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On the Set of Non Normal Points of an Analytic Set

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Introduction. In the present paper we shall consider the set of non normal points in an analytic set and discuss under which condition an analytic set is normal[1][2].

First of all let us recall definitions ([8], p. 260) which are fundamental for our arguments. Let M be an analytic set in a domain D of the space of n complex variables $C^n(z_1, \cdots, z_n)$, i. e., the set which is locally expressible as common zeros of a finite number of holomorphic functions. A function f on M is called *holomorphic*, when the following conditions are satisfied: (1) f is uniquely defined at every regular point of M, (2) for every regular point x of M, f coincides in a neighborhood of x with some holomorphic function in the ambiant space, and (3) for every point x of M, f is bounded in a neighborhood of x. A function is called *holomorphic at a point* x of M when it is holomorphic in a neighborhood on M of x. For a holomorphic function f on M we shall denote by $S_N(f)$ the set of those points of M in any neighborhood on M of which f is not the restriction of a holomorphic function in the ambiant space. By S_N we shall mean the set of those points of M at each of which some holomorphic function in the intersection of M with a neighborhood of this point can not be expressed as restriction of a holomorphic function in the ambiant space. At a point of M not belonging to S_N, M is called *normal* ([3] Exposé XIV, this is called "*la propriété* (H)" in [8]). Similarly for a holomorphic function f on M, we call M *normal with respect to* f at a point of M not belonging to $S_N(f)$ ("*la propriété* (H)" of f in [8]).

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[1] The author was inspired to study this subject, when he attended Prof. K. Oka's seminar at Kyoto University.

[2] After having prepared this paper, the following two papers appeared quite recently:
W. Thimm: Über Moduln und Ideale von holomorphen Funktionen mehrerer Variablen, Math. Ann., 139 (1959).
W. Thimm: Untersuchungen über das Spurproblem von holomorphen Funktionen auf analytischen Mengen, ibid,
in which the problem treated in this paper and related ones are thoroughly studied; Theorem 3 of the present paper is included as a special case in Satz 9 of the second paper. But it seems to the author of the present paper that his approach to this theorem is different from Thimm's.

In the theory of functions of several complex variables one often encounters the problem to extend functions given on an analytic set to the ambiant space, which was first treated by H. Cartan [1]. As to this, fundamental theorems of a holomorpically convex domain ([9] Théorème V and VII) or of a Stein manifold ([3] Théorème A and B) say: A holomorphic function f on an analytic set M of a holomorphically convex domain (or a Stein manifold) X is extendable to the whole space X, if and only if the set $S_N(f)$ is empty — precisely we should say that there exists a continuation $\tilde{f}$ of f such that $S_N(\tilde{f})$ is empty, but for the brevity we write in this way throughout this paper. From this point of view it is important to see under which condition the set $S_N(f)$ is empty and generally to decide the structure of $S_N(f)$ and S_N.

When an analytic set M is of 1 codimension, and if codim. $S_N(f) \geqq 3$, then $S_N(f)$ is empty ([8] Lemme 1, p. 261). About the set S_N some facts are known ([4] Fxposé X and XI, and [6] §7).

After explanation in §1 about the notations and conventions used in this paper, in §2 we shall show the analyticity of the set $S_N(f)$ from which that of the set S_N ([4] Théorème 3 bis of Exposé X) is directly derived on the basis of "Lemme fondamental" ([8], p. 275); next we shall discuss about the structure of the set S_N. In §3 for an analytic set M of 2 codimensions we give two theorems, Theorem 3 and Theorem 3 bis, as a generalization of Lemme 1 mentioned above. For lower dimensional cases the corresponding results are also expected, though we have not obtained them here[3].

§1. Preliminaries.

Throughout this paper we denote by M an analytic set of a domain D in the space of n complex variables C^n. Since about the definition of a holomorphic function on an analytic set and about that of the sets S_N and $S_N(f)$ we wrote in Introduction, we do not repeat them here.

When we indicate for an analytic set M its dimension or its codimension ($=n$-dim. M), we always mean that M is of pure dimension, i. e., all components of M are of the same dimension.

When we speak of a neighborhood of a point of an analytic set, the neighborhood is meant by an open set in the ambiant space.

Let $\mathcal{J}$ be an analytic ideal on a domain D; a system of a finite number of holomorphic functions $(F_1, \cdots, F_m)$ at a point $x \in D$ is called a *pseudo-base* of $\mathcal{J}$ at x, if at any point y of a neighborhood of x $F_1, \cdots, F_m$ generate the stalk $\mathcal{J}_y$ of $\mathcal{J}$ ([7], p. 6). An analytic ideal is coherent, if and only if it has a pseudo-base at every point of D by Théorème 4 of [8].

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[3] These are affirmatively resolved by Satz 9 of the second paper mentioned in 2).

To an analytic set M in a domain D corresponds an analytic ideal $\mathcal{J}(M)$ on D such that the stalk $\mathcal{J}(M)_x$, $x \in D$, consists of function-germs which are represented by holomorphic functions in a neighboshood U of x vanishing on $U \cap M$ identically. The analytic ideal $\mathcal{J}(M)$ is coherent ([8] Théorème de H. Cartan); hence, $\mathcal{J}(M)$ has a pseudo-base at every point of D, and in particular when D is a relatively compact subdomain of a holomorphically convex domain (or a Stein manifold) X, there are a finite number of holomorphic functions in X which constitute a pseudo-base of $\mathcal{J}(M)$ at every point of D ([9] Théorème V).

Let $\mathcal{O}$ be the sheaf of holomorphic functions in a domain D; the stalk $\mathcal{O}_x$, $x \in D$, consists of all function-germs which are represented by holomorphic functions at x. For an analytic set M we can consider the quotient sheaf $\mathcal{O}/\mathcal{J}(M)$, whose restriction to M is an analytic coherent sheaf on M. This sheaf is denoted by $\mathcal{A}(M)$; an element of the stalk $\mathcal{A}(M)_x$, $x \in M$, is a germ represented by the restriction to M of a holomorphic function at x in the ambiant space; that is, $\mathcal{A}(M)$ is the sheaf of holomorphic functions on M which are induced on M by holomorphic functions in the ambiant space.

Let M be an analytic set of k dimensions in a domain D of C^n. When we limit our consideration within a neighborhood of a point on M, after a linear transformation, we can choose the coordinates $z_1, \cdots, z_n$ such that M spreads (*"ausgebreiten"*; see, for example, [5] p. 255) over a domain of the space $C^k(z_1, \cdots, z_k)$; further, we can consider the complex space M^* (*"domaine multiple"* in [8]) corresponding to M which spreads over the same domain of $C^k(z_1, \cdots, z_k)$. In these cases we shall say simply that M and M^* spread over $C^k(z_1, \cdots, z_k)$.

Since the problems treated in this paper are essentially local, it is sufficient to prove propositions only in a neighborhood of an arbitrary point of an analytic set M of a domain D, and accordingly we may assume that the domain D is a small neighborhood of a point. In the course of proofs we often denote the intersection $U \cap M$ of a neighborhood U of a point with M simply by M, when there is no fear of confusion.

§2. **Theorem 1.** *Let M be an analytic set of a domain D in $C^n(z_1, \cdots, z_n)$. Then, for a holomorphic function f on M, the set $S_N(f)$ of M is an analytic set of D*[4].

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[4] Mr. E. Ohnishi showed: Let f be a holomorphic function on a 1 codimensional analytic set M, and let σ be a 2 codimensional analytic subset of M. On every component of σ if there is at least one regular point of σ which does not belong to $S_N(f)$, all regular point sof σ also do not belong to $S_N(f)$. From this fact, by using Lemme 1 of [8], we get Theorem 1 for the codimensional case.

Proof. Suppose, in a neighborhood U of $x \in M$, M be common zeros of

$$(1)\qquad \begin{cases} \Phi_1(z_1, \cdots, z_n) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Phi_l(z_1, \cdots, z_n) = 0, \end{cases}$$

where $\Phi_i (i=1, \cdots, l)$ are holomorphic functions in U. By assumption we can set $|f(y)| < m$ $(m > 0)$ for regular points y of M. As is well known, for the function f there exists a unitary polynomial $P(z_1, \cdots, z_n, w)$ in w, whose coefficients are holomorpoic functions in U, such that $P(y, f(y)) = 0$ for all regular points y of M. Consider the analytic set $\tilde{M}$ determined by

$$(2)\qquad \begin{cases} \Phi_1(z_1, \cdots, z_n) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Phi_l(z_1, \cdots, z_n) = 0, \\ P(z_1, \cdots, z_n, w) = 0 \end{cases}$$

in the set $\tilde{U} = \{(z_1, \cdots, z_n, w) \mid (z_1, \cdots, z_n) \in U, |w| < m\}$ of the space $C^{n+1}(z_1, \cdots, z_n, w)$; moreover, on M and $\tilde{M}$ consider the sheaves $\mathcal{A}(M)$ and $\mathcal{A}(\tilde{M})$. It is easily seen that the analytic mapping $\pi: \tilde{M} \rightarrow M$ determined by the projection $(z_1, \cdots, z_n, w) \rightarrow (z_1, \cdots, z_n)$ is non degenerate and proper. Hence, the π-image of $\mathcal{A}(\tilde{M})$, denoted by $\mathcal{A}^*$, is a coherent analytic sheaf on M ([8] Théorème 1, and [6] Satz 27,). Then the sheaf $\mathcal{A}(M)$ is a subsheaf of $\mathcal{A}^*$ and the quotient shaef $\mathcal{A}^*/\mathcal{A}(M)$ is also coherent. Now, a point y of M is included in $S_N(f)$, if and only if the stalk $(\mathcal{A}^*/\mathcal{A}(M))_y$ of $\mathcal{A}^*/\mathcal{A}(M)$ does not vanish. In order to see this, it is sufficient to observe the structure of $\mathcal{A}^*$. Let (y, w) be a point of M and let $\pi^{-1} \circ \pi[(y, w)]$ consist of $(y, w^{(1)}), \cdots, (y, w^{(m)})$; where $w^{(1)} = w$. An element of $\mathcal{A}(M)_{(y, w^{(i)})}$ is a germ represented by a restriction of $\phi^{(i)}(z, w)$ to M, which is a holomorphic function of (z, w) at $(y, w^{(i)}) (i=1, \cdots, m)$. Then, $\mathcal{A}^*_y$ is the module consisting of all vectors of the form $\{\phi^{(1)}(z, f^{(1)}(z)), \cdots, \phi^{(m)}(z, f^{(m)}(z))\}$ where $f^{(i)}$ is a branch of a root of $P(z, w) = 0$ passing through $(y, w^{(i)})$, and where z runs over M in a neighborhood of y.[5] On the other hand, the set of points $y \in M$ for which $(\mathcal{A}^*/\mathcal{A}(M))_y \neq 0$ is analytic ([4] lemme 1 of Exposé X), which completes our proof.

Corollary ([4] Théorème 3 bis of Exposé X). *Let M be an analytic set of a domain D of $C^n(z_1, \cdots, z_n)$. Then the set S_N of M is an analytic set of D.*

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[5] There is no fear of misunderstanding though we do not distinguish here a germ from its representative.

Proof. Let x be a point of M and U its neighborhood. Then, by "Lemme fondamental" of [8], we can construct a finite number of holomorphic functions in U, $\Psi_1(z), \cdots, \Psi_p(z)$ and $u_0(z)$, in the following way: For every holomorphic function f at any point $y \in M \cap U$ there exists a holomorphic function $F(z)$ in V, a neighborhood of y contained in U, such that F is a linear combination of Ψ_i's, where the coefficients are holomorphic functions in V, and the restriction of F to M is equal to fu_0. Further, the lemma says that the functions $f_i = \Psi_i/u_0 (i=1, \cdots, p)$ are holomorphic functions on M. From both facts, it follows $S_N = S_N(f_1) \cup \cdots \cup S_N(f_p)$. Since $S_N(f_i)$ are the analytic sets by Theorem 1, it follows that S_N is also an analytic set. (q. e. d.)

In the following, by the singularity of an analytic set M we understand the subset of non regular points of M which is also an analytic set.

As is well known ([4] Exposé X and XI, and [6] Satz 21), if an analytic set M of k dimensions is normal, the singularity S of M is of at most $k-2$ dimensions and M is locally irreducible. We remark that the converse is also true. Suppose an analytic set M have these properties, and let $z^0=(z_1^0, \cdots, z_n^0)$ be a point of M and U a neighborhood of z^0. Further consider a complex space M^* correspoding to M spread over the space $C^k(z_1, \cdots, z_k)$, then using $n-k$ (one-valued) holomorphic functions $f_{k+1}, \cdots, f_n$ on M^*, M is representable in U as $(z_1, \cdots, z_k, f_{k+1}, \cdots, f_n)$. Since M is irreducible at every point of a neighborhood of z^0, we can determine one and only one point x^* of M^* over $(z_1^0, \cdots, z_k^0)$ with the following properties: a) There exists no point y^* of M^* over $(z_1^0, \cdots, z_k^0)$ other than x^* which satisfies $f_j(y^*) = f_j(x^*) = z_j^0 (j=k+1, \cdots, n)$, that is, for some j_0, x^* is not a "*point équivoque*" with respect to f_{j_0} ([8], p. 264); for, otherwise, M would be reducible at z^0 and the singularity would pass through z^0. b) For every ramification variety σ^* of M^* through x^* there exists at least one f_{j_0} with respect to which σ^* is of "*première espèce*" ([8], p. 264), that is, when f_{j_0} is developed around σ^* at a point corresponding to a regular point of an analytic subset of M corresponding to σ^*, then the term of the first degree of f_{j_0} does not vanish on σ^* identically[6]; for, otherwise, M would have a singularity of $k-1$ dimensions through z^0. Then, we can choose coefficients $c_{k+1}, \cdots, c_n$ $(c_{k+1} \neq 0)$ such that, when we set $f = c_{k+1}f_{k+1} + \cdots + c_nf_n$, a) x^* is not a "*point équivoque*" with respect to f, and b) every ramification variety through x^* is of "*première espèce*" with respect to f; thus, after transforming M from the space $C^n(z_1, \cdots, z_n)$ into the space $C^n(z_1', \cdots, z_n')$ by

$$(3) \qquad \begin{cases} z_i' = z_i & (i = 1, \cdots, k) \\ z_{k+1}' = c_{k+1}z_{k+1} + \cdots + c_nz_n, & \\ z_j' = z_j & (j = j+1, \cdots, n), \end{cases}$$

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[6] We mean that, when f_{j_0} is developed as $\overline{G}_2$ in (12), $a_1 \neq 0$.

by the projection $(z'_1, \cdots, z'_n) \to (z'_1, \cdots, z'_{k+1})$, we can map M biholomorphically onto an analytic set M' of k dimensions in the $k+1$ dimensional space, which has no $k-1$ dimensional singularity. Now, for a holomorphic function f_0 on M in the neighborhood of z^0, we can determine the image f'_0 on M' which is holomorphic except at points of an at most $k-2$ dimensional analytic set of M'; because M has no singularity of $k-1$ dimensions, and at regular points of M, f_0 is locally expressed as holomorphic function of $(z'_1, \cdots, z'_{k+1})$. Since Lemme 1 of [8] is applicable to M' and f'_0, there exists a holomorphic function of $(z'_1, \cdots, z'_{k+1})$ which induces the function f'_0 on M' and this induces also the function f_0 on M.

From this remark we can conclude:

Theorem 2. *The set S_N of a k dimensional analytic set M of a domain D in C^n $(n > k \geqq 0)$ consists of:*

1) *$k-1$ dimensional components of the singularity of M,*

2) *the closure of the set of all reducible points of M.*

Proof. Let S' be the union of these sets. S' is evidently an analytic set. It is easily seen that, for any regular point x of S', there exists an at least holomorphic function at x on M with respect to which M is not normal at x; hence, the every regular points of S' are contained in S_N. Since S_N is an analytic set by Corollary to Theorem 1, and hence a closed set, so non regular points of S' are also contained in S_N. On the other hand, S_N has no component other than those of S', which was shown by the above remark. Thus, S_N coincides with S'. (q. e. d.)

§3[7]. In this paragraph we consider the 2 codimensional case of Lemme 1 of [8].

Theorem 3. *Let M be a 2 codimensional analytic set in a domain D of C^n and suppose M be expressed as common zeros of holomorphic functions $F(z)$ and $G(z)$ in D. Moreover, we assume (F, G) be a pseudo-base of the ideal $\mathfrak{J}(M)$ at every point of D. Then, if, for a holomorphic function f on M, codim. $S_N(f) \geqq 4$, $S_N(f)$ is empty.*

REMARK 1. The minimum number of generators of an ideal corresponding to a 2 codimensional analytic set is not always 2 even locally[8]: for example, considering in the space (x, y, z) a curve determinad by the equations $x=t^3$, $y=t^4$ and $z=t^5$, we can show that its pseudo-base at the origin consists of at least three functions.

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[7] Results of this paragraph were obtained during the term mentioned in 1).

[8] Though this fact may be well known, we illustrate it here; because it seems that for the analytic case there is no explicit explanation about this matter in bibliography. Of this example the author was informed by Mr. E. Ohnishi.

REMARK 2. Though our proof proceeds on a similar line as in the Lemme 1 often cited hitherto, we note that for our case we use "Théorème du reste" ([7], p. 15).

Proof. Taking an arbitrary point possibly belonging to $S_N(f)$, we may assume it to be the origin (0). Owing to Weierstrass, after a linear transformation of coordinates, we can choose a neighborhood U of (0) in the space $(z_1, \cdots, z_{n-2}, w_1, w_2)$ satisfying the following conditions :

1) $U = Z \times W$, where $Z = \{(z_1, \cdots, z_{n-2}) \mid |z_j| < r \ (j = 1, \cdots, n-2)\}$ and $W = \{(w_1, w_2) \mid |w_k| < r' \ (k = 1, 2)\}$ $(r, r' > 0)$.
2) In U, M spreads over Z.
3) $F(z_1, \cdots, z_{n-2}, w_1, w_2)$ and $G(z_1, \cdots, z_{n-2}, w_1, w_2)$ are unitary polynomials in w_2 whose coefficients are holomorphic functions for $(z_1, \cdots, z_{n-2}) \in Z$ and w_1, $|w_1| < r'$.
4) For any (z', w_1'), $z' \in Z$ and $|w_1'| < r'$, all the roots w_2 of the equation $F(z', w_1', w_2') = 0$ are smaller than r' in modulus, and the same holds for the equation $G(z', w_1', w_2) = 0$.
5) $S_N(f)$ does not meet $V_1 \cup V_2 \cup V_3$, where

$$V_1 = \{(z_1, \cdots, z_{n-2}, w_1, w_2) \mid |z_j| < r \quad (j = 1, \cdots, n-4, n-2), \\ \rho < |z_{n-3}| < r,\ |w_k| < r' \quad (k = 1, 2\}),$$

$$V_2 = \{(z_1, \cdots, z_{n-2}, w_1, w_2) \mid |z_j| < r \quad (j = 1, \cdots, n-4, n-3), \\ \rho < |z_{n-2}| < r,\ |w_k| < r' \quad (k = 1, 2\}) \text{ and}$$

$$V_3 = \{(z_1, \cdots, z_{n-2}, w_1, w_2) \mid |z_j| < r \quad (j = 1, \cdots, n-3, n-2), \\ \rho' < |w_1| < r',\ |w_2| < r'\} \ (0 < \rho < r,\ 0 < \rho' < r').$$

By 2) over each point z of Z there are at most a finite number of points of M. Let λ and λ' be the degrees of F and G respectively as unitary polynomials in w_2. 5) is surely realized because $\operatorname{codim.} S_N(f) \geqq 4$.

Since $V_1 \cap S_N(f)$ is empty, by Théorème 2 of [7] there exists a holomorphic function $F_1(z, w)$ in V_1 which induces f on M; similarly there exist $F_2(z, w)$ and $F_3(z, w)$ in V_2 and V_3 respectively. The functions $F_i - F_j$ belong to $\mathcal{J}(M)$ at every point of $V_i \cap V_j$; hence, there exist holomorphic functions $a_k(z, w)$ and $b_k(z, w)$ in $V_i \cap V_j$ ((i, j, k) is a cyclic interchange of (1, 2, 3,)) such that

$$(4) \qquad \begin{cases} F_1 - F_2 = a_3 F + b_3 G & \text{in } V_1 \cap V_2, \\ F_2 - F_3 = a_1 F + b_1 G & \text{in } V_2 \cap V_3, \\ F_3 - F_1 = a_2 F + b_2 G & \text{in } V_3 \cap V_1 \end{cases}$$

([7] Théorème 1). By 4) we can apply a_k to and F "Théorème du reste" in $V_i \cap V_j$, and we get

$$(5) \qquad a_k = a_k' + a_k'' G ,$$

where a_k' and a_k'' are holomorphic functions in $V_i \cap V_j$, and moreover a_k' are polynomials in w_2 of degree $\lambda' - 1$. In $V_1 \cap V_2 \cap V_3$, summing up both sides of (4), we have, by means of 5),

$$(6) \qquad (a_1' + a_2' + a_3')F = -[(a_1'' + a_2'' + a_3'')F + b_1 + b_2 + b_3]G$$

For any $z' = (z_1', \cdots, z_{n-2}')$ of Z, on account of 2), there are only a finite number of common zeros of $F(z', w_1, w_2) = 0$ and $G(z', w_1, w_2) = 0$ in W; we denote them by $(w_1^{(1)}, w_2^{(2)}), \cdots, (w_1^{(\nu)}, w_2^{(\nu)})$. If w_1' whose modulus is smaller than r is not equal to any $w_1^{(i)} (i = 1, \cdots, \lambda')$ and if the discriminant of G does not vanish at $(z_1', \cdots, z_{n-2}', w_1)$, then $G(z_1', \cdots, z_{n-2}', w_1', w_2) = 0$ has exactly λ' roots smaller than r' in modulus, which we denote by $w_2^{(1)}, \cdots, w_2^{(\lambda')}$. We have $F(z_1', \cdots, z_{n-2}', w_1', w_2^{(i)}) \neq 0$ $(i = 1, \cdots, \lambda')$. Especially, when we take $(z_1', \cdots, z_{n-2}', w_1')$ such that $(z_1', \cdots, z_{n-2}', w_1', w_2^{(i)})$ for all i are contained in $V_1 \cap V_2 \cap V_3$, and substitute them into (6), its right hand side has λ' roots as a polynomial in w_2, while its left hand side has at most λ'-1 zeros. Hence, $a_1'(z_1', \cdots, z_{n-2}', w_1', w_2) + a_2'(z_1', \cdots, z_{n-2}', w_1', w_2) + a_3'(z_1', \cdots, z_{n-2}', w_1', w_2) = 0$ for all w_2 whose modulus is smaller than r'. Since $(z_1', \cdots, z_{n-2}', w_1')$ is arbitrarily taken, provided keeping away from an at least 2 codimensional analytic set, this means

$$a_1'(z_1, \cdots, z_{n-2}, w_1, w_2) + a_1'(z_1, \cdots, z_{n-2}, w_1, w_2) + a_3'(z_1, \cdots, z_{n-2}, w_1, w_2) = 0$$

in $V_1 \cap V_2 \cap V_3$ identically. Hence, in $V_1 \cap V_2 \cap V_3$

$$(7) \qquad (a_1'' + a_2'' + a_3'')F) + (b_1 + b_2 + b_3) = 0$$

identically. Then, by applying to b_k and F "Théorème du reste" in $V_i \cap V_j$ as we did to a_k,

$$(8) \qquad b_k = b_k' + b_k'' F \qquad (k = 1, 2, 3)$$

where b_k and b_k'' are holomorphic functions in $V_i \cap V_j$ and b_k' are polynomials in w_2 of degree $\lambda - 1$. From (7) and (8), we have in $V_1 \cap V_2 \cap V_3$

$$(9) \qquad b_1' + b_2' + b_3' = -(a_1'' + a_2'' + a_3'' + b_1'' + b_2'' + b_3'')F$$

By a similar consideration as above, we obtain $b_1' + b_2' + b_3' = 0$ in $V_1 \cap V_2 \cap V_3$.

Thus, we have the following three identities in $V_1 \cap V_2 \cap V_3$

$$
(10) \qquad \begin{cases} a_1' + a_2' + a_3' = 0, \\ b_1' + b_2' + b_3' = 0, \\ a_1'' + b_1'' + a_2'' + b_2'' + a_3'' + b_3'' = 0\,; \end{cases}
$$

and we get from (4)

$$
(4') \qquad \begin{cases} F_1 - F_2 = a_3'F + b_3'G + (a_3'' + b_3'')FG & \text{in } V_1 \cap V_2, \\ F_2 - F_3 = a_1'F + b_1'G + (a_1'' + {}_1''b)FG & \text{in } V_2 \cap V_3, \\ F_3 - F_1 = a_2'F + b_2'G + (a_2'' + b_2'')FG & \text{in } V_3 \cap V_1. \end{cases}
$$

Now we have reached the situation where we can apply Cartan's lemma [2] as in Lemme 1 of [8]; that is, by applying Cartan's lemma to each system of (a_1', a_2', a_3'), (b_1', b_2', b_3') and $(a_1'' + b_1'', a_2'' + b_2'', a_3'' + b_3'')$, and by modifying $F_k (k = 1, 2, 3)$ and finally by using Hartogs theorem of analytic continuation, we obtain a holomorphic function in U which induces f on M. We omit details, because these are only repetitions of the last part of the proof of Lemme 1 in [8]. (q. e. d.)

Next we shall examine under which condition the assumptions of Theorem 3 are fulfilled. For this purpose we define the *intersection number* of two 1 codimensional analytic sets. This is already done in p. 314 of [6], but for the present use we do it in the following way. Let $\sum_1$ and $\sum_2$ be 1 codimensional analytic sets in a domain D whose ideals are generated by F_1 and F_2 respectively, and set

$$
(11) \qquad \begin{cases} G_1 = F_1 + \alpha F_2, \\ G_2 = F_1 + \beta F_2, \end{cases}
$$

where α, $\beta (\alpha \neq \beta)$ are complex parameters. We consider the analytic set $\sum$ determind by $G_1 = 0$ and also the complex space $\sum^*$ corresponding to $\sum$. We can assume that $\sum^*$ is spread over the space $(z_1, \cdots, z_{n-1})$, because, after a linear transformation of coordinates, this is surely possible by Satz 9 of [5]. Let us take a component σ of $\sum_1 \cap \sum_2$, and let us determine the intersection number of $\sum_1 \cap \sum_2$ on σ. To σ corresponds the set σ^* on $\sum^*$ which is a component of the surface given by $\bar{G}_2 = 0$, where $\bar{G}_2$ is a restriction of G_2 to $\sum$. In these circumstances, except a special pair (α, β), the order of zeros of $\bar{G}_2$ on σ^* ([8], p. 269 and p. 270) is a uniquely determined value, which we call the intersection number of $\sum_1 \cap \sum_2$ on σ. Here we must make clear what we mean by "special pair (α, B)". Let x_0 be a regular point of σ^*, and let its coordinates be (0) for convenience, and further, let σ^* be given by $z_{n-1} = 0$ in a neighborhood of (0), where $\bar{G}_2$ is developed in t $(t = z_{n-1}^{1/\nu}\,;$ ν is the index of ramification of $\sigma^*)$ as follows:

$$\bar{G}_2 = a_k t^k + a_{k+1} t^{k+1} + \cdots \qquad (a_k \not\equiv 0,\ k \geqq 0) \tag{12}$$

where the coefficients $a_j = a_j(z_1, \cdots, z_{n-2}, \alpha, \beta)$ $(j = k, k+1, \cdots)$ are holomorphic in (z) in a neighborhood of (0) and rational in (α, β). The pairs (α, β) satisfying $a_k \equiv 0$ identically for $(z_1, \cdots, z_{n-2})$ form an analytic set in the space of pairs (α, β) because they are common zeros of infinitely many equations in (α, β). These pairs (α, β) are what we mean by special pairs (α, β). Thus, we can determine the intersection number of $\sum_1 \cap \sum_2$ for every component of $\sum_1 \cap \sum_2$ unless the pair (α, β) belongs to an exceptional set of first category in the space of pairs (α, β). This definition is obviously symmetric for F_1 and F_2, and morever it is independent of the particular choice of generators F_1, and F_2, because, even if we rewrite (11) as

$$\begin{cases} G_1 = F_1 + \alpha\omega F_2, \\ G_2 = F_1 + \beta\omega' F_2, \end{cases} \tag{11'}$$

using everywhere non zero holomorphic functions ω, ω' in D, we obtain the same values, which is shown by a direct calculation of a_j's in (12).

Lemma. *If an analytic set M of 2 codimension in a domain D of C^n is the intersection of two 1 codimensional analytic sets $\sum_1$ and $\sum_2$ whose intersection number is 1 on every component, then M is not contained in the singularity of $\sum_1$ and that of $\sum_2$.*

Proof. Let G_1 and G_2 be generators of the ideals corresponding to $\sum_1$ and $\sum_2$ respectively. If $\partial G_1/\partial z_j$ and $\partial G_2/\partial z_j$ $(j = 1, \cdots, n)$ are all identically zero on M, then it is easily verified that the coefficient a_1 in (12) is also identically zero on M for all (α, β) This contradicts the assumption.

Theorem 3 bis. *Let M be a 2 codimensional analytic set in a domain D of $C^n(z_1, \cdots, z_n)$, and suppose M be the intersection of two 1 codimensional analytic sets whose intersection number is 1. Then, if codim. $S_N(f) \geqq 4$ for a holomorphic function f on M, $S_N(f)$ is empty.*

Proof. By assumption M is expressed as common zeros of $F(z_1, \cdots, z_n) = 0$ and $G(z_1, \cdots, z_n) = 0$, where F and G are holomorphic functions in D and the intersection number of two 1 codimensional analytic sets defined by $F = 0$ and $G = 0$ respectively is 1. It will be sufficient to show that at every point of D (F, G) is a pseudo-base of an ideal $\mathcal{J}(M)$; that is, if U is a neighborhood of a point z^0 of D, and if $\Phi(z)$ is holomorphic function in U and vanishes on $U \cap M$, then $\Phi(z)$ is a linear combination of F and G whose coefficients are holomorphic functions in U. For simplicity let z^0 be the origin (0). By the above Lemma the intersection of M with the singularity of the analytic set defined by $F=0$ is an analytic set of at least 3 codimensions and besides, non regular points of the analytic set determined by $\bar{G}=0$ on the analytic set defined by $F=0$ form also an analytic set of at least 3 codimensions. We denote by M_0 the union of these special sets. M_0 is of at least 3 codimensions. It is obvious that at each point of $U-M_0$ our assertion is already satisfied. Here we suppose that $U=\{(z_1,\cdots,z_n)\mid |z_j|<r\ (j=1,\cdots,n)\}$ $(r>0)$, and that F and G are unitary polynomials in z_n; and we set $V_1=\{(z_1,\cdots,z_n)\mid |z_j|<r$ $(j=1,\cdots,n-3,n-1,n)$, $r'<|z_{n-2}|<r\}$ and $V_2=\{(z_1,\cdots,z_n)\mid |z_j|<r$ $(j=1,\cdots,n-2,n)$, $r'<|z_{n-1}|<r\}$ $(0<r'<r)$ such that $M_0\cap(V_1\cup V_2)$ is empty, These are possible by applying a linear transformation of coordinates and by substituting a smaller neighborhood of (0) for U, if necessary. By Théorème 1 of [7] there exist holomorphic functions $a_1(z)$, $b_1(z)$ in V_1 and $a_2(z)$, $b_2(z)$ in V_2 such that

$$\begin{cases} \Phi = a_1F+b_1G & \text{in } V_1, \\ \Phi = a_2F+b_2G & \text{in } V_2. \end{cases} \tag{13}$$

By using "Théorème du reste" as in the proof of Theorem 2, we get

$$\begin{cases} a_1 = a_1'+a_1''G \\ b_1 = b_1'+b_1''G \end{cases} \text{in } V_1 \text{ and } \begin{cases} a_2 = a_2'+a_2''F \\ b_2 = b_2'+b_2''F \end{cases} \text{in } V_2, \tag{14}$$

where a_1', b_1' are polynomials of degree $\lambda'-1$ in z_n and a_2', b_2' are similarly of degree $\lambda-1$ (λ, λ' are the degrees of F and G respectively). Thus, (13) becomes

$$\begin{cases} \Phi = a_1'F+b_1'G+(a_1''+b_1'')FG & \text{in } V_1 \\ \Phi = a_2'F+b_2'G+(a_2''+b_2'')FG & \text{in } V_2, \end{cases} \tag{13'}$$

and in $V_1\cap V_2$

$$0=(a_1'-a_2')F+(b_1'-b_2')G+[(a_1''+b_1'')-(a_2''+b_2'')]FG$$

Then, by a similar consideration as in the proof of Theorem 3, we obtain in $V_1\cap V_2$ $a_1'=a_2'$, $b_1'=b_2'$ and $a_1''+b_2''=a_1''+b_2''$; therefore there exist holomorphic functions $A(z)$, $B(z)$ and $C(z)$ in $V_1\cap V_2$ such that

$$A(z) = a_i'(z),\ B(z) = b_i'(z) \quad \text{and} \quad C(z) = a_i''(z)+b_i''(z)$$

in V_i $(i=1, 2)$. Thus,

$$\Phi = AF+BG+CFG \quad \text{in } V_1\cup V_2. \tag{15}$$

Since the holomorphic envelope of $V_1\cup V_2$ coincides with U, the holomorphic functions A, B and C in $V_1\cap V_2$ are holomorphically continued onto

U and (15) holds also there. This completes the proof.

Corollary. *Let M be a 2 codimensional analytic set in a domain D of C^n which satisfies the conditions of Theorem 3 or Theorem 3 bis. Then all reducible points of M are contained in the 3 codimensional singularity of M.*

Proof. Let x be a reducible point of M through which the intersection of several components of M passes, and assume the codimension of the intersection be larger than 3. Then, there exists a holomorphic function f at x whose $S_N(f)$ is not empty and is of larger codimension than 3. This contradicts Theorem 3 or Theorem 3 bis. (q. e. d.)

For lower dimensions we do not know if the analog to Theorem 3 holds or not[9], but, for example, when the analytic set M of $n-k$ codimensions of a domain D in $C^n(z_1, \cdots, z_n)$ $(n>k\geqq 0)$ is given by

$$
\begin{cases}
F_1(z_1, \cdots, z_k, z_{k+1}, \cdots, z_n) = 0, \\
F_2(z_1, \cdots, z_k, z_{k+1}, \cdots, z_{n-1}) = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
F_{n-k-1}(z_1, \cdots, z_k, z_{k+1}, z_{k+2}) = 0, \\
F_{n-k}(z_1, \cdots, z_k, z_{k+1}) = 0,
\end{cases}
\tag{16}
$$

where $F_i(z_1, \cdots, z_k, z_{k+1}, \cdots, z_{n-i+1})$ $(i=1, \cdots, n-k)$ are unitary polynomials in z_{n-i+1} whose coefficients are holomorphic functions in D, and moveover, if $(F_1, \cdots, F_{n-k})$ form a pseudo-base of the ideal $\mathcal{J}(M)$ at every point of D, then, a for holomorphic function f on M the codimension of whose $S_N(f)$ is larger than 3, $S_N(f)$ is empty. This is shown by repeting the procedure used in the proof of Theorem 3.

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