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The Thickening of Combinatorial n-Manifolds in $(n+1)$-Space

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1. Introduction

The Schönflies conjecture for dimension n is the following statement: *Let a combinatorial $(n-1)$-sphere S^{n-1} be piecewise linearly imbedded in Euclidean n-space R^n. Then the closure of the bounded component of R^n-S^{n-1} is a combinatorial n-cell.* For $n\leqq 3$ this has been affirmatively proved, see Alexander [1], Graeub [2] and Moise [5].

The purpose of this paper is to prove the following (Theorem 3 in section 6): *Let a combinatorial, closed (=compact and without boundary), orientable n-manifold M^n be imbedded as a subcomplex of a combinatorial, orientable $(n+1)$-manifold W^{n+1} without boundary. Let $U(M^n, W^{n+1})$ be a regular neighborhood of M^n in W^{n+1}. Suppose that the Schönflies conjecture is true for dimension $\leqq n$. Then there is a piecewise linear homeomorphism into θ: $M^n\times J\rightarrow W^{n+1}$ such that $\theta(x, 0)=x$ for all $x\in M^n$ and such that $\theta(M^n\times J)=U(M^n, W^{n+1})$, where J is the interval $-1\leqq s\leqq 1$.* (The regular neighborhood $U(M^n, W^{n+1})$ in this paper is necessarily a closed neighborhood of M^n in W^{n+1} in the sense of the set-theory, see Definition 1 in section 3. The simplicial subdivision of M^n gives, in the usual way [3], p. 35, a simplicial subdivison of $M^n\times J$; and the mapping θ is to be piecewise linear relative to such an induced simplicial subdivision of $M^n\times J$.)

In fact, the above theorem is a consequent of the following main theorem (Theorem 2 in section 5): *Let a combinatorial, closed n-manifold M_i^n be imbedded as a subcomplex of a combinatorial, oriented (=orientable, oriented) $(n+1)$-manifold W_i^{n+1} without boundary, $i=1, 2$. Let $U(M_i^n, W_i^{n+1})$ be a regular neighborhood of M_i^n in W_i^{n+1}, and ϕ: $M_1^n\rightarrow M_2^n$ be a piecewise linear homeomorphism onto. Suppose that the Schönflies conjecture is true for dimension $\leqq n$. Then there is a piecewise linear homeomorphism onto ψ: $U(M_1^n, W_1^{n+1})\rightarrow U(M_2^n, W_2^{n+1})$ such that $\psi|M_1^n=\phi$, and such that the oriented image of oriented $U(M_1^n, W_1^{n+1})$ is the oriented $U(M_2^n, W_2^{n+1})$, where the orientation of $U(M_i^n, W_i^{n+1})$ is induced by that of W_i^{n+1}.* Another application of Theorem 2 is Theorem 4 in section 6.

In the proofs of these theorems, we shall make extensive use of combinatorial methods and results of J. H. C. Whitehead [7] and V. K. A. M. Gugenheim [3], [4]. In particular, the following (Theorem 1 in section 3) is a modification of results of Whitehead. *Let a finite polyhedron P be imbedded as a subcomplex of a combinatorial manifold W without boundary, and let $U_i(P, W)$ be regular neighborhoods of P in W, $i=1, 2$. Then there is a piecewise linear homeomorphism onto $\psi: W\rightarrow W$ such that $\psi(U_1(P, W))=U_2(P, W)$ and such that $\psi|P=$identity where ψ is an orientation preserving piecewise linear homeomorphism onto if W is orientable.*

The expositions is as follows: In section 2 Definitions and notation will be explained. In section 3 a modification of the regular neighborhood of Whitehead and Theorem 1 will be given. Section 4 will prepare the preliminary lemmas and notation needed in the latter. Section 5 will be devoted to prove Theorem 2. In section 6 applications of Theorem 2 will be stated.

In his delightful paper "Embeddings of spheres", Bull. Amer. Math. Soc., vol. 65 (1959), pp. 59-65, Professor B. Mazur mentioned an unpublished lemma of mine. The present paper is the revised version of the manuscript in question.

It is a pleasure to express m y graditude to Professor V. K. A. M. Gugenheim for many useful suggestions, and to Professors E. E. Moise and J. R. Stallings for their advices during revising the paper.

2. Definition and Notation

By a *simplex* we shall always mean a closed Euclidean simplex and the word *complex* will mean a closed, rectilinear, locally finite, simplicial complex of some Euclidean space. If K is a complex, then $|K|$ denotes the point-set which is the union of the simplices of K. Such a set $|K|$ will be called a *polyhedron*, and K will be called a simplicial subdivision of the polyhedron $|K|$. K' and K'' will stand for the first and second barycentric subdivisions of K. Let K be a q-complex. We say that K is homogeneous (see, [6], p. 48), if every p-simplex ($p<q$) is a face of at least one q-simplex. Then ∂K denotes its *boundary* (modulo 2), that is, the totality of all $(q-1)$-simplices which are incident to an odd number of q-simplices; and if $P=|K|$, then ∂P denotes the polyhedron $|\partial K|$. The point set $P-\partial P$ will be called the *interior* of P, and will be denoted by *Int* P. A polyhedron will be called *finite* if it has a simplicial subdivision which is a finite complex.

Let K be a complex and Δ one of its simplex. The set of all simplices of K having Δ as a face is called the *star* of Δ in K, whose polyhedron is denoted by $St(\Delta, K)$ and is called the *star set* of Δ in K. The set of simplices of K which are faces opposite Δ in some simplex of the star of Δ in K is called the *link* of Δ in K, whose polyhedron is denoted by $Lk(\Delta, K)$ and is called the *link set* of Δ in K. Let x be a point of $|K|$. We denote by $St(x, K)$ the point set of points of all simplices of K containing x and by $Lk(x, K)$ the point set of points of simplices of $St(x, K)$ not containing x. If x is a vertex of K, these definitions coincide with those given just above, see [4], p. 134. Let L be a subcomplex of K. Then $N(L, K)$ will stand for the point set of points of all simplices of K meeting $|L|$, and will be called the *star neighborhood* of L in K, see [7], p. 251.

As usual two complexes K_1, K_2 are combinatorially equivalent if K_1 and K_2 have isomorphic simplicial subdivisions L_1, L_2. In this case, we shall say that the polyhedra $|K_1|$ and $|K_2|$ are *equivalent*. By a *q-cell* we shall mean a polyhedron equivalent to q-simplex, by a *q-sphere* one equivalent to the boundary of $(q+1)$-simplex. When polyhedra $|K_1|$ and $|K_2|$ are equivalent, there is a homeomorphism

$$\phi: \ |K_1| \leftrightarrow |K_2|$$

which maps each simplex of L_1 linearly onto the corresponding simplex of L_2 This ϕ is simplicial relative to L_1 and L_2, and *piecewise linear* relative to the original complexes K_1 and K_2. All mappings used in this paper will be piecewise linear homeomorphisms. *Thus, whenevre we mention a homeomorphism, it should be understood that we mean a piecewise linear homeomorphism.* If the mapping is onto, this will be indicated by a double-headed arrow, as in the displayed formula above. If P, Q are polyhedra and $\phi: P \to Q$ is a homeomorphism of P into Q, and ∂P is well defined, then $\partial\phi$ denotes the homeomorphism $\phi | \partial P$.

A complex K is called the *combinatorial q-manifold* if for each point x of $|K|$, $St(x, K)$ is the q-cell, alternatively $Lk(x, K)$ is the $(q-1)$-cell if $x \in |\partial K|$ and $Lk(x, K)$ is the $(q-1)$-sphere if $x \in Int\ |K|$. (See [3], p. 31) A polyhedron is called the *combinatorial q-manifold* if it has a simplicial subdivision which is a combinatorial q-manifold. *Whenever we mention a manifold, it should be understood that we mean a combinatorial, connected manifold.* We shall call a finite manifold *closed* if it has no boundary. For the sake of convenience, *a polyhedron P in a polyhedron Q* will stand for *the polyhedron P being piecewise linearly imbedded as a subcomplex of a simplicial subdivision of the polyhedron Q.*

If a polyhedron M is an orientable q-manifold, we shall denote by $\langle M \rangle$ the oriented manifold obtained by assigning one of the possible orientations; M with the opposite orientation will be denoted by $-\langle M \rangle$. As a matter of convention, $1\langle M \rangle$, $-1\langle M \rangle$ will mean $\langle M \rangle$, $-\langle M \rangle$ respectively. If $N \subset M$ is an orientable q-manifold, we shall write

$\langle N\rangle\subset\langle M\rangle$ if $\langle N\rangle$ has been oriented by giving to each of q-simplices the orientation of $\langle M\rangle$. If ∂M is not empty and orientable, by $\partial\langle M\rangle$ we shall denote the oriented ∂M obtained by giving to each of its $(q-1)$-simplices the orientation coherently induced by that of the oriented q-simplex $\langle\Delta\rangle\subset\langle M\rangle$ which is incident to the former. Let $\langle M\rangle$, $\langle N\rangle$ be oriented q-manifolds and $\phi: M\rightarrow N$ be a homeomorphism. If the orientation of $\langle N\rangle$ and the orientation induced by ϕ and that of $\langle M\rangle$ are identical, we shall write $\phi: \langle M\rangle\rightarrow\langle N\rangle$, and denote the oriented image of M by $\phi\langle M\rangle$.

Let $P, Q\subset M$ be polyhedra, M be an orientable manifold and $\phi: M\leftrightarrow M$ be an orientation preserving homeomorphism such that $\phi P=Q$. In this case, we shall say that P, Q are *congruent* in M.

By I and J we shall denote the linear intervals $0\leqq t\leqq 1$ and $-1\leqq s\leqq 1$ respectively. We shall denote by $Cl_Y X$ or $Cl\ X$ the closure of X in Y. Let X, Y be point sets of some Euclidean space. We shall denote by $XY=YX$ the *join* of X and Y, that is, the set of points $tx+(1-t)y$ where $x\in Y$, $y\in Y$ and $t\in I$, using vector notation.

3. The Regular Neighborhood

Let P be a finite polyhedron in an m-manifold V. The regular neighborhood of P in V, defined by Whitehead [7], p. 297, is an m-manifold $U(P, V)$ contained in V and containing P, which contracts geometrically into P. The following results of Whitehead are necessary in this paper, see [7], pp. 293-296.

(1) $N(K'', L'')$ is a regular neighborhood of P in V where K, L are simplicial subdivisions of P, V and where K is a subcomplex of L.

(2) If P is a cell, then $U(P, V)$ is an m-cell.

The regular neighborhood defined above is not necessarily a neighborhood in the point-set theoretic sense and Theorem 1 in this section does not hold for this regular neighborhood. Therefore we shall put some restrictions to it as follows.

DEFINITION 1. Let P be a finite polyhedron in an m-manifold W *without boundary*. The *regular neighborhood* $U(P, W)$ of P in W means an m-manifold contained in W and containing P *in the interior*, which contracts geometrically into P.

In sections 3 and 4 however we shall use the regular neighborhood defied by Whitehead which will be called the *regular neighborhood in the weak sense* there.

Lemma 1. *The properties* (1) *and* (2) *above mentioned still hold for the regular neighborhood.*

Proof. Since the regular neighborhood is also the regular neighborhood in the weak sense, it is enough to prove (1) that $N(K'', L'')$ contains P in the interior where K, L are simplicial subdivisions of P, W and where K is a subcomplex of L. Let x be a point P. Then $St(x, L'')$ is an m-cell containing x in the interior, for W is an m-manifold without boundary. Since $Int\ St(x, L'')$ is open in W and contained in $Int\ N(K'', L'')$, P is contained in $Int\ N(K'', L'')$. The property (2) follows immediately from the property (2) for the regular neighborhood in the weak sense.

Let N be a q-manifold and C a q-cell such that

$$N \cap C = \partial N \cap \partial C = F,$$

a $(q-1)$-cell. We shall say that N and C have *regular contact* in F. In this situation, a transformation

$$N \Rightarrow N \cup C,$$

or the resultant of a finite sequence of such transformations will be called the *regular expansion* of N, see [7], p. 291. Then, suppose that N is in an m-manifold W without boundary. Let $D \subset N$ be a q-cell such that

$$\partial N \cap \partial D \supset F.$$

Let $G \subset W$ be an m-cell containing $C \cup D$ in the interior, and

$$\theta : G \to R \text{ be a homeomorphism such that } \theta(C \cup D) = \Delta,$$

a q-simplex in Euclidean m-space R. Then we call C a *flat attachment* to N, see [3], p. 33.

Lemma 2. *Let N be an m-manifold in an m-manifold W without boundary, and N and an m-cell $C \subset W$ have regular contact in an $(m-1)$-cell F. Then C is a flat attachment to N.*

Proof. Let D be a regular neighborhood $U(F, N)$ in the weak sense. By the property (2) of Whitehead, D is an m-cell in N and $\partial N \cap \partial D \supset F$. Since C and D have regular contact in F, C and $C \cup D$ are equivalent, see [3], p. 35, and $C \cup D$ is an m-cell. By (2) of Lemma 1, a regular neighborhood $U(C \cup D, W) = G$, say, is an m-cell containing $C \cup D$ in the interior. Let $\theta' : G \to R$ be a homeomorphism. This is possible, for G is an m-cell. By Theorem 3 in [3], p. 32, there is a homeomorphism $\phi : R \leftrightarrow R$ such that $\phi\theta'(C \cup D) = \Delta$, an m-simplex in R. Therefore $\theta = \phi\theta' : G \to R$ is a homeomorphism such that $\theta(C \cup D) = \Delta$. Hence C is a flat attachment to N.

Lemma 3. *Let P be a finite polyhedron in an m-manifold W without boundary. Let $U_1(P, W)$ and $U_2(P, W)$ be regular neighborhoods of P in W such that $U_1(P, W)$ expands regularly into $U_2(P, W)$. Then there is a homeomorphism $\psi: W \leftrightarrow W$ such that*

$$\psi(U_1(P, W)) = U_2(P, W) \text{ and } \psi | P = \text{identity}$$

where ψ is an orientation preserving homeomorphism if W is orientable.

Proof. Let $N_1, \cdots, N_k$ be a sequence of m-manifolds in W such that $N_1 = U_1(P, W)$, $N_k = U_2(P, W)$ and $N_{i-1} \Rightarrow N_i = N_{i-1} \cup C_i$ is a regular expansion where N_{i-1} and an m-cell C_i have regular contact in an $(m-1)$-cell F_i $(i=2, \cdots, k)$. By Lemma 2, C_i is a flat attachment to N_{i-1}. Namely there are m-cell $G_i \subset W$ containing C_i and $D_i = U(F_i, N_{i-1})$ in the interior, and a homeomorphism $\theta_i: G_i \to R$ such that $\theta_i(C_i \cup D_i) = \Delta$, an m-simplex. By Theorem 6 in [3], pp. 48–49, there is a homeomorphism

$$\eta_i : \theta_i G_i \leftrightarrow \theta_i G_i$$

such that

$$\eta_i | \theta_i(\partial G_i \cup (Cl(N_{i-1} - D_i) \cap G_i)) = \text{identity and } \eta_i \theta_i D_i = \Delta .$$

Then $\psi_i : W \leftrightarrow W$ defined by taking

$$\psi_i | Cl(W - G_i) = \text{identity and } \psi_i | G_i = \theta_i^{-1} \eta_i \theta_i$$

is a homeomorphism such that

$$\psi_i N_{i-1} = N_i \text{ and } \psi_i | Cl(N_{i-1} - D_i) = \text{identity} ,$$

where ψ_i is an orientation preserving homeomorphism if W is orientable.

In this situation, $D_i = U(F_i, N_{i-1})$ will be taken so that D_i does not meet P. This is possible, because $P \subset Int\, N_1 \subset Int\, N_{i-1}$ and by the property (2) of Whitehead if we give a sufficiently fine simplicial subdivision to N_{i-1}, then D_i may be arbitrarily near F_i which is contained in ∂N_{i-1}. Then $P \subset Cl(N_{i-1} - D_i)$ and $\psi_i | P = $ identity.

Hence $\psi: W \leftrightarrow W$ defined by taking

$$\psi = \psi_k \cdots \psi_2$$

is the required homeomorphism.

Theorem 1. *Let P be a finite polyhedron in an manifold W without boundary. Then for any two regular neighborhoods $U_1(P, W)$ and $U_2(P, W)$ of P in W there is a homeomorphism $\psi: W \leftrightarrow W$ such that*

$$\psi(U_1(P, W)) = U_2(P, W), \psi | P = \text{identity} ,$$

where ψ is an orientation preserving homeomorphism if W is orientable.

Proof. Let K, L be simplicial subdivisions of P, W where K is a subcomplex of L and where each of $U_i(P, W)$, considering it as subcomplex of L, contracts formally into K. Then by Whitehead [7], p. 296, we have the following

$$U_i''(P, W) \Rightarrow N(U_i''(P, W), L'') \Leftarrow N(K'', L''),$$

where $i=1, 2$ and $\Rightarrow$ means the regular expansion.

By the property (1) of Whitehead and Definition 1, $N(U_i''(P, W), L'')$ is a regular neighborhood of $U_i(P, W)$ in W and a regular neighborhood of P in W. By Lemma 3 we have homeomorphisms $\psi_i, \rho_i: W \leftrightarrow W$ such that

$$\psi_i U_i''(P, W) = N(U_i''(P, W), L''),$$
$$\rho_i N(U_i''(P, W), L'') = N(K'', L'')$$

and

$$\psi_i | P = \rho_i | P = \text{identity},$$

where ψ_i, ρ_i are orientation preserving homeomorphisms if W is orientable. Therefore

$$\psi = \psi_2^{-1}\rho_2^{-1}\rho_1\psi_1 : W \leftrightarrow W$$

is the required homeomorphism.

4. Preliminaries for Thickening

Let M be a closed n-manifold in an m-manifold W without boundary, where $n < m$.

NOTATION 1. By K and L we shall denote simplicial subdivisions of M and W respectively where K is a subcomplex of L. By Δ we shall denote a simplex of L' and then x will denote the barycenter of Δ. If Δ is an $(n-q)$-simplex of K', we shall denote by ∇ the q-cell dual to Δ in K' with the simplicial subdivision Y which is a subcomplex of K'', and by $\square$ we shall denote the $q+(m-n)$-cell dual to Δ in L' with the simplicial subdivision Z which is a subcomplex of L''. Let us denote the q-skeleton of K' by $(K')^q$ where $(K')^{-1}$ means the empty set. By $\mathfrak{K}^q$ we shall denote the polyhedron of the q-cellcomplex which consists of all the dual cells ∇ and by $\mathfrak{N}^{q+(m-n)}$ the polyhedron of the $q+(m-n)$-cellcomplex which consists of all the dual cells $\square$, where Δ ranges over $K'-(K')^{n-q-1}$.

Lemma 4. *Let* Δ *be an* $(n-q)$*-simplex of* K'. *Then*

$$\bigcup_j \square_j = N(\partial Y, \partial Z)$$

and $\bigcup_j \square_j$ *is a regular neighborhood of the* $(q-1)$*-sphere* $\partial\nabla$ *in the* $(q+(m-n)-1)$*-sphere* $\partial\square$, *where* Δ_j *ranges over the* $(n-q+1)$*-simplices of* K' *incident to* Δ.

Proof. As a matter of convenience N will stand for $N(\partial Y, \partial Z)$. If $\Delta_a, \cdots, \Delta_\alpha$ are simplices of K' which have Δ as a proper face and $\Delta_a \subset \cdots \subset \Delta_\alpha$, then by the proof of Theorem II of [6], p. 230, the join $x_a \cdots x_\alpha$ is a simplex of ∂Y and conversely every simplex of ∂Y is such a join. Similarly a join $x_b \cdots x_\beta$ is a simplex of ∂Z if and only if the simplices $\Delta_b, \cdots, \Delta_\beta$ are in L', which have Δ as a proper face and $\Delta_b \subset \cdots \subset \Delta_\beta$.

By the definition of N, a simplex $B = x_b \cdots x_\beta$ of ∂Z is in N if and only if there is a simplex $A = x_a \cdots x_\alpha$ of ∂Y such that AB is a simplex of ∂Z. Let $\Delta_a \subset \cdots \subset \Delta_\alpha$. Then Δ_a is a simplex of K' having Δ as a proper face, and there is an $(n-q+1)$-simplex Δ_j of K', incident to Δ_b, which is a face of Δ_a. Then the simplex $x_j x_b \cdots x_\beta$ is in the complex Z_j, and $\bigcup_j \square_j \supset N$. Conversely every $(q+(m-n)-1)$-simplex C of $\square_j$ is written by $x_j x_1 \cdots x_{q+(m-n)-1}$ where Δ_i is an $(n-q+1)+i$-simplex of L', $1 \leqq i \leqq q+(m-n)-1$, such that $\Delta_j \subset \Delta_1 \subset \cdots \subset \Delta_{q+(m-n)-1}$. Since x_j is a vertex of ∂Y, C is in N. Since $Z_j \subset L''$ is a $(q+(m-n)-1)$-homogeneous complex, $\bigcup_j \square_j \subset N$. Therefore $\bigcup_j \square_j = N$.

Let p be a point in $\partial\nabla$. Then $St(p, \partial Z)$ is a $(q+(m-n)-1)$-cell containing p in the interior, for $\partial\square$ is a $(q+(m-n)-1)$-sphere. Since $Int\ St(p, \partial Z) \subset Int\ N$, we have $\partial\nabla \subset Int\ N$.

It remains to prove that N is a regualr neighborhood of $\partial\nabla$ in $\partial\square$ in the weak sense. To show this we first prove the following three assertions (see, [7], p. 293).

(a) None of the simplices and its interior of $\partial Z - \partial Y$ has all its vertices in $\partial\nabla$.

(b) If a simplex A of ∂Z does not meet $\partial\nabla$, then $\partial\nabla \cap Lk(A, \partial Z)$ is a cell (possibly the empty set).

(c) If B is a simplex of ∂Z, then the complexes $\partial Y \cap Lk(B, \partial Z)$ and $Lk(B, \partial Z)$ also satisfy the conditions (a) and (b).

Proof of (a). If a simplex $x_c \cdots x_d$ of ∂Z or its interior has all its vertices in $\partial\nabla$, then the simplices $\Delta_c, \cdots, \Delta_d$ are in K'. Hence the simplex $x_c \cdots x_d$ and its interior are in $\partial\nabla$, proving (a).

Proof of (b). Let $A = x_a \cdots x_\alpha$ where $\Delta_a, \cdots, \Delta_\alpha$ are simplices of L'

having Δ as a proper face and $\Delta_a \subset \cdots \subset \Delta_\alpha$. Since A does not meet $\partial\nabla$, there does not exist i among $a, \cdots, \alpha$ such that Δ_i is in K'. In particular Δ_a is not in K'.

Suppose that $\partial\nabla \cap Lk(A, \partial Z)$ is not empty. Since $Lk(A, \partial Z) = \bigcup_B Lk(A, B)$ where B ranges over all simplices of ∂Z having A as a face, there is a B for which $\partial\nabla \cap Lk(A, B)$ is not empty. Let $Lk(A, B) = C$, a simplex of ∂Z. If $\Delta_s \subset \Delta_t$ and Δ_t is in K', then Δ_s is also in K'. Then $C = x_c \cdots x_e x_f \cdots x_\gamma$, where $\Delta_c, \cdots, \Delta_e$ are simplices of K' having Δ as a proper face and $\Delta_f, \cdots, \Delta_\gamma$ are not simplices of K' but simplices of L' such that $\Delta_c \subset \cdots \subset \Delta_e \subset \Delta_f \subset \cdots \Delta_\gamma$. Then $\partial\nabla \cap Lk(A, B) = \partial\nabla \cap C = x_c \cdots x_e$ which is not empty. Since B is a simplex of L'' having A, C as faces, Δ_c is a face of Δ_a, which is in K' and has Δ as a proper face.

Let $p \geqq n-q+1$ be the dimension of a face of Δ_a as follows. There is a p-face Δ^p having Δ as a proper face, which is in K', and there is no s-face Δ^s $(s > p)$ having Δ as a proper face, which is in K'. This is possible, because Δ_c is in K' and Δ_a is not in K', and both of which have Δ as a proper face. Suppose that there is an r-face Δ^r of Δ_a, in K', having Δ as a proper face, and that none of Δ^p and Δ^r is a face of the other. Then all vertices of the simplex $\Delta^p\Delta^r$, a face of Δ_a, is in K' and the simplex $\Delta^p\Delta^r$ is in L'. The dimension of $\Delta^p\Delta^r$ is at least $p+1$. By the maximum property of p, $\Delta^p\Delta^r$ is not in K'. This contradicts the well known result [7], p. 294, that no simplex of L' has all its vertices in K'. Therefore every face of Δ_a which is in K' and has Δ as a proper face is a face of Δ^p. Therefore every simplex of $\partial\nabla \cap Lk(A, \partial Z)$ is the join $x_g \cdots x_h$ where $\Delta \subset \cdots \subset \Delta_g \subset \cdots \subset \Delta_h \subset \cdots \subset \Delta^p$, the dimension of $\Delta_g \geqq n-q+1$ and $p \leqq n$, and conversely.

By Δ_u we denote the $(p-n+q-1)$-simplex such that $\Delta^p = \Delta\Delta_u$. Then every simplex $x_g \cdots x_h$ is in $Lk(x, x\Delta_u)$, where $x\Delta_u$ will be thought of as a subcomplex of K''. Conversely every simplex of $Lk(x, x\Delta_u)$ is such a join. Hence $\partial\nabla \cap Lk(A, \partial Z) = Lk(x, x\Delta_u)$ which is a $(p-n+q-1)$-cell, because $x\Delta_u$ is the $(p-n+q)$-simplex containing x on the boundary, proving (b).

Proof of (c). For (a) is obviously satisfied. If A is a simplex of $Lk(B, \partial Z)$ not meeting $\partial\nabla \cap Lk(B, \partial Z)$, then AB is a simplex in ∂Z not meeting $\partial\nabla$ and $\partial\nabla \cap Lk(AB, \partial Z)$ is a cell, by (b). Since $Lk(AB, \partial Z) = Lk(A, L(B, \partial Z))$ and $Lk(AB, \partial Z) \subset Lk(B, \partial Z)$,

$$\partial\nabla \cap Lk(B, \partial Z) \cap Lk(A, L(B, \partial Z)) = \partial\nabla \cap Lk(AB, \partial Z),$$

a cell, satisfying (b) and also proving (c).

Finally we shall prove that N is a regualr neighborhood of $\partial\nabla$ in

$\partial\square$ in the weak sense (see, [7], p. 293-294). This will be proved by induction on the dimension $q+(m-n)-1$ of $\partial\square$. This is trivial if $q+(m-n)-1=0$. By (a) and the definition of N, N is a normal neighborhood of ∂Y (see, [7], p. 250). Since $\partial\nabla\subset N$ and $Lk(A, N)=Lk(A, \partial Z)\cap N$, we have that $\partial\nabla\cap Lk(A, N)=\partial\nabla\cap Lk(A, \partial Z)$, which is a cell, by (b), where A is a simplex of N not meeting $\partial\nabla$. Then N is a contractible neighborhood of ∂Y, [7], p. 250. By Theorem 2 of [7], p. 250, N contracts into ∂Y. It remains to prove that N is a manifold. Let b be a vertex in N. If b is in ∂Y, then $Lk(b, N)=Lk(b, \partial Z)$ which is a $(q+(m-n)-2)$-sphere, for $\partial\square$ is a $(q+(m-n)-1)$-sphere. Suppose that b is not in ∂Y. A simplex Ab of ∂Z meets $\partial\nabla$ if and only if A meets $\partial\nabla$. Therefore $Lk(b, N)=N(\partial Y\cap Lk(b, \partial Z), Lk(b, \partial Z))$. By the hypothesis of induction and (c), $Lk(b, N)$ is a regular neighborhood of the cell $\partial\nabla\cap Lk(b, \partial Z)$ in $Lk(b, \partial Z)$ in the weak sense, and $Lk(b, N)$ is a $(q+(m-n)-2)$-cell, by the property (2) of Whitehead in section 3. Therefore N is a manifold, and a regular neighborhood of $\partial\nabla$ in $\partial\square$ in the weak sense, completing the proof of Lemma 4.

DEFINITION 2. Let us take a finite sequence $\alpha=\Delta_1, \cdots, \Delta_a$ of successively incident simplices of K' such that $\Delta_1=\Delta^*$, a fixed n-simplex. We call α the *way* in K' to Δ_a. By $\langle\Delta^*\rangle$ we shall denote the oriented n-simplex. Since $\Delta_1=\Delta^*$, we have the well defined oriented simplex, written $\langle\Delta_a\rangle_\alpha$, inductively such that $\langle\Delta_i\rangle$ is either $\partial\langle\Delta_i\rangle\supset\langle\Delta_{i-1}\rangle$ or $\langle\Delta_i\rangle\subset\partial\langle\Delta_{i-1}\rangle$ according the case.

Let M_i be a closed n-manifold in an oriented m-manifold $\langle W_i\rangle$ without boundary where $n<m$ and $i=1, 2$. Let $\phi: M_1\leftrightarrow M_2$ be a homeomorphism.

NOTATION 2. Using Notation 1, suppose that ϕ is simplicial relative to the complexes K_1 and K_2 which are isomorphic under the isomorphism induced by ϕ. Then ϕ is also simplicial relative to K_1' and K_2', and relative to K_1'' and K_2''. From now on by Δ_i, Δ_{ij} we denote simplices of K_i' satisfying $\phi\Delta_1=\Delta_2$, $\phi\Delta_{1j}=\Delta_{2j}$. Then $\phi\nabla_1=\nabla_2$ and thus, ϕ will induce a homeomorphism onto between the polyhedra $\mathfrak{K}_1^n$ and $\mathfrak{K}_2^n$. The correspondence between cells of $\mathfrak{N}_1^m$ and cells of $\mathfrak{N}_2^m$ induced by the correspondence between $\square_1$ and $\square_2$ is one-to-one. By $\langle\Delta_i^*\rangle$ we shall denote n-simplices with orientations such that $\phi\langle\Delta_1^*\rangle=\langle\Delta_2^*\rangle$, which will keep fixed in the rest of the paper. Let α be a way in K_1' to Δ_1, then the simplices of K_2' corresponding the simplices of the way in K_1' will be naturally thought of as a way in K_2' to Δ_2, which will be again denoted by α, and these are called the *ways* to Δ_i. It is well known [6], p. 249, that for the oriented simplex $\langle\Delta_i\rangle_\alpha$ in the oriented manifold $\langle W_i\rangle$ there is the oriented dual cell, written $\langle\square_i\rangle_\alpha$, whose orientation is uniquely determined such that the intersection number of $\langle\Delta_i\rangle_\alpha$ and $\langle\square_i\rangle_\alpha$ is equal to 1 in $\langle W_i\rangle$.

Lemma 5. *Let α, β be the ways to Δ_i. If $\langle\square_1\rangle_\alpha=\epsilon\langle\square_1\rangle_\beta$. Then $\langle\square_2\rangle_\alpha=\epsilon\langle\square_2\rangle_\beta$, and if Δ_i is a vertex and $\langle\square_1\rangle_\alpha\subset\epsilon\langle W_1\rangle$, then $\langle\square_2\rangle_\alpha\subset\epsilon\langle W_2\rangle$, where $\epsilon=1$ or -1.*

Proof. If $\langle\square_1\rangle_\alpha=\epsilon\langle\square_1\rangle_\beta$, then $\langle\Delta_1\rangle_\alpha=\epsilon\langle\Delta_1\rangle_\beta$. Since K_1' and K_2' are isomorphic under the correspondence $\phi\Delta_1=\Delta_2$, $\langle\Delta_2\rangle_\alpha=\epsilon\langle\Delta_2\rangle_\beta$ and then $\langle\square_2\rangle_\alpha=\epsilon\langle\square_2\rangle_\beta$. If $\langle\square_1\rangle_\alpha\subset\epsilon\langle W_1\rangle$ then $\langle\Delta_1\rangle_\alpha=\epsilon\Delta_1$. By the same reason mentioned above and $\phi\langle\Delta_1^*\rangle=\langle\Delta_2^*\rangle$ we have that $\langle\Delta_2\rangle_\alpha=\epsilon\Delta_2$, and that $\langle\square_2\rangle_\alpha\subset\epsilon\langle W_2\rangle$.

5. The Proof of Theorem 2

Lemma 6. *Let T be a $(q-1)$-sphere in a q-sphere S and $U(T,S)$ a regular neighborhood of T in S. Suppose that the Schönflies conjecture is true for dimension q. Then there is a homeomorphism $\theta: T_0\times J\to S$ such that*

$$\theta(T_0\times J)=U(T,S) \text{ and } \theta(T_0\times 0)=T\,,$$

where T_0 is a $(q-1)$-sphere.

Proof. Let Δ_a, Δ_0 and Δ_b be q-simplices in S similarly situated with respect to a center of similitude in $Int\,\Delta_a$ such that $\Delta_a\subset Int\,\Delta_0$ and $\Delta_0\subset Int\,\Delta_b$. By Corollary to Theorem 8 of [7], p. 260, $Cl(\Delta_b-\Delta_a)$ is a regular neighborhood of $\partial\Delta_0$ in S. There is a homeomorphism

$$\phi:\ \partial\Delta_0\times J\leftrightarrow Cl(\Delta_b-\Delta_a)$$

such that $\phi(\partial\Delta_0\times 0)=\partial\Delta_0$. By the assumption and Theorems 3 and 4 of [3], p. 32, there is an orientation preserving homeomorphism

$$\psi_1:\ S\leftrightarrow S$$

such that $\psi_1\partial\Delta_0=T$. It is immediate that $\psi_1(Cl(\Delta_b-\Delta_a))$ is a regular neighborhood of T in S. By Theorem 1 in section 3 there is an orientation preserving homeomorphism

$$\psi_2:\ S\leftrightarrow S$$

such that $\psi_2\psi_1(Cl(\Delta_b-\Delta_a))=U(T,S)$ and $\psi_2|T=$identity. Putting $T_0=\partial\Delta_0$ and $\theta=\psi_2\psi_1\phi$, it completes the proof.

Lemma 7. *Let $\langle S_i\rangle$ be an oriented q-sphere and $T_i\subset S_i$ a $(q-1)$-sphere where $i=1,2$. Suppose that the Schönflies conjecture is true for* *dimension q, and that there is a homeomorphism*

$$\phi: \langle U(T_1, S_1)\rangle \leftrightarrow \langle U(T_2, S_2)\rangle$$

such that $\phi T_1 = T_2$ *where* $\langle U(T_i, S_i)\rangle \subset \langle S_i\rangle$. *Then there is a homeomorphism*

$$\psi: \langle S_1\rangle \leftrightarrow \langle S_2\rangle$$

such that $\psi | U(T_1, S_1) = \phi$.

Proof. By Lemma 6 there are a $(q-1)$-sphere T_0 and homeomorphisms $\theta_i: T_0 \times J \leftrightarrow U(T_i, S_i)$ such that $\theta_i(T_0 \times 0) = T_i$. By the assumption the $(q-1)$-spheres $\theta_i(T_0 \times 1)$ and $\theta_i(T_0 \times -1)$ are congruent to the boundary of q-simplex in S_i. Then $Cl(S_i - U(T_i, S_i))$ consists of two q-cells C_i and D_i such that $\partial C_i = \theta_i(T_0 \times 1)$ and $\partial D_i = \theta_i(T_0 \times -1)$. If we put $\rho_c = \phi | \partial C_1$, then $\rho_c(\partial C_1)$ is either ∂C_2 or ∂D_2, say ∂C_2. If we put $\rho_d = \phi | \partial D_1$, then $\rho_d(\partial D_1) = \partial D_2$. By $\phi\langle U(T_1, S_1)\rangle = \langle U(T_2, S_2)\rangle$, we have that

$$\rho_c: \partial\langle C_1\rangle \leftrightarrow \partial\langle C_2\rangle \text{ and } \rho_d: \partial\langle D_1\rangle \leftrightarrow \partial\langle D_2\rangle,$$

where $\langle C_i\rangle$, $\langle D_i\rangle \subset \langle S_i\rangle$.
By Lemma in 3.12 of [3], p. 37, there are homeomorphisms

$$\eta_c: \langle C_1\rangle \leftrightarrow \langle C_2\rangle, \text{ and } \eta_d: \langle D_1\rangle \leftrightarrow \langle D_2\rangle$$

such that $\partial\eta_c = \rho_c$ and $\partial\eta_d = \rho_d$. Then $\psi: \langle S_1\rangle \leftrightarrow \langle S_2\rangle$ defined by taking

$$\psi | U(T_1, S_1) = \phi, \quad \psi | C_1 = \eta_c \text{ and } \psi | D_1 = \eta_d$$

is the required homeomorphism.

Lemma 8. *Let* M_i^n *be a closed n-manifold in an oriented* $(n+1)$-*manifold* $\langle W_i^{n+1}\rangle$ *without boundary,* $i=1, 2$. *Using Notation 2, let*

$$\phi: M_1^n \leftrightarrow M_2^n$$

be a homeomorphism which is simplical relative to K_1 *and* K_2. *Suppose that the Schönflies conjecture is true for dimensin* $\leqq n$. *Then there is a homeomorphism*

$$\psi: \langle \mathfrak{N}_1^{n+1}\rangle \leftrightarrow \langle \mathfrak{N}_2^{n+1}\rangle \text{ such that } \psi | M_i^n = \phi \text{ where } \langle \mathfrak{N}_i^{n+1}\rangle \subset \langle W_i^{n+1}\rangle.$$

To prove the lemma we first prove the following;

(0). *Let a homeomorphism* $\phi: M_1^n \leftrightarrow M_2^n$ *be simplicial relative to* K_1 *and* K_2. *Then there is a homeomorphism*

$$\psi^0: \mathfrak{N}_1^1 \leftrightarrow \mathfrak{N}_2^1 \text{ such that } \psi_0 | \mathfrak{K}_1^0 = \phi \text{ and } \psi^0\langle\square_1\rangle_\alpha = \langle\square_2\rangle_\alpha$$

for each n-simplex Δ_i *of* K_i' *and each way* α *to* Δ_i.

Proof of (0). Since Δ_i is an n-simplex, $\partial\Box_i$ is a 0-sphere and we have a homeomorphism

$$\psi''_\alpha : \partial\langle\Box_1\rangle_\alpha \leftrightarrow \partial\langle\Box_2\rangle_\alpha \quad \text{for a way } \alpha .$$

Since ∇_i is the point such that $\Box_i = \nabla_i(\partial\Box_i)$, we have a homeomorphism

$$\psi'_\alpha : \langle\Box_1\rangle_\alpha \leftrightarrow \langle\Box_2\rangle$$

such that

$$\partial\psi'_\alpha = \psi''_\alpha \text{ and } \psi'_\alpha\nabla_1 = \nabla_2, \text{ by 3.11 of [3], p. 36.}$$

Let β be another way to Δ_i, then $\langle\Box_1\rangle_\beta = \in\langle\Box_1\rangle_\alpha$ implies $\langle\Box_2\rangle_\beta = \in\langle\Box_2\rangle_\alpha$, by Lemma 5. Therefore we have that

$$\psi'_\alpha\langle\Box_1\rangle_\beta = \langle\Box_2\rangle_\beta .$$

Thus we can put $\psi' = \psi'_\alpha$. Then $\psi^0 : \mathfrak{N}_1^1 \leftrightarrow \mathfrak{N}_2^1$ defined by taking $\psi^0 | \Box_1 = \psi'$ is a homeomorphism such that

$$\psi^0 | \mathfrak{K}_1^0 = \phi \text{ and } \psi^0\langle\Box_1\rangle_\alpha = \langle\Box_2\rangle_\alpha$$

for Δ_i and α to Δ_i, proving (0).

Next we shall prove the following;

$(q-1)\rightarrow(q)$. *Suppose that there is a homeomorphism*

$$\psi^{q-1} : \mathfrak{N}_1^q \leftrightarrow \mathfrak{N}_2^q$$

such that

$$\psi^{q-1} | \mathfrak{K}_1^{q-1} = \phi \textit{ and } \psi^{q-1}\langle\Box_1\rangle_\gamma = \langle\Box_2\rangle_\gamma$$

for each $(n-q+1)$*-simplex* Δ_i *of* K'_i *and for each way* γ *to* Δ_i, *and suppose that the Schönflies conjecture is true for dimension q. Then there is a homeomorphism*

$$\psi^q : \mathfrak{N}_1^{q+1} \leftrightarrow \mathfrak{N}_2^{q+1}$$

such that

$$\psi^q | \mathfrak{K}_1^q = \phi \textit{ and } \psi^q\langle\Box_1\rangle_\alpha = \langle\Box_2\rangle_\alpha$$

for each $(n-q)$*-simplex* Δ_i *of* K'_i *and for each way* α *to* Δ_i.

Proof of $(q-1)\rightarrow(q)$. By Δ_{ij} we denote an $(n-q+1)$-simplex of K'_i incident to an $(n-q)$-simplex Δ_i. By γ we denote the way to Δ_{ij} which is obtained from a way α to Δ_i adding Δ_{ij} as the final term. Then $\langle\Box_{ij}\rangle_\gamma \subset \partial\langle\Box_i\rangle_\alpha$. Since $\psi^{q-1}\langle\Box_{1j}\rangle_\gamma = \langle\Box_{2j}\rangle_\gamma$, we have that

$$\psi^{q-1}\langle\textstyle\bigcup_j\Box_{1j}\rangle_\alpha = \langle\bigcup_j\Box_{2j}\rangle_\alpha, \text{ where } \langle\bigcup_j\Box_{ij}\rangle_\alpha \subset \partial\langle\Box_i\rangle_\alpha .$$

By Lemma 4, $\bigcup_j \square_{ij}$ is a regular neighborhood of the $(q-1)$-sphere $\partial\nabla_i$ in the q-sphere $\partial\square_i$. Then by the assumption and Lemma 7 there is a homeomorphism

$$\psi''_\alpha : \partial\langle\square_1\rangle_\alpha \leftrightarrow \partial\langle\square_2\rangle_\alpha$$

such that

$$\psi''_\alpha | \bigcup_j \square_{1j} = \psi^{q-1}.$$

Let x_i be the barycenter of Δ_i, then $\square_i = x_i(\partial\square_i)$ and $\nabla_i = x_i(\partial\nabla_i)$, by Theorem II of [6], p. 230. By 3.11 of [3], p. 36, we have a homeomorphism

$$\psi'_\alpha : \langle\square_1\rangle_\alpha \leftrightarrow \langle\square_2\rangle_\alpha \text{ such that } \partial\psi'_\alpha = \psi''_\alpha ,$$

and $\psi'_\alpha | \nabla_1$ is simplicial relative to Y_1 and Y_2, see Notation 1. Let β be another way to Δ_i, then we have that

$$\psi'_\alpha\langle\square_1\rangle_\beta = \langle\square_2\rangle_\beta , \text{ by Lemma 5.}$$

Thus we can put $\psi'_\alpha = \psi'$. Then $\psi^q : \mathfrak{N}_1^{q+1} \leftrightarrow \mathfrak{N}_2^{q+1}$ defined by taking

$$\psi^q | \square_1 = \psi'$$

is a homeomorphism such that

$$\psi^q | \mathfrak{R}_1^q = \phi \text{ and } \psi^q\langle\square_1\rangle_\alpha = \langle\square_2\rangle_\alpha$$

for each $(n-q)$-simplex Δ_i and α to Δ_i, proving $(q-1)\to(q)$.

Proof of Lemma 8. By assertions (0) and $(q-1)\to(q)$ there is a homeomorphism

$$\psi^n : \mathfrak{N}_1^{n+1} \leftrightarrow \mathfrak{N}_2^{n+1}$$

such that

$$\psi^n | \mathfrak{R}_1^n = \phi \text{ and } \psi^n\langle\square_1\rangle_\alpha = \langle\square_2\rangle_\alpha$$

for each 0-simplex Δ_i and each way α to Δ_i. By Lemma 5 we have that if $\langle\square_1\rangle_\alpha \subset\in \langle W_1^{n+1}\rangle$ then $\langle\square_2\rangle_\alpha \subset\in \langle W_2^{n+1}\rangle$. Therefore $\psi^n\langle\square_1\rangle = \langle\square_2\rangle$, and

$$\psi^n\langle\mathfrak{N}_1^{n+1}\rangle = \langle\mathfrak{N}_2^{n+1}\rangle, \text{ where } \langle\square_i\rangle \subset \langle W_i^{n+1}\rangle .$$

If we put $\psi^n = \psi$, then this completes the proof of the lemma.

Theorem 2. *Let M_i^n be a closed n-manifold in an oriented $(n+1)$-manifold $\langle W_i^{n+1}\rangle$ without boundary where $i=1, 2$ and let*

$$\phi : M_1^n \leftrightarrow M_2^n$$

be a homeomorphism. Suppose that the Schönflies conjecture is true for dimension $\leqq n$. Then for any regular neighborhoods $U(M_i^n, W_i^{n+1})$ there is a homeomorphism

$$\psi : \langle U(M_1^n, W_1^{n+1})\rangle \leftrightarrow \langle U(M_2^n, W_2^{n+1})\rangle \quad \textit{such that}$$
$$\psi | M_1^n = \phi \quad \textit{where} \ \langle U(M_i^n, W_i^{n+1})\rangle \subset \langle W_i^{n+1}\rangle .$$

Proof. Let K_i, L_i be simplicial subdivisions of M_i^n, W_i^{n+1} where K_i is a subcomplex of L_i and ϕ is simplicial relative to K_1 and K_2. By Lemma 8 there is a homeomorphism

$$\psi' : \langle \mathfrak{N}_1^{n+1}\rangle \leftrightarrow \langle \mathfrak{N}_2^{n+1}\rangle$$

such that

$$\psi' | M_1^n = \phi \ \text{where} \ \langle \mathfrak{N}_i^{n+1}\rangle \subset \langle W_i^{n+1}\rangle .$$

Let Δ_{ij} be a 0-simplex of K_i', then $\square_{ij} = N(\Delta_{ij}, L_i'')$. On the other hand it is well known [7], p. 294, that $N(K_i'', L_i'') = \bigvee_j \square_{ij}$. Therefore we have that $N(K_i'', L_i'') = \mathfrak{N}_i^{n+1}$. Hence ψ' is a homeomorphism

$$\psi' : \langle N(K_1'', L_1'')\rangle \leftrightarrow \langle N(K_2'', L_2'')\rangle$$

such that

$$\psi' | M_1^n = \phi \ \text{where} \ \langle N(K_i'', L_i'')\rangle \subset \langle W_i^{n+1}\rangle .$$

By Theorem 1 there are orientation preserving homeomorphisms

$$\psi_i : W_i^{n+1} \leftrightarrow W_i^{n+1}$$

such that

$$\psi_i(U(M_i^n, W_i^{n+1})) = N(K_i'', L_i'') \ \text{and} \ \psi_i | M_i^n = \text{identity}.$$

Then $\psi = \psi_2^{-1}\psi' \psi_1$ is the required homeomorphism.

6. Applications

Theorem 3. *Let M^n be an orintable, closed n-manifold in an orientable $(n+1)$-manifold W^{n+1} without boundary. Let $U(M^n, W^{n+1})$ be a regular neighborhood of M^n in W^{n+1}. Suppose that the Schönflies conjecture is true for dimension $\leq n$. Then there is a homeomorphism*

$$\theta : M^n \times J \to W^{n+1}$$

where J is the linear interval $-1 \leq s \leq 1$, such that

$$\theta(x, 0) = x \ \textit{for each point} \ x \ \textit{of} \ M^n$$

and such that

$$\theta(M^n \times J) = U(M^n, W^{n+1}) .$$

Proof. Let us consider the Cartesian product $M^n \times R$ where R is a Euclidean 1-space containing J. Then $M^n \times R$ is an orientable $(n+1)$-manifold without boundary. By Theorem 8 of [7], p. 260, and Definition

1, $M^n \times J$ is a regular neighborhood of $M^n \times 0$ in $M^n \times R$. A map $\phi: M^n \times 0 \to M^n$ defined by $\phi(x, 0) = x$ for each x is a homeomorphism onto. If we give orientations to $M^n \times R$ and W^{n+1}, then by Theorem 2 we have a homeomorphism

$$\theta: M^n \times J \to W^{n+1}$$

which satisfies the theorem.

Theorem 4. *Let M^n be an orientable, closed n-manifold in an orientable $(n+1)$-manifold W^{n+1} without boundary. Let*

$$\phi: M^n \leftrightarrow M^n$$

be a homeomorphism which is onto isotopic to the identity (see, [3], p. 30). *Suppose that the Schönflies conjecture is true for dimension $\leqq n$. Then there is an orientation preserving homeomorphism*

$$\psi: W^{n+1} \leftrightarrow W^{n+1} \text{ such that } \psi | M^n = \phi .$$

Proof. By Theorem 3 each point of a regular neighborhood $U(M^n, W^{n+1})$ will be denoted by a pair (x, s) where x is a point of M^n and $s \in J$ and $(x, 0) = x$. Let $\phi_t: M^n \leftrightarrow M^n$, $t \in I$, be an onto isotopy between $\phi_0 = \phi$ and $\phi_1 =$ identity. Then $\psi: W^{n+1} \leftrightarrow W^{n+1}$ defined by taking

$$\psi(z) = z, \quad \text{if a point } z \in Cl(W^{n+1} - U(M^n, W^{n+1}))$$

and

$$\psi(z) = (\phi_{|s|}(x), s), \text{ if } z \in U(M^n, W^{n+1}) \text{ and } z = (x, s)$$

is the required homeomorphism.

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