W. Fulks and J. S. Maybee
Osaka Math. J.
12 (1960), 1–19.

A Singular Non-Linear Equation

By W. Fulks[1] and J. S. Maybee[2]

1. Introduction

To begin with, we wish to illustrate the physical problem which leads to the following mathematical work.

Let R be a region of three dimensional space occupied by an electrical conductor. Then each point in R becomes a source of heat as a current is passed through R. Let $u(x, t)$ be the temperature at the point $x \in R$ and at time t, and suppose that a function $E(x, t)$ which describes the local voltage drop in R is given as a function of position and time. Then if $\sigma(u)$ is the electrical resistivity which is, in general, a function of the temperature u, the rate of generation of heat at any point x at time t is $E^2(x, t)/\sigma(u)$. Let c and κ be the specific heat and thermal conductivity of R, respectively, which we take to be constant. Then the temperature satisfies the equation,

$$cu_t - \kappa\Delta u = E^2(x, t)/\sigma(u)\,,$$

in the simplest case $\sigma(u) = \alpha u$ where α is a positive constant. More generally σ can be assumed to be a positive function of u which is increasing with u and which tends to zero with u. Thus the differential equation is singular in the sense that the right hand side becomes unbounded at $u=0$.

This physical problem leads naturally then to the study of the differential equation

$$u_t - \Delta u = F(x, t, u)$$

where Δ is the Laplace operator in E^N. We will write $Hu = u_t - \Delta u$ and call H the heat operator. Our equation then becomes

$$Hu = F(x, t, u)\,. \tag{1}$$

---

1) Work on this paper was supported by the Office of Naval Research, Contract Nonr 710 (16) at the University of Minnesota.

2) Part of the results contained in this paper were obtained under the sponsorship of the National Science Foundation, Contract NSF-G 6331, New York University. Reproduction in whole or in part permitted for any purpose of the United States Government.

We study this equation in a bounded region (open connected set—but not necessarily simply connected) R in E^N, or rather the space time cylinder built on R. We consider both the case where the time interval is finite and the steady state case where the time becomes infinite. In the latter case we adjoin the improper point $t=\infty$ to the t-axis and carry out our study in the resulting compactified space time cylinder.

We set up some standard notations which we will use consistently:

B is the boundary of R,

$R_T = R\otimes(0, T)$ $T<\infty$, and

$R_\infty = R\otimes(0, \infty]$

$B_T = \{B\otimes[0, T]\}\cup\{R\otimes(0)\}$, $0<T\leq\infty$.

B_T will be called the *lower boundary* of R_T. In general the lower boundary of a space time cylinder will be the union of the bottom and the sides.

In the steady state cases time dependent functions, (e.g. $F(x, t, u)$) which have values at $t=\infty$ will have the t variable suppressed at $t=\infty$: $F(x, \infty, u)=F(x, u)$. And by $Hu(x, t)=F(x, t, u(x, t))$ at $t=\infty$ we will mean $\Delta u(x)=-F(x, u(x))$.

Throughout the paper we will make the following assumptions on $F(x, t, u)$. These will be referred to as assumptions A:

A: (a) For no $t_0>0$ is $F(x, t, u)\equiv 0$ for $0<t<t_0$.

(b) $F(x, t, u)\geq 0$ $(x, t)\in R_T$, $u>0$.

(c) $F(x, t, u)$ is locally Hölder continuous for $(x, t)\in R_T$, $u>0$. (In case $T=\infty$ local Hölder continuity means what it says for finite t, and that $F(x, u)$ is locally Hölder continuous for $x\in R$, $u>0$.)

(d) For each $c>0$, there is an $M(c)$ for which

$$-M(c)\leq\frac{F(x, t, u)-F(x, t, v)}{u-v}\leq 0 \qquad \text{for all} \quad u\geq c,\ v\geq c.$$

Finally in case $T=\infty$ we also assume

(e) For each $c>0$, $\varepsilon>0$ there are two positive numbers $\delta(\varepsilon, c)$, $T(\varepsilon, c)$ so that

$$|F(x, t, u)-F(x, v)|<\varepsilon$$

if

$$|u-v|<\delta, \qquad t>T,\ u\geq c,\ v\geq c.$$

In particular it should be noted that these assumptions place no restrictions on the rate of growth of F near $u=0$. For example

$$F(x, t, u) = \zeta(x, t)u^{-n}, \qquad n > 0$$

and

$$F(x, t, u) = \zeta(x, t)e^{1/u}$$

where ζ satisfies reasonable regularity conditions are perfectly admissible functions. On the other hand F need not grow at all, in fact F can be zero in large portions of the space time cylinder. (The assumption A(a) is of course a normalization rather than a real restriction.) Or of course the growth can be mixed: of a mild even bounded character in parts of R_T, and of a wild unrestrained sort in other parts.

Let $f=f(x, t)$ be a given continuous function defined on B_T:

$$f(x, t) = \begin{cases} \phi(x) & x \in \bar{R},\ t = 0 \\ \psi(x, t) & \begin{cases} x \in B,\ 0 \leq t < T,\ T < \infty \\ \text{or} \\ x \in B,\ 0 \leq t \leq \infty,\ T = \infty . \end{cases} \\ \phi(x) = \psi(x, 0) & x \in B . \end{cases}$$

We seek a solution to the Dirichlet problem

$$\text{I} \qquad \begin{aligned} Hu &= F \qquad (x, t) \in R_T \\ u &= f \qquad (x, t) \in B_T . \end{aligned}$$

That is, we seek a function $u(x, t)$ continuous in $R_T \cup B_T$ which satisfies conditions I. In case $T = \infty$, this means we specify that $f(x, t)$ shall converge uniformly to $f(x)$ on B as $t \to \infty$, and we ask for the solution $u(x, t)$ to converge uniformly to the solution $u(x)$ of the problem

$$\begin{aligned} \Delta u &= -F \qquad x \in R \\ u &= f \qquad x \in B . \end{aligned}$$

This will be handled by proving continuity of $u(x, t)$ in the compact space time cylinder $\bar{R}_\infty = R_\infty \cup B_\infty$.

The region R will be called *regular* for Laplace's equation ($\Delta u = 0$) if for every continuous function $f(x)$ given on B, the Dirichlect problem for $\Delta u = 0$ has a (unique) solution, that is, if for each continuous f on B there is a function $u(x)$ continuous in $\bar{R}$ for which

$$\begin{aligned} \Delta u &= 0 \qquad x \in R , \\ u &= f \qquad x \in B . \end{aligned}$$

We can now state our "*Hauptsatz*"

Theorem. *If R is regular for Laplace's equation, and if $F(x, t, u)$*

satisfies assumptions A, then problem I:

$$\begin{aligned} Hu &= F \qquad (x, t) \in R_T \\ u &= f \qquad (x, t) \in B_T \end{aligned}$$

has a unique solution for every continuous non-negative data function f given on B_T for all T, $0 < T \leq \infty$.

In the case $T = \infty$ this theorem contains the convergence to the steady state and thereby the existence of the solution of the problem

$$\begin{aligned} \Delta u &= -F(x, u) \qquad x \in R \\ u &= f \qquad\qquad x \in B \end{aligned}$$

if f is continuous on B, and if F satisfies (b), (c), (e) of assumptions A. For (a) is clearly superfluous and (d) will be automatically satisfied. If we define $\phi(x)$ to be e.g. the solution to

$$\begin{aligned} \Delta u &= 0 \qquad x \in R \\ u &= f \qquad x \in B, \end{aligned}$$

and

$$f(x, t) = \begin{cases} \phi(x) & x \in R,\ t = 0 \\ f(x) & x \in B,\ t \geq 0, \end{cases}$$

then the steady state of the problem

$$\begin{aligned} Hu &= F \qquad x \in R_\infty \\ u &= f \qquad x \in B_\infty \end{aligned}$$

leads to the desired solution of the problem posed at the beginning of this paragraph.

Similar problems have been recently considered by Friedman [3][1], but he does not consider the case where F becomes singular.

2. Uniqueness and Preliminary Existence Theorem

Let u and v be two solutions of $Hu = F$. Denote by w their difference: $w = u - v$. Then w satisfies

$$Hw = F(x, t, u) - F(x, t, v) = \frac{F(x, t, u) - F(x, t, v)}{u - v} w .$$

that is, w satisfies a linear equation

$$\text{II} \qquad Hw = cw, \qquad c \leq 0,$$

from which it follows (Nirenberg [5]) that w cannot attain a positive

1) Numbers in brackets refer to the bibliography at the end of the paper.

interior maximum nor negative interior minimum without being constant for all previous time. Hence if, as $(x, t) \to B_T$, we have for $m, k \geq 0$

$$-m \leq \liminf w(x, t) \leq \limsup w(x, t) \leq k$$

then

$$-m \leq w(x, t) \leq k, \qquad (x, t) \in R_T.$$

In case $T=\infty$, the previous inequality holds for all $t<\infty$, and hence for $t=\infty$ also, by passing to the limit.

This maximum principle establishes uniqueness of the solutions of boundary value problems for R_T in the class of functions which do not vanish in the interior. For if u and v were two such solutions then w would achieve zero boundary and inital values, so m and k could both be 0.

REMARK: By virtue of the fact that $F \geq 0$ any solution of $Hu=F$ is a super-parabolic function. And it is a well known property of such functions that they cannot attain a minimum at an interior *point* without being identically constant for all previous time. If by super-parabolic in R_∞ we mean super-parabolic for $t<\infty$ and super-harmonic at $t=\infty$, then the minimum cannot be attained on the plane $t=\infty$ without being identically constant for all t, $0<t\leq\infty$. This is an easy extension of the theorem for finite t. In particular this implies then that all solutions of $Hu=F$ in R_T are larger than or equal to the infimum of their boundary limits. (See Doob [2] for some of the ideas mentioned here.)

We turn now to existence of the solution in the finite case where $T<\infty$, and where the data function f is bounded away from zero.

Let $v(x, t)$ be the unique solution of

$$\begin{aligned} Hv &= 0 \qquad (x, t) \in R_T \\ v &= f \qquad (x, t) \in B_T \end{aligned}$$

where f is a given continuous data function on B_T. v exists since R is regular for Laplace's equation and hence for the heat equation (Tychonoff [6], or Fulks [3]). From the regularity of R for $Hu=0$ it also follows that Green's function for the heat equation exists for R.

LEMMA 1. *Let F satisfy assumptions A, and suppose that f is continuous and $f \geq \alpha > 0$ on B_T (α constant). Then the solution of the Dirichlet problem I is equivalent to the solution of the integral equation*

$$\text{III} \qquad u(x, t) = v(x, t) + \int_0^t \int_R G(x, y, t-\tau) F(y, \tau, u(y, \tau))\, dy d\tau$$

where $dy = dy_1\, dy_1 \cdots dy_N$, and $v(x, t)$ is defined above.

This is a standard result.

LEMMA 2. *Under the conditions of Lemma 1, the integral equation III has a unqiue solution $u(x, t)$ for $0<t<\tau_0$ where τ_0 depends only upon α, if R is regular for Laplace's equation.*

Proof: We will need the following well known properties of Green's function $G(x, y, t)$ for the heat equation for R:

(i) $H_{xt}G(x, y, t) = H_{yt}G(x, y, t) = 0$ for $x \in R$, $y \in R$, $t>0$

(ii) $G(x, y, t) = 0$ if either $x \in B$ or $y \in B$

(iii) $0 \leq G(x, y, t) \leq k(x-y, t)$ for $x \in R$, $y \in R$, $t>0$

where k is the fundamental solution for the heat equation:

$$k(x-y, t) = (4\pi t)^{-N/2} \exp \{-[\sum_1^N (x_j-y_j)^2]/4t\}$$

We solve the integral equation III by iteration. Let $u_0=v$, and for $n \geq 0$ set

$$(*) \qquad u_{u+1} = v + \int_0^t \int_R G(x, y, t-\tau) F(y, \tau, u_n(y, \tau))\, dy d\tau .$$

Now since our boundary values f are bounded away from zero: $f \geq \alpha_0 > 0$, so is $v \geq \alpha$ and hence so is $u_n \geq \alpha > 0$ for all n.
Denote $\sup\limits_{\substack{x\in R \\ 0\leq t\leq t_0}} |u_n - u_{n-1}|$ by $||u_n - u_{n-1}||_{t_0}$

Then

$$|u_{n+1} - u_n| \leq \int_0^t \int_R G|F(y, \tau, u_n) - F(y, \tau, u_{n-1})|\, dy d\tau$$

$$\leq M(\alpha) \int_0^t \int_R G|u_u - u_{n-1}|\, dy d\tau .$$

Hence for any $t_0 < T$

$$||u_{n+1} - u_n||_{t_0} \leq ||u_n - u_{n-1}||_{t_0} M(\alpha) \int_0^{t_0} \int_R G dy d\tau$$

But $\int_R G dy \leq \int_R k dy \leq \int_{EN} k dy = 1$, so that

$$||u_{n+1} - u_n||_{t_0} \leq ||u_n - u_{n-1}||_{t_0} M(\alpha) \cdot t_0 .$$

Hence if we take t_0 to be any fixed number $<\tau_0 \equiv 1/M(\alpha)$ we will have uniform convergence for $0 \leq t \leq t_0$. The limit function u is clearly a solution to the integral equation III for $0 \leq t < \tau_0$, for one can pass to the limit in (*) since F is bounded because $u_n \geq \alpha$.

This proves the existence. The uniqueness has already been established.

Lemma 3. *If F satisfies assumption A, and if f is continuous and $f \geq \alpha > 0$ on B_T (α constant), then problem I has a unique solution for $T < \infty$.*

If $T \leq \tau_0$ of Lemma 2 the proof is complete. If $T > \tau_0$ then let $u_1(x, t)$ be solution defined for $0 \leq t < \tau_0$. Now choose any $t_0 < \tau_0$ and pose the problem

$$\begin{aligned} &Hu_2 = F(x, t, u_2) && t > t_0 \\ &u_2(x, t_0) = u_1(x, t_0) && \\ &u_2(x, t) = f(x, t) && x \in B,\ t \geq t_0 . \end{aligned}$$

By Lemma 2 this has a solution for $t_0 \leq t < t_0 + \tau_0$. By iterating this procedure we build up a function, continuous in $R_T \cup B_T$, in a finite number of steps. It obviously takes on the appropriate boundary vlues, and initial values. But is it a solution of the equation? In each layer this is clear. But to see that it is a solution compare u_1 and u_2 in the interval $t_0 < t < \tau_0$. They achieve the same values at $t = t_0$, and they achieve the same boundary values. Hence by the uniqueness argument given earlier they are identical, so that u_2 is a continuation of u_1 into the second layer in such a manner as to make the resulting function a solution of $Hu = F$ across the plane $t = \tau_0$. By induction the argument proceeds from one layer to the next.

We wind up this section with the following:

Lemma 4. *Let $u_1, u_2, \cdots, u_n, \cdots$, be a sequence of solutions of $Hu = F$ in a space time cylinder R_T, and suppose that on each compact subset of R_T we have u_n converging uniformly to a non-vanishing limit function u, then u is also a solution of $Hu = F$ in R_T.*

Proof: Let (x_0, t_0) be a point in R_T (if $T = \infty$ the case $t_0 = \infty$ will be handled later). Then let S be a sphere centered at x_0 with $\bar{S}$ entirely in R. We consider the space time cylinder built on S from time $t_0 - \varepsilon$ to $t_0 + \varepsilon$, where ε is sufficiently small that the closed cylinder lies in R_T. On the lower boundary of this cylinder, i.e., for $x \in S$, $t = t_0 - \varepsilon$ and for $x \in \partial S$, $t_0 - \varepsilon \leq t < t_0 + \varepsilon$, u_n defines a continuous function. Let v_n be the solution of the problem $Hv = 0$ in the interior and $v = u_n$ on the lower boundary of the cylinder. Now since v_n converges uniformly on the boundary, then by the maximum principle it must do so in the interior. And if $G'(x, y, t)$ is the Green's function for S for the heat equation, then

$$u_n = v_n + \int_{t_0 - \varepsilon}^{t} \int_S G'(x, y, t - \tau) F(y, \tau, u_n(y, \tau))\, dy\, d\tau .$$

Now since u_n is bounded away from zero in this cylinder $F(y, \tau, u_n)$

converges uniformly to $F(y, \tau, u)$ so passing to the limit on both sides we get

$$u = v + \int_{t_0-\varepsilon}^{t} \int_S G'(x, y, t-\tau) F(y, \tau, u(y, \tau)) dy d\tau$$

so that u is a solution of $Hu=F$ at (x_0, t_0), and hence for all (x, t) in R_T for finite time.

If T is infinite we discuss the surface $t=\infty$ separately. There u_n solves the equation $\Delta u = -F(x, u)$. Then the same sort of analysis based on the equation

$$u_n(x) = v_n(x) + \int_S G'(x, y) F(y, u_n(y)) dy$$

where G' is the Green's function for Laplaces equation for the sphere S, yields the same result. This proves the lemma.

3. Certain Steady State Problems

We begin with the study of the steady state solution of

$$Hu = g(x, t) . \tag{2}$$

This is covered by recent work of A. Friedman [3] but the proof presented here is of a more elementary nature.

We assume that $g(x, t)$ is given as a bounded continuous function in R_∞, and that as $t \to \infty$ we have $g(x, t)$ converging uniformly to $g(x, \infty) = g(x)$ for $x \in R$. We also assume that $g(x, t)$ is locally Hölder continuous in R_∞. By this we mean that it is locally Hölder continuous in R_T for every finite T and that $g(x)$ is locally Hölder continuous in R. We then can prove

LEMMA 5. *Let R be regular for Laplace's equation, and let f be continuous on B_∞. Then the Dirichlet problem*

$$\begin{aligned} Hu &= g \qquad (x, t) \in R_\infty \\ u &= f \qquad (x, t) \in B_\infty \end{aligned}$$

has a unique solution in B_∞, if g satisfies the conditions of the previous paragraph.

For finite t the solution is given by

$$u(x, t) = v(x, t) + \int_0^t \int_R g(x, y, t-\tau) g(y, \tau) dy d\tau$$

where, as before, v is the solution of

$$\begin{aligned} Hu &= 0 \qquad (x, t) \in R_\infty , \\ u &= f \qquad (x, t) \in B_\infty . \end{aligned}$$

For $t=\infty$ the solution is given by

$$u(x) = v(x) + \int_R G(x, y)\, g(y)\, dy ,$$

where $v(x)$ is the solution of

$$\begin{aligned} \Delta u &= 0 \qquad x \in R \\ u &= f \qquad x \in B . \end{aligned}$$

and $G(x, y)$ is Green's function for Laplace's equation for R.

Our task is to prove that as $t \to \infty$, $u(x, t)$ converges uniformly to $u(x)$.

It is known (Tychonoff [6]) that $v(x, t)$ converges uniformly to $v(x)$, and, from the same source, that

$$\int_0^\infty G(x, y, t)\, dt = G(x, y) .$$

Actually Tychonoff proved this relationship between $G(x, y, t)$ and $G(x, y)$ only for $N=3$. But the proof is valid without change for $N>3$ and is not difficult to establish for $N=2$ and $N=1$.

It is therefore sufficient to establish the uniform convergence to zero of

$$I = \int_0^t \int_R G(x, y, t-\tau)\, g(y, \tau)\, dy\, d\tau - \int_R G(x, y)\, g(y)\, dy .$$

We write this in the form

$$\begin{aligned} I &= \int_R \left\{ \int_0^t G(x, y, t-\tau)\, g(y, \tau)\, d\tau - g(y) \int_0^t G(x, y, t-\tau)\, d\tau \right\} dy \\ &\quad + \int_R g(y) \left\{ \int_0^t G(x, y, t-\tau)\, d\tau - G(x, y) \right\} dy \\ &= I_1 + I_2 \text{ respectively.} \end{aligned}$$

Now

$$\int_R g(y) \int_0^t G(x, y, t-\tau)\, d\tau\, dy = \int_R g(y) \int_0^t G(x, y, \tau)\, d\tau\, dy ,$$

and the integrand of the outer integral, namely

$$g(y) \int_0^t G(x, y, \tau)\, d\tau$$

is monotone increasing. By the monotone convergence theorem we can pass to the limit under the integral sign to obtain

$$\int_R g(y) \int_0^t G(x, y, \tau)\, d\tau\, dy \to \int_R g(y)\, G(x, y)\, dy$$

for each fixed x. To see that the convergence is uniform we note that the left hand side is converging monotonically to the right hand side. Both sides are continuous functions of x in $\bar{R}$. Thus by a theorem of Dini (Courant-Hilbert Vol. 1 [1]) the convergence is uniform. That is, I_2 converges uniformly to zero.

We now consider the integral I_1. Let

$$M = \max_{x \in R} \int_R G(x, y)\, dy .$$

and given $\varepsilon > 0$ choose T so large that

$$|g(y, t) - g(y)| < \frac{\varepsilon}{2M} \qquad \text{for all } y \in R \text{ if } t > T .$$

Then for $t > T$ we have

$$\begin{aligned} I_1 &= \int_R \int_0^{t-T} G(x, y, s)[g(y, t-s) - g(y)]\, ds\, dy \\ &\quad + \int_R \int_{t-T}^{t} G(x, y, s)[g(y, t-s) - g(y)]\, ds\, dy \\ &= I_3 + I_4 \text{ respectively.} \end{aligned}$$

Now

$$\begin{aligned} |I_3| &\leq \frac{\varepsilon}{2M} \int_R \int_0^{t-T} G(x, y, s)\, ds\, dy \leq \frac{\varepsilon}{2M} \int_R G(x, y)\, dy \\ &\leq \frac{\varepsilon}{2} \text{ uniformly for } x \in \bar{R}. \end{aligned}$$

If M' is a bound for $g(x, t)$ in R_∞ and if V is the volume of R, then

$$\begin{aligned} |I_4| &\leq \int_R \int_{t-T}^{t} G(x, y, s)\, 2M'\, dy\, ds \\ &\leq 2VM' \int_{t-T}^{t} k(x-y, s)\, ds \\ &\leq 2VM' \int_{t-T}^{t} (4\pi s)^{-N/2}\, ds \\ &\leq \text{const.} \begin{cases} \log t/(t-T) & \text{if } N = 2 \\ \dfrac{1}{t^{N/2-1}} - \dfrac{1}{(t-T)^{N/2-1}} & \text{if } N \geq 3 . \end{cases} \end{aligned}$$

These two estimates prove uniform convergence by the Cauchy criterion. This establishes the lemma for $N \geq 2$. For $N=1$ it is not difficult to see that $\int_0^t G(x, y, t)\,dt$ converges uniformly as $t \to \infty$ for $x \in R$ (which is now an interval). This is sufficient by the Cauchy criterion to again guarantee that I_4 can be made uniformly small for t large. This secures the lemma for the case $N=1$.

Let us now return to equation (1): $Hu=F$. We want to examine the steady state, i.e., the solution of the Dirichlet problem in R_∞. We assume of course that F satisfies assumptions A and that $f \geq \alpha > 0$ on B_∞. We examine the equivalent integral equation III and iterate as before, taking $u_0 = v$ and

$$u_{n+1} = v + \int_0^t \int_R G\{F(y, \tau, u_n)\,dy\,d\tau\,.$$

These iterates are each defined in R_T for every $T < \infty$ (though convergence of the sequence was proved only for $0 \leq t < \tau_0$ for τ_0 sufficiently small). It is the behavior of these functions in R_∞ that we will now examine.

Now as remarked before $u_0 = v(x, t)$ converges to a steady state solution (Tychonoff [6]). That is u_0 is the unique solution of

$$\begin{aligned} Hu_0 &= 0 \qquad (x, t) \in R_\infty \\ u_0 &= f \qquad (x, t) \in B_\infty\,. \end{aligned}$$

Hence $F(x, t, u_0(x, t))$ is defined, locally Hölder continuous, bounded (since $u_0 \geq \alpha$) and $F(x, t, u_0(x, t))$ converges uniformly to $F(x, u_0(x))$ as $t \to \infty$ by (e) of assumptions A and by the uniform convergence of $u_0(x, t)$ to $u_0(x)$. Thus $F(x, t, u_0(x, t))$ is an admissible $g(x, t)$ for Lemma 5, hence by this lemma the problem

$$\begin{aligned} Hu &= F(x, t, u_0) \qquad (x, t) \in R_\infty \\ u &= f \qquad (x, t) \in B_\infty \end{aligned}$$

has a unique solution. In other words $u_1(x, t)$ is continuous in $\bar{R}_\infty = R_\infty \cup B_\infty$. It is clear that by induction $u_{n+1}(x, t)$ is continuous in $\bar{R}_\infty$ and solves the problem

$$\begin{aligned} Hu_{n+1} &= F(x, t, u_n) \qquad (x, t) \in R_\infty \\ u_{n+1} &= f \qquad (x, t) \in B_\infty \end{aligned}$$

for each $n = 0, 1, 2, \cdots$.

Now let $\|u_n - u_{n-1}\| \doteq \|u_n - u_{n-1}\|_\infty = \sup_{\bar{R}_\infty} |u_n - u_{n+1}|$ and observe

$$|u_{n+1}-u_n| \leq \int_0^t \int_R G(x, y, t-\tau) \left| \frac{F(y, \tau, u_n)-F(y, \tau, u_{n-1})}{u_n-u_{n-1}} \right| |u_n-u_{n-1}| \, dy \, d\tau$$

$$\leq \|u_n-u_{n-1}\| M(\alpha) \cdot \int_R \int_0^t G(x, y, s) \, ds \, dy .$$

Thus

$$\|u_{n+1}-u_n\| \leq \|u_n-u_{n-1}\| M(\alpha) \cdot \int_R G(x, y) \, dy .$$

Hence the sequence converges uniformly in R_∞ if the volume of R is so small that

$$M(\alpha) \int_R G(x, y) \, dy < 1 .$$

It is clear that this condition can be achieved restricting only the volume of R, for if a be the radius of a sphere with the same volume as R, then

$$\int_R G dy \leq \frac{1}{\Omega} \int_R \frac{1}{r^{N-2}} r^{N-1} dr \, d\omega = \int_0^a r \, dr = a^2/2 .$$

(If $N=2$ or 1, the calculations are slightly different.)

We have thus proved

LEMMA 6. *Suppose F satisfies assumptions A, and that $f \geq \alpha > 0$ and is continuous on B_∞, then there is a number $v(\alpha)$ depending only on α so that problem I for $T=\infty$:*

$$\begin{aligned} Hu &= F \qquad (x, t) \in R_\infty \\ u &= f \qquad (x, t) \in B_\infty , \end{aligned}$$

has a unique solution if R is regular for Laplace's equation and if the volume of R is less than $v(\alpha)$.

Suppose R is a bounded region, regular for Laplace's equation whose boundary B is the union of two disjoint closed sets B' and B''. Then the problem

$$Hw = 0 \qquad (x, t) \in R_\infty$$

$$w = \begin{cases} 0 & x \in R, \ t=0 \\ 0 & x \in B', \ 0 \leq t \leq \infty \\ 1 & x \in B'', \ 0 < t \leq \infty \end{cases}$$

has a unique solution, which we will call w, in R_∞. It is of course not continuous at the points $x \in B''$, $t=0$, but it is continuous elsewhere in $\bar{R}$, and lies everywhere between 0 and 1 (see Tychonoff [6]).

Now $w(x, t)$ cannot attain the value 1 at any point $(x, t) \in R_\infty$: if t were finite then $w(x, t)$ would be identically 1 for all previous time, and if t were ∞ then $w(x) = w(x, \infty)$ would be identically 1 in $\bar{R}$. But neither of these can obtain since $w(x, t) = 0$ if $x \in B'$, $0 \leq t \leq \infty$. Thus for any compact subset D of R there is a number q, $0 < q < 1$ for which

$$0 \leq w < q \qquad (x, t) \in D \otimes [0, \infty].$$

This is because the set $D \otimes [0; \infty]$ is a compact set on which w is continuous, non-negative, and everywhere less than 1.

LEMMA 7. *Let R be a region regular for Laplace's equation whose boundary B is the union of two disjoint closed sets B' and B''. Let w satisfy equation II*:

$$Hw = cw \qquad c \leq 0$$

in R_∞. Suppose further that

$$\lim_{(x,t) \to (\xi,\tau)} w(x, t) = 0 \qquad \textit{if} \quad (\xi, \tau) \in R \times \{(0)\}$$
$$\textit{or if} \quad \xi \in B',\ 0 \leq \xi \leq \infty$$

and that

$$m = \sup \left[\lim_{(x,t) \to (\xi,\tau)} |w(x, t)| \right]$$

where $\xi \in B''$ and $0 \leq \tau \leq \infty$ and the sup *is taken over all such points (ξ, τ). Then for each compact subset D of R there exists a number q, $0 < q < 1$, depending only on D for which*

$$|w| \leq mq \qquad \textit{for} \quad (x, t) \in D \otimes [0, \infty].$$

Proof: Let

$$\nu = w - mw$$

where w is the function defined above. Now

$$H\nu = Hw - mHw = cw = c\nu + cmw .$$

But $cmw \leq 0$, $c \leq 0$ and the boundary values of $\nu \leq 0$. We want to show that $\nu \leq 0$. For suppose it were positive at some point. Then it would assume a positive maximum at some point in R_T. And by Nirenberg's maximum principle it would be constant on a set which reaches out to the boundary of the set where it is positive, but by continuity it must vanish there. Hence we have a contradiction. Thus

$$\nu \leq 0$$
$$\text{or} \quad w \leq mw .$$

Similarly, by considering $\mu = w + mw$ one sees that $\mu \geq 0$ or

$$w \geq -mw,$$

so that

$$|w| \leq mw.$$

Then on $D \otimes [0, \infty]$ we have

$$|w| \leq mw \leq mq.$$

We can now prove

LEMMA 8. *Let R be a bounded region regular for Laplace's equation and suppose F satisfies assumptions A. Suppose also that f is a given continuous function on B_∞ with $f \geq \alpha > 0$ (α constant). Then the problem I with $T = \infty$:*

$$\begin{aligned} Hu &= F \qquad (x, t) \in R_\infty \\ u &= f \qquad (x, t) \in B_\infty \end{aligned}$$

has a unique solution.

Proof: Our region R is a bounded open set and therefore has finite volume. We can then represent R as the union of a finite number of open sets R^j, $j = 0, 1, \cdots, k$, with the following properties:

(i) $R^{j+1} \supset R^j$ (ii) $R^k \equiv R$

(iii) each point of the boundary of R_j is a regular point for both the exterior and the interior. (iv) if V_j is the volume of R^j, then

$$V_j < V(\alpha)(1+j)/2.$$

The R^j can be constructed, for example, by taking appropriate unions of spheres.

We proceed by induction: we suppose that we can solve the Dirichlet problem for R^j, $j \geq 1$, and we show that we can then solve it for R^{j+1}. Since we can clearly solve it for R^1, we are led in a finite number of steps to R itself. We use a variation of Schwartz's alternating method.

First we note that the "rings" of the form $R^{j+1} - R^{j-1}$ are regular for Laplace's equation. (These "rings" may of course have high connectivity, but we may visualize them as annuli.) This is because the boundary of the ring consists of the two disjoint sets B^{j+1} and B^{j-1}, the boundaries of R^{j+1} and R^{j-1} respectively. Now each point of B^{j+1} is a regular point for the interiors of R^{j+1} and hence for the ring, and each point of B^{j-1} is a regular point for the exterior of R^{j-1} and hence for the ring.

Let us denote $R^{j+1}-\bar{R}^{j-1}$ by S and its boundary by C:

$$S = R^{j+1}-R^{j-1}$$
$$C = B^{j+1}\cup B^{j-1}.$$

Now suppose f given on B_∞^{j+1}, $f\geq\alpha$, and f continuous. We want to show that we can solve the Dirichlet problem with these boundary values. Define a boundary function f over the lower boundary (see introduction) of S, i.e., over C_∞: for $x\in S$, $t=0$, and for $x\in B^{j+1}$, $0\leq t\leq\infty$ we retain the given values of f, but for $x\in B^{j-1}$, $0\leq t\leq\infty$ we define f_1 by

$$f_1(x, t) = f(x, 0)$$

i.e., we extend f up the generators of the cylinder so as to be constant on each generator. Then f_1 is continuous in C_∞ and $f_1\geq\alpha$. Now S is a regular region whose volume is $V_{j+1}-V_{j-1}<V(\alpha)$, so that we can solve the problem

$$\begin{aligned} Hu &= F \qquad (x, t)\in S_\infty \\ u &= f_1 \qquad (x, t)\in C_\infty . \end{aligned}$$

Let u_1 be the solution of this problem. u_1 is now continuous in $\bar{C}_\infty$ and coincides with f for $x\in S$, $t=0$. Hence the function ϕ_1 defined on B_∞^j by

$$\phi_1 = \begin{cases} f & x\in R^j,\ t=0 \\ u_1 & x\in B^j,\ 0\leq t\leq\infty \end{cases}$$

is continuous and $\geq\alpha$, and by hypothesis the solution of the problem

$$\begin{aligned} Hu &= F \qquad (x, t)\in R_\infty^j \\ u &= \phi_1 \qquad (x, t)\in B_\infty^j \end{aligned}$$

exists. It will be called v_1. Now $v_1\geq\alpha$ since it is super-parabolic in R_T^j for every T, and super-harmonic in R for $t=\infty$.

The function f_2 defined to be f for $x\in S$, $t=0$ and $x\in B^{j+1}$, $0\leq t\leq\infty$ and to be v_1 for $x\in B^{j-1}$, $0\leq t\leq\infty$ is then continuous and $\geq\alpha$. The problem

$$\begin{aligned} Hu &= F \qquad (x, t)\in S_\infty \\ u &= f_2 \qquad (x, t)\in C_\infty \end{aligned}$$

has then a solution u_2 in $\bar{S}_\infty$. This gives rise to a ϕ_2:

$$\phi_2 = \begin{cases} f & x\in R^j,\ t=0 \\ u_2 & x\in B^j,\ 0\leq t\leq\infty . \end{cases}$$

which leads to a solution v_2 of the problem

$$Hu = F \qquad (x, t) \in R^j_\infty$$
$$u = \phi_2 \qquad (x, t) \in B^j_\infty .$$

By continuing this alternate solving of problems for S_∞ and R^j_∞ we define sequences v_n and u_n of solutions of $Hu=F$ in R^j_∞ and R_∞ respectively. Each v_n is continuous in $\bar{R}^j_\infty$ and each u_n is continuous in $\bar{S}_\infty$. Also v_n coincides with f for $t=0$, $x \in R^j$, and u_n coincides with f for $x \in S$, $t=0$, and for $x \in B^{j+1}$, $0 \leq t \leq \infty$.

Let us now examine $w_n \equiv u_{n+1} - u_n$ in S:

$$Hw_n = \frac{F(x, t, u_{n+1}) - F(x, t, u_n)}{u_{n+1} - u_n} w_n .$$

Hence $Hw_n = c_n w_n$, $c_n \leq 0$. Now w_n vanishes for $x \in S$, $t=0$, and for $x \in B^{j+1}$, $0 \leq t \leq \infty$, and is bounded (by M_n say) for $x \in B^{j-1}$, $0 \leq t \leq \infty$. Then by Lemma 7 there is a $q<1$, independent of n, for which

$$|w_n| \leq qM_n \qquad \text{for} \quad x \in B^j,\ 0 \leq t \leq \infty .$$

We next examine $z_n \equiv v_{n+1} - v_n$ in R^j_∞. Now z_n vanishes for $t=0$, $x \in B^j$, and coincides with w_n for $x \in B^j$, $0 \leq t \leq \infty$. And z_n also satisfies a linear equation of the form II:

$$Hz_n = c'_n z_n , \qquad c'_n \leq 0 ,$$

so, by the maximum principle, $|z_n| \leq M_n q$ in $\bar{R}^j_\infty$. But for $x \in B^{j-1}$, $0 \leq t \leq \infty$, we have $w_{n+1} = z_n$, and on the rest of C_∞ we have $w_{n+1} = 0$. Hence

$$M_{n+1} \leq qM_n .$$

So, by induction

$$|w_n| = |u_{n+1} - u_n| \leq q^{n-1} M_1 \qquad (x, t) \in S_\infty$$

and

$$|z_n| = |v_{n+1} - v_n| \leq q^n M_1 \qquad (x, t) \in \bar{R}_\infty .$$

Consequently the sequences u_n and v_n converge uniformly in their respective domains of definition. The limit functions, denoted by u and v respecively, are clearly solutions of the differential equation $Hu=F$. And $u=f$ for $x \in S$, $t=0$, and $x \in B^{j+1}$, $0 \leq t \leq 0$ and $v=f$ for $x \in R^j$, $t=0$.

We consider now the difference $u_n - v_n$ in its domain of definition, namely $\bar{R}^j_\infty \cap \bar{S}_\infty$. For $t=0$ or for $x \in B^j$, $0 \leq t \leq \infty$ the difference is zero. For $x \in B^{j-1}$, $0 \leq t \leq \infty$, $u_n \equiv v_{n+1}$, so there the difference is bounded by

$M_1 q^{n-1}$. From this it follows that $u \equiv v$ in this domain. That is the function $\bar{u}$ defined by

$$\bar{u} = \begin{cases} u & (x, t) \in \bar{S}_\infty \\ v & (x, t) \in \bar{R}_\infty^j \end{cases}$$

satisfies both the differential equation and the boundary values. That is $\bar{u}$ is the solution of the problem

$$\begin{aligned} Hu &= F \qquad (x, t) \in R_\infty^{j+1} \\ u &= f \qquad (x, t) \in B_\infty^{j+1} . \end{aligned}$$

This completes the proof of Lemma 8 and gives our general existence theorem for boundary values bounded away from zero.

4. The Proof of the Theorem

We can now give the proof of the theorem stated in the introduction. Accordingly we suppose that F satisfies assumptions A, that f is continuous on B_T ($0 < T \leq \infty$), that $f \geq 0$, and of course that R is regular for Laplace's equation. Let α_n be a decreasing sequence of constants with limit zero. Each of the problems

$$\begin{aligned} Hu &= F \qquad (x, t) \in R_T \\ u &= f + \alpha_n \qquad (x, t) \in B_T \end{aligned}$$

has a unique solution in R_T by Lemma 3 or 8 according as $T < \infty$ or $T = \infty$.

The differences $u_n - u_{n+k}$ satisfy an equation of type II, so that by the maximum principle

$$0 \leq u_n - u_{n+k} \leq \alpha_n - \alpha_{n+k} < \alpha_n .$$

From this we can assert that the sequence $\{u_n\}$ is non-increasing and converges uniformly in $R_T \cup B_T$ to a limit function u which must therefore be non-negative (since each $u_n \geq \alpha_n > 0$) and continuous.

We now show that u satisfies the proper boundary values. Consider for arbitrary n:

$$0 \leq u_n - u = \lim_{k \to \infty} (n_n - u_{n+k}) \leq \lim_{k \to \infty} (\alpha_n - \alpha_{n+k}) = \alpha_u .$$

Let (x, t) tend to an arbitrary point $(\xi, \tau) \in B_T$. Then

$$0 \leq f + \alpha_n - \overline{\lim}\, u \leq f + \alpha_n - \underline{\lim}\, u \leq \alpha_n ,$$

and n tends to infinity we get

$$0 \leq f - \overline{\lim}\, u \leq f - \overline{\lim}\, u \leq 0 .$$

This implies $\lim u = f$ at each point of B_T.

Next we prove that u does not vanish in R_T. To see this we observe that u is a continuous super-parabolic function since it is the uniform limit of continuous super-parabolic functions. Hence, by the remark in section 2, if it vanishes at a point (x_0, t_0) it must vanish for all previous time, and thus vanishes at some finite point (x_1, t_1), (for t_0 may be infinite) and for all, t, $0 < t < t_1$.

By assumption $A(a)$, $F(x, t, u) \not\equiv 0$ for $0 < t < t_1$, so that there is a point $(x_2\, t_2) \in R_{t_1}$ and a number $\varepsilon > 0$ for which $F(x_2, t_2, \varepsilon) > 0$. Thus by continuity there is a neighborhood of (x_2, t_2), lying in R_{t_1}, and a number $\eta > 0$ for which $F(x, t, \varepsilon) > \eta$ in the neighborhood. Since the sequence u_n converges monotonically and uniformly to 0 in R_{t_1}, there is an n for which

$$u_{n+k} \leq u_n \leq \varepsilon \qquad (x, t) \in R_{t_1} .$$

Then

$$F(x, t, u_{n+k}) \geq F(x, t, u_n) \geq F(x, t, \varepsilon) \qquad (x, t) \in R_{t_1} .$$

Let v_j be the solution to the problem

$$\begin{aligned} Hv &= 0 \qquad && (x, t) \in R_T \\ v &= f + \alpha_j \qquad && (x, t) \in B_T . \end{aligned}$$

Then

$$\begin{aligned} u_{n+k}(x_1, t_1) &= v_{n+k}(x_1, t_1) + \int_0^{t_1} \int_R G(x_1, y, t_1 - \tau) F(y, \tau, u_{n+k})\, dy\, d\tau \\ &\geq \int_0^{t_1} \int_R GF(y, \tau, u_{n+k})\, dy\, d\tau \\ &\geq \int_0^{t_1} \int_R GF(y, \tau, \varepsilon)\, dy\, d\tau > 0 , \end{aligned}$$

the last inequality being true since the integrand is positive in a neighborhood.

As k tends to infinity this gives

$$u(x, t) \geq \int_0^t \int_R GF(y, \tau, \varepsilon)\, dy\, d\tau > 0$$

which yields the contradiction.

The proof is completed by invoking Lemma 4 to prove that u is a solution of $Hu = F$ in R_T.

(Received December 1, 1959)

Bibliography

[1] Courant, R. and Hilbert D., Methods of mathematical physics, Vol. 1, New York (1953).

[2] Doob, J. L., A probability approach to the heat equation, Trans. Amer. Math. Soc. **80** (1955).

[3] Friedman, A., Convergence of solutions of parabolic equations to a steady state. J. Math. and Mech. **8** (1959).

[4] Fulks, W., Regular regions for the heat equation, Pac. J. Math. 7 (1957).

[5] Nirenberg, L., A strong maximum principle for parabolic equations. Comm. Pure Appl. Math. **6** (1953).

[6] Tychonoff, A., Sur l'equation de la chaleur le plusieurs variables, Bull. Univ. d'Etat de Moscou I (1938).