $u_2>0$. Then:

(i) If $k=0$, then $v(t_{ijk})>0$. (ii) If $i\geq 1$, $k=1$, then $v(t_{ijk})>0$. (iii) For $i=0$, $j=4$, $k=1$, the value $v(t_{ijk})$ is non-negative if and only if $3u_2\geq u_1$. (iv) For $i=3$, $j=0$, $k=2$, the value $v(t_{ijk})$ is non-negative if and only if $u_1\geq 2u_2$. (v) For the other (i, j, k), the value $v(t_{ijk})$ is negative. Therefore Q must be in F_1 and Q' must be in F_3. (3) the case $u_2<0$ is the same as above (2) with opposite sign, and we see that Q must be in F_3 and Q' must be in F_1.

Thus we have proved that if Q, Q' are in U', $(P, P^g)\rightarrow(Q, Q')$ being a specialization, then Q' must be in the orbit of Q.

Assume now that $Q=f_3(x, y)+zf_4(x, y)+az^2x^3$. Then by the same specialization as above in the case where $u_1\geq u_2>0>u_3$ and $3u_2>u_1>2u_2$, we see that $(Q, Q^g)\rightarrow(Q, 0)$ is a specialization. Thus, if Q is in F_1, then the closure of the orbit of Q contains the origin, hence in particular, if Q_1 and Q_2 are in F_1, then the closures of orbits of Q_1 and Q_2 meet. Therefore, if f is a G-invariant rational function on V which is regular at one point of F_1, then f is regular at all points of F_1 and the value of f on F_1 is constant.

We shall show that $F=F_1\cap U'=F_1-(F_1\cap(F_2\cup F_3))$ carries infinitely many orbits. For each element of V, there corresponds uniquely a plane curve of degree 5. If Q is in either F_1 or F_2, then the curve defined by Q has a triple point, hence it has no more triple point unless it has a line as a component, i.e., $Q\in F_3$. Therefore we see that $Q=f_5(x, y)+zf_4(x, y)+az^2x^3$ $(\in F_1)$ is not in $F_2\cup F_3$ unless it satisfies one of the following three conditions: (a) $a=0$, (b) $f_4(x, y)$ is divisible by x, (c) Q has a linear factor. Thus we see that F contains a non-empty open subset of F_1. The dimension of the set of forms of type $f_5(x, y)+zf_4(x, y)+az^2x^3$ is 12, hence $\dim F_1\geq 13$. Since $\dim G=8$, each orbit in U has dimension 8, whence there are infinitely many orbits in F. This completes the proof of our example.

REMARK 4.4. In the above V, if we take a G-admissible open subset U'' of U such that $U''\cap F_2$ is not empty, then the set of (Q, Q^g) $(Q\in U'', g\in G)$ is not closed in $U''\times U''$, as is easily seen.

We now want to apply 4.2 to the case of projective variety.

Theorem 4.5. *Assume that V is a projective variety such that its homogeneous coordinate ring $\bar{R}$ is a unique factorization domain. If G is* *a semi-simple group acting on V such that whose operation can be lifted to the operation on the representative cone $\bar{V}$ of V (not necessarily uniquely) and if K is of characteristic zero, then the set of P_G is the set of localities of points of a quasi-projective variety.*

Proof. The operation of G on V induces, by our assumption, an operation of a group $\bar{G}$ on $\bar{V}$ so that $\bar{G}$ contains the multiplicative group G_0 of K in its center such that $a \in G_0$ transforms points $(a_0, \cdots, a_n)$ of $\bar{V}$ to $(aa_0, \cdots, aa_n)$ and $\bar{G}/G_0$ is isomorphic to G. Then $\bar{G}$ contains a semi-simple group G_1 such that $G_0G_1 = \bar{G}$. The structure shows that L_G is the field of $\bar{G}$-invariants in the function field $\bar{L}$ of $\bar{V}$. Let L^* be the field of G_1-invariants in $\bar{L}$. Then the set of $\bar{P}_{G_1} = \bar{P} \cap L^*$ $(\bar{P} \in \bar{V})$ is the set of localities of points of the affine variety defined by the ring $\bar{R}_{G_1}$ of G_1-invariants in $\bar{R}$. The operation of G_0 on $\bar{R}_{G_1}$ satisfies the condition in Theorem 2.2, and we complete the proof.

Remark 4.6. Theorem 4.1 shows that under the assumption there, the set of points Q of V which have the same Q_G contains generically only one orbit of maximal dimension. But this does not imply that orbits are generically closed as is easily seen by some examples.

5. Normal varieties.

The purpose of this section is to prove the following

Theorem 5.1. *Even if G is a simple group, K is of characteristic zero and V is normal, then ring P_G $(P \in V)$ is not necessarily a Noetherian ring.*

In order to prove this, we use the following two lemmas, whose proofs are found in our lecture note [3].

Lemma 5.2. *Let W be a subvariety of an affine variety V and let H be a subgroup of G. Assume that G is connected, H operates on W and that each H-orbit on W is the intersection of a G-orbit on V with W. Then the ring R' of H-invariant regular rational functions on W is the homomorphic image of a ring R_G'' consisting of G-invariant rational functions on the closure W'' of the union W^G of G-orbits of points of W such that they have no pole at any point of W^G.* (Lemma of Seshadri).

Lemma 5.3. *Assume that K is of characteristic zero. If a representation ρ of the additive group G_a of K in $GL(n, K)$ is given, then there is a representation $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ of $SL(2, K)$ such that* (i) *$\mu(g) = g$ for any $g \in SL(2, K)$ and* (ii) *$\lambda\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \rho(t)$.*

Now we shall prove Theorem 5.1, by showing an example. By virtue of our counter example to the 14-th problem of Hilbert (see [3]) and also by Theorem 4.1, we see that there is an affine ring R_1 over K which is a unique factorization domain whose invertible elements are only elements of K, G_a operates on R_1 and the ring R_{1G_a} of G_a-invariants in R_1 has a maximal ideal $\mathfrak{m}'$ such that $(R_{1G_a})_{\mathfrak{m}'}$ is not Noetherian. Let ρ be a representation of G_a in $GL(n, K)$ with give the operation of G_a on R_1 (cf. [3]). Now consider the group $G=\left\{\begin{pmatrix}\lambda(g) & 0\\ 0 & \mu(g)\end{pmatrix}\middle| g\in SL(2, K)\right\}$ given by Lemma 5.3. Let $\mathfrak{a}'$ be the ideal such that $R_1=K[x_1, \cdots, x_n]/\mathfrak{a}'$ and let W be the subvariety of the affine $(n+2)$-space V defined by $\mathfrak{a}=\mathfrak{a}'R+(x_{n+1}-1)R+x_{n+2}R$ $(R=K[x_1, \cdots, x_{n+2}])$. Then R_1 can be identified with $R/\mathfrak{a}$. Since $x_{n+1}=1$ and $x_{n+2}=0$ on W, (i) W is G_a admissible and (ii) no element of G outside of G_a transforms any point of W to any point of W. Therefore the condition in Lemma 5.2 is satisfied by our case with $H=G_a$. The same can be applied to the derived normal variety W^* of W'', because W is a normal variety and a generic point of W'' is a generic transform of a generic point of W. So, we may assume that W'' is normal. Thus R_{1G_a} is the homomorphic image of the ring R''_G of rational functions on W'' which are regular on W^G. Consider now the maximal ideal $\mathfrak{m}'$ of R_{1G_a} and let P' be a point of W such that $\mathfrak{m}_{P'}\cap R_{1G_a}=\mathfrak{m}'$. Let P be the point P' as a point on W''. By our choice of R_1, $(R_{1G_a})_{\mathfrak{m}'}$ is the set of G_a-invariant rational functions on W which are regular at P' by virtue of Theorem 4.1. Therefore the homomorphic image of P_G in the function field of W is contained in $(R_{1G_a})_{\mathfrak{m}'}$. The converse inclusion follows immediately from the above consequence of Lemma 5.2. Thus P_G has a homomorphic image which is not Noetherian, hence P_G itself is not Noetherian.

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