Wada, Junzo
Osaka Math. J.
11 (1959), 173–185.

Positive Linear Functionals on Ideals of Continuous Functions

By Junzo WADA

Let N be the set of all continuous functions on a compact Hausdorff space or the set of all continuous functions whose carriers are compact on a locally compact Hausdorff space. Then any positive linear functional T on N has an integral representation (Kakutani [12] and Halmos [8]), so any T has the condition (MA'), i.e. $T(f_n)$ converges to $T(f)$ for any $f \in N$ and for any sequence $\{f_n\} \subset N$ with $f_n \uparrow f$. Let X be a locally compact space and let Y be the one-point compactification of X ([1], p. 93). Then we can regard the set of all continuous functions whose carriers are compact on X as an ideal ($=l$-ideal. § 1) of $C(Y)$, the set of all real-valued continuous functions of Y. V. S. Varadarajan [16] raised the following question: Let X be a compact Hausdorff space and let N be an ideal of $C(X)$. When can we say that all non-negative linear functionals on N satisfy the condition (MA')[1]? An ideal N is said to satisfy the property (A) if T satisfies the condition (MA') for any non-negative linear functional T on N (§ 1). In this paper we consider more generalized problems. After some preliminaries in § 1 we consider in § 2 the above problem in the case where X is a completely regular space. We characterize ideals which satisfy the property (A) under some conditions (Theorem 4). In § 3 we prove that any m-closed (ring-) ideal satisfies (A) (Theorem 5), and in § 4 we show that an α-ideal satisfies the stronger property (B) (§ 1) if it satisfies (A) in the case where X is a normal Q-space (Theorem 6).

§ 1. Preliminaries.

Throughout this paper, spaces are always completely regular Hausdorff spaces.

For a space X, a subset N in $C(X)$ will be called an *l-ideal*[2] (or, briebly, an *ideal*) if the following conditions are satisfied:

(i) if $f, g \in N$, then $f+g \in N$,

(ii) if $f \in N$ and t is any real number, then $tf \in N$,

(iii) if $f \in N$, $|g|$[3] $\leq f$, then $g \in N$.

---

1) See, Bourbaki [3]. Varadarajan [16] used the term "σ-smooth" in place of "the condition (MA')". Numbers in bracket refer to the references cites at the end of the paper.

2) See, Birkhoff [2].

3) For any function f, $|f|(x) = |f(x)|$.

Let X be a space and let f be in $C(X)$. Then we put

$$Z(f)=\{x \mid x \in X, \quad f(x)=0\},$$
$$P(f)=\{x \mid x \in X, \quad f(x)>0\},$$
$$\mathfrak{P}(X)=\{P(f) \mid f \in C(X)\}.$$

Let N be an ideal. Then we put

$$Z(N)=\bigcap_{f \in N} Z(f),$$
$$\mathfrak{P}(N)=\{P(f) \mid f \in N\}.$$

Let N be an ideal and let T be a non-negative linear functional. Then T is said to satisfy the *condition* (MA') (resp. (MA)) if $T(f_n)$ (resp. $T(f_\alpha)$) converges to $T(f)$ for any $f \in N$ and for any sequence $\{f_n\} \subset N$ (resp. for any directed set $\{f_\alpha\} \subset N$) with $f_n \uparrow f$ (resp. $f_\alpha \uparrow f$)[4]. An ideal N is said to satisfy the *property* (A) (resp. (B)) if T satisfies the condition (MA') (resp. (MA)) for any non-negative linear functional T on N.

Let N be an ideal. Then we put

$$K=\{f \mid f \in C(X), \quad \varphi_{P(|f|)} \leqslant \text{some } h \in N\},$$
$$K^*=K \frown C^*(X),$$

where $C^*(X)$ is the set of all bounded continuous functions. We denote by φ_A the characteristic function of a set A. We easily see that K and K^* are ideals and both are contained in N. If X is compact, then we have that $K=K^*=\{f \mid f \in C(X)$, the carrier of f is contained in some compact subset of $Y=X-Z(N)\}$.

Let N be an ideal. Then N is called an *α-ideal* if $f \in N$ for any $f \in C(X)$ with $|f| \wedge n \in N$ $(n=1, 2, 3, \cdots)$. If X is compact (or pseudo compact), then any ideal is an α-ideal, and if X is locally compact, the set of all continuous functions whose carriers are compact on X is an α-ideal. If N is an α-ideal and if $f \in K$, then we have that $K \supset \{g \mid g \in C(X), P(|g|) \subset P(f)\}$.

Let X be any space. Then E. Hewitt [10] introduced a Baire measure on $\mathfrak{P}(X)$. Let N be an α-ideal and let T be a non-negative linear functional. Similary, we can introduce a countably additive measure on $\mathfrak{P}(K)$ as follows.

Let G be any set in $\mathfrak{P}(K)$. We define the measure $\gamma(G)$ as sup $T(f)$,

---

4) Let A be a directed system. Then $\{f_\alpha\}_{\alpha \in A}$ is said to be a directed set if for any pair α_1, α_2 with $\alpha_1 \geqslant \alpha_2$, $f_{\alpha_1} \geqslant f_{\alpha_2}$. "$f_\alpha \uparrow f$" means that $\lim_\alpha f_\alpha(x)=f(x)$ for any x. We see that a non-negative linear functional T on N satisfies (MA') (resp. (MA)) if $T(f_n)$ (resp. $T(f_\alpha)$) converges to $T(f)$ for any $f(\geqq 0) \in N$ and for any sequence $\{f_n\} \subset N$ (resp. for any directed set $\{f_\alpha\} \subset N$) such that $f_n \uparrow f$ (resp. $f_\alpha \uparrow f$) and $f_n \geqq 0$ for any n (resp. $f_\alpha \geqslant 0$ for any α) (Cf. [14]).

where f runs through the set of all functions in K such that $0 \leqq f \leqq \varphi_G$. By the similar method as Hewitt [10], we have

(1) a) $G \subset H$ implies that $\gamma(G) \leqq \gamma(H)$,
b) $0 \leqslant \gamma(G) < +\infty$
c) $\gamma(0)=0$,
G and H being arbitrary sets in $\mathfrak{P}(K)$.

(2) $\gamma(G \cup H) \leqq \gamma(G)+\gamma(H)$ for any G, H in $\mathfrak{P}(K)$.

(3) If $G, H \in \mathfrak{P}(K)$ and $G \cap H=0$, then $\gamma(G \cup H)=\gamma(G)+\gamma(H)$.

(4) Let G_n, G be in $\mathfrak{P}(K)$ and let $G \subset \bigcup_{n=1}^{\infty} G_n$. Then $\gamma(G) \leqq \sum_{n=1}^{\infty} \gamma(G_n)$.

For any subset $A \subset X$, we put

$\gamma^*(A)=\inf \{\sum_{n=1}^{\infty} \gamma(G_n),\ A \subset \bigcup_{n=1}^{\infty} G_n,\ G_n \in \mathfrak{P}(K)\}$

if this set is non-empty, and $\gamma^*(A)=+\infty$ otherwise.

Then we have

(5) a) $0 \leqq \gamma^*(A)$ for any $A \subset X$,
b) $\gamma^*(A) \leqq \gamma^*(B)$ if $A \subset B$,
c) $\gamma^*(\bigcup_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} \gamma^*(A_n)$ for all $\{A_1, A_2, \cdots, A_n, \cdots\}$,
d) $\gamma^*(G)=\gamma(G)$ for any $G \in \mathfrak{P}(K)$.

(6) Every set in $\mathfrak{P}(K)$ is measurable with respect to the outer measure γ^*.

(7) The outer measure γ^* is countably additive on the family $\overline{\mathfrak{P}(K)}$, where $\overline{\mathfrak{P}(K)}$ is the smallest family which contains $\mathfrak{P}(K)$ and closed under the formation of complements and of countable unions.

(8) For any non-negative function $f \in K$, there exists some $a>0$ such that $\gamma[x \mid x \in X,\ 0<f(x) \leqq a]=\gamma(P(f))$.

If X is a locally compact space and if N is the set of all continuous functions on X whose carriers are compact, then we easily see that $\gamma(G)=\mu(G)$ for any $G \in \mathfrak{P}(N)=\mathfrak{P}(K)$, whose μ is the measure introduced by Halmos ([8]. p. 247, Theorem 8).

By the similar method as Hewitt, we have that for any α-ideal N and for any $f \in K$, $T(f)=\int f(x)d\gamma(x)$. If T satisfies the condition (MA') and if f is a non-negative function in N, then $g_n=f-f \wedge n^{-1} \uparrow f$ and $g_n \in K$, so $T(f)=\int f(x)d\gamma(x)$. Therefore we have

Let N be an α-ideal. Then a non-negative linear functional T satisfies the condition (MA') if and only if there exists a countably additive measure γ on $\overline{\mathfrak{P}(K)}$ for which

$$T(f) = \int f(x)d\gamma(x) \quad (f \in N).$$

Let X be a space and let $\mathfrak{O}(X)$ be the set of all open subsets in X. By a Borel measure, we shall mean a real-valued function γ defined on $\overline{\mathfrak{O}(X)}$ which is countably additive, where $\overline{\mathfrak{O}(X)}$ is the smallest family which contains $\mathfrak{O}(X)$ and closed under the formation of complements and of countable unions.

Let N be a set of continuous functions such that (i) N is a linear lattice, (ii) if $f \in N$, then $1 \wedge f \in N$ and (iii) for any closed subset F and for any point p with $p \notin F$, there is an $f \in N$ such that $f(F)=0$, $f(p)=1$ and $0 \leqslant f(x) \leqslant 1$. Then Ishii [11] proved the following: Let T be a positive linear functional on N having the condition (MA). Then there is a reducible[5] Borel measure γ on X such that $T(f)=\int f(x)dr(x)$ $(f \in N)$.

Similarly, we have

Let N be an ideal and let T be a positive linear functional on N having the condition (MA). Then there is a reducible Borel measure γ on $Y=X-Z(N)$ such that

$$T(f) = \int_Y f(x)d\gamma(x) \qquad (f \in N).$$

§ 2. Property (A).

We first prove the following lemmas.

Lemma 1. *Let N be an α-ideal and let T be a non-negative functional on N. Then the restriction T_0 on K of T satisfies the condition (MA').*

Proof. By § 1, there is a measure γ such that for any $f \in K$ $T_0(f)=T(f)=\int f(x)d\gamma(x)$, so the lemma is clear.

This lemma can also be proved directly.

Lemma 2. *If N is an α-ideal, then it is a ring, i.e. if $f, g \in N$, then $fg \in N$.*

Proof. Let $f \in N$ $f \geqq 0$ and let m be a natural number. Then $mf-(f^2 \wedge m) \geqq mf - f(f \wedge m) = f(m - f \wedge m) \geqq 0$, or $mf \geqq f^2 \wedge m$. Since $f \in N$ and N is an α-ideal, $f^2 \in N$. If $f, g \in N$, then $fg \in N$ since $(|f|+|g|)^2 \geqq 4|fg|$.

We can prove the following theorem.

Theorem 1. *Let N be an α-ideal. Then the following conditions are equivalent* :

5) A measure γ on X is said to be reducible if there is a closed subsets F in X such that F is measurable and $\gamma(X-F)=0$. (Cf. [13]).

(1) *N satisfies the property* (A).
(2) *If* T *is a non-negative functional on* N *such that* $T(K^*)=0$, *then* T *is identically zero.*
(3) *If* T *is a non-negative functional on* N *such that* $T(K)=0$, *then* T *is identically zero.*

Proof. (1)→(2). Suppose that there is a positive functional T on N such that $T(K^*)=0$ and $T(f)=1$ for som $f\in N$, $f\geqq 0$. Put $f_n=(f-n)\vee(f\wedge n^{-1})$. Then $f_n\downarrow 0$. We easily see that $\varphi_{p(f-f_n)}\leqslant nf\in N$ and $0\leqslant f-f_n\leqslant n$, so $f-f_n\in K^*$ and $T(f)-T(f_n)=0$, or $T(f_n)=T(f)=1$ for any n. This shows that (1) does not hold.

(2)→(3). Clear.

(3)→(1). Let T be a non-negative linear functional on N. For any $f\in N$ $f\geqq 0$, we put $T'(f)=\inf\lim\limits_{n\to\infty} T(f_n)$, where $f_n\geqq 0$ $(n=1, 2, 3, \cdots)$ and $f_n\uparrow f$, and the infininum is taken for all sequences $\{f_n\}$ such that $f_n\uparrow f$, $f_n\geqq 0$ and $f_n\in N$. Then we have that for any $f, g\in N$ $f, g\geqq 0$, $T'(f+g)=T'(f)+T'(g)$ and for any real number $t\geqq 0$, $T'(tf)=tT'(f)$. For any arbitrary function $f\in N$, we define $T'(f)=T'(f^+)-T'(f^-)$, where f^+ and f^- denotes $f\vee 0$ and $(-f)\vee 0$ respectively. Then T' is a linear functional on N and $T\geqq T'$. Put $T''=T-T'$, then T'' is non-negative linear functional on N. But, by Lemma 1, $T''(K)=0$ and by (2) $T''\equiv 0$, so $T'=T$. This shows that N has the property (A)[4].

If N is an ideal which is not an α-ideal, we can easily see that Theorem 1 does not always hold.

Definition. An ideal $N(=N_{f_0})$ will be called a *principal ideal* if there exist a non-negative function $f_0\in N$ such that $N=\{g\,|\,g\in C(X),\ |g|\leqq\alpha f_0$ for some $\alpha\geqq 0\}$. An ideal N will be called a 0-*principal* (resp. ∞-*principal*) if there exists a non-negative s-function (resp. an unbounded function) f_0 such that $N=\{g\,|\,g\in C(X),\ |g(x)|\leqslant\alpha f_0(x)$ on U_m for some $\alpha>0$ and some natural number $m\}$ (resp. $N=\{g\,|\,g\in C(X),\ |g(x)|\geqslant\alpha f_0(x)$ on V_m for some $\alpha>0$ and $m\}$), where $U_m=\{x\,|\,x\in X, 0<f(x)<m^{-1}\}$ and $V_m=\{x\,|\,x\in X, f(x)>m\}$. A positive function f is said to be an *s-function* if it admits any small value, i.e. U_m is not empty for any m. If X is compact, then any 0-principal ideal is principal, but it is not true in general.

Theorem 2. (1) *A principal ideal* N $(=N_{f_0})$ *fulfills the condition* (A) *if and only if* $Z(f_0)$ *is open,* $Y=X-Z(f_0)$ *is pseudo-compact*[6] *and* $N=\{f\,|\,f\in C(X),\ f(Z(f_0))=0\}$ (*it is lattice-isomorphic to* $C(Y)$).

6) A topological space X is said to be pseudo-compact if any continuous function on X is bounded.

(2) *Any* 0-*principal* (*or* ∞-*principal*) *ideal* $N(=N_{f_0})$ *does not fulfill the condition* (A).

Proof. (1) Suppose that N fulfills (A). Then we put $U_n=\{x \mid x\in X, 0<f_0(x)<n^{-1}\}$. If for any n U_n is not empty, we can select a point x_n in U_n. We put $M=\{g \mid g\in N, \lim_{n\to\infty} g(x_n)/f_0(x_n) \text{ exists}\}$. For any $g\in M$, we define $T(g)=\lim_{n\to\infty} g(x_n)/f_0(x_n)$. Then T is a positive linear functional on M. For any $g\in N_{f_0}$ there exists an $m>0$ such that $|g|\leqslant mf_0$. Since $mf_0\in M$, T is extended to a positive linear functional on N_{f_0} (Cf. [4] p. 20). We denote it again with T. If $f_n=f_0\wedge 1/n$, we have that $f_n\downarrow 0$ and $T(f_n)=1$ for any n. Since T satisfies (MA'), it is a contradiction. This fact shows that U_m is empty for some m, or $f(x)\geqq m^{-1}$ for any x with $f(x)\neq 0$. Therefore $Z(f_0)$ is open, so $Y=X-Z(f_0)$ is open and closed. Let f' be the restriction of f on Y. Then $N_{f'}(\subset C(Y))$ satisfies the property (A). For any non-negative linear functional T^* on $C^*(Y)$ and for any $h\in N_{f'}$, we define $T_1(h)=T^*(h/f')$. Then $T^*(g)=T_1(f'g)$ for any $g\in C^*(Y)$. We easily see that $C^*(Y)$ satisfies (A). By Glucksberg [5], Y is pseudo-compact and $N_{f'}=C^*(Y)=C(Y)$. The converse is clear by [5].

(2) We define U_n, M and T as (1). Then T is a positive linear functional on M. For any $g\in N$, there are a positive integer m and $\alpha>0$ such that $|g(x)|\leqq\alpha f_0(x)$ on U_m. We put $h=\alpha f_0\vee|g|$. Then $h\in M$ and $|g|\leqslant h$, so T is extended to a positive functional on N. If $f_n=f_0\wedge n^{-1}$, then $T(f_n)=1$ for any n and $f_n\downarrow 0$. This is a contradiction.

REMARK. If X is an infinite (completely regular) space, then there is an α-ideal in $C(X)$ which does not satisfy (A). For, if X is infinite, then there is an s-function $f\in C(X)$, so the 0-principal ideal N_f does not satisfy (A) (Theorem 2. (2)). We easily see that N_f is an α-ideal.

DEFINITION. A directed set[4)] $\{f_\alpha\}_{\alpha\in A}$ of positive functions ($\subset N$) is called a *base* of an ideal N if for any $f\in N$ there is an f_α such that $|f|\leqq mf_\alpha$ for some m.

Let f be a positive s-function in $C(X)$ and let g be any function in $C(X)$. Then we define

$$\overline{\lim_{f\to 0}}\, g/f = \lim_{n\to\infty}\sup_{U_n} g(x)/f(x)\,,$$

$$\underline{\lim_{f\to 0}}\, g/f = \lim_{n\to\infty}\inf_{U_n} g(x)/f(x)\,,$$

where $U_n=\{x \mid x\in X, 0<f(x)<n^{-1}\}$.

If $\overline{\lim_{h\to 0}}\, g/f=\underline{\lim_{f\to 0}}\, g/f$, we write simply $\lim_{f\to 0} g/f$ (admits $+\infty$).

Theorem 3. *Let N be an α-ideal and let $\{f_\alpha\}_{\alpha\in A}$ be a base in N. If for any s-function f_α there is an f_β such that $\underline{\lim}_{f_\alpha\to 0} f_\beta/f_\alpha=\infty$, then N satisfies the property* (A).

Proof. Suppose that N does not satisfy (A). By Theorem 1 there exists a positive functional T such that $T(K)=0$ and $T(f)=1$ for some positive function $f\in N$. Since $\{f_\alpha\}_{\alpha\in A}$ is a base in N, there is an f_α and a positive constant c such that $0\leqslant f\leqslant cf_\alpha$. Now let f_α be an s-function. Then by the hypothesis, there is an f_β such that $\underline{\lim}_{f_\alpha\to 0} f_\beta/f_\alpha=\infty$. Therefore, for any positive number M there is an m such that $f_\beta(x)\geqslant Mf_\alpha(x)$ if $x\in U_m$. We set $W_m=\{x\,|\,x\in X,\ 0\leqq f_\alpha(x)<m^{-1}\}$ and $F=X-W_m$. Then if $x\in W_m$, $f_\beta(x)\leqq Mf_\alpha(x)$. Let h be a function in K^* such that $h(F)=1$. Then we easily see that $Mf_\alpha h+f_\beta\geqq Mf_\alpha$, or $cMf_\alpha h+cf_\beta\geqq cMf_\alpha\geqq Mf$. Since $f_\alpha h\in K$, $T(f_\alpha h)=0$, so $cT(f_\beta)\geqq M$. But M is an arbitrary positive number. This is a contradiction.

Next, let f_α be not an s-function. Then if $f_\alpha(x)\neq 0$, $f_\alpha(x)\geqq\delta$ for some positive number δ. The set $P=\{x\,|\,x\in X,\ f_\alpha(x)>0\}$ is open and closed and $N\supset\{f\,|\,f\in C^*(X),\ f(Z(f_\alpha))=0\}$. Since N is an α-ideal, $N\supset N_0=\{f\,|\,f\in C(X),\ f(Z(f_\alpha))=0\}$ and $K\supset N_0$. Since $T(K)=0$, $T(N_0)=0$. But $f\in N_0$ and $T(f)=1$. This is a contradiction.

Finally, we characterize ideals which satisfy the property (A) under some conditions. We see that these conditions are necessary as the later example shows.

Theorem 4. *Let N be an α-ideal and let it have a base $\{f_\alpha\}$ such that for any s-function f_α and for any f_β with $\beta\geqslant$ some α' (α' depends on α), $\lim_{f_\alpha\to 0} f_\beta/f_\alpha$ exists (admits $+\infty$). Then N satisfies the property* (A) *if and only if N is not* 0*-principal.*

Proof. If N satisfies (A), then by Theorem 2. (2), N is not 0-principal. Conversely, suppose that N is not 0-principal. Then for any f_α which is an s-function, there exists an f_β such that $\overline{\lim}_{f_\alpha\to 0} f_\beta/f_\alpha=\infty$. For, otherwise, there would exist an s-function f_α such that for any $f_\gamma\in\{f_\alpha\}$ $\overline{\lim}_{f_\alpha\to 0} f_\gamma/f_\alpha\leqslant$ some $M_\gamma<+\infty$, i.e. if $x\in U_m$, then $f_\gamma(x)\leqslant M'_\gamma f_\alpha(x)$ for some m and $M'_\gamma>0$, so N would be a 0-principal ideal N_{f_α}. This is a contradiction. We can here assume that for any α the above $\beta\geqslant\alpha'$. Therefore, by the hypothesis, for any s-function f_α there is an f_β such that $\underline{\lim}_{f_\alpha\to 0} f_\beta/f_\alpha=\infty$. By Theorem 4 N satisfies (A).

Let X be a locally compact space and let N be the set of all continuous functions on X whose carriers are compact. Then $N^+=\{f\,|\,f\in N,$

$f \geqslant 0\}$ forms a base which satisfies the hypothesis of Theorem 4. The ordering of the directed system for the base can be defined as follows: $\alpha > \beta$ if $f_\alpha \geqslant \varphi_{P(f_\beta)}$ for any f_α, f_β in N^+.

EXAMPLE. The hypothesis in Theorem 4 is necessary. The following example shows it. Let X be the closed interval $[0, 1]$ and let N be an ideal having a base $\{f_n\}$. For any n we define: $f_n(x) = x$ if $x = 2^{-2m}$ or $x = 0$ $(m = 0, 1, 2, \cdots)$, $f_n(x) = x^{1/n}$ if $x = 2^{-(2m+1)}$ $(m = 0, 1, 2, \cdots)$ and it is linear on the intervals $[2^{-(m+1)}, 2^{-m}]$ $(m = 0, 1, 2, \cdots)$. We see that N is an α-ideal (since X is compact) and is not 0-principal. But N does not satisfy (A). For, Put $M = \{f \mid f \in N, \lim_{n\to\infty} 2^{2n} f(2^{-2n}) \text{ exists}\}$. Define $T(f) = \lim_{n\to\infty} 2^{2n} f(2^{-2n})$ for any $f \in M$. T is extended to a positive linear functional on N (Cf. [4]. p. 20) Set $g_m = f_1 \wedge m^{-1}$. Then we have that $g_m \downarrow 0$ and $T(g_m) = 1$ for any m, so N does not satisfy (A).

§ 3. Ring-ideals.

A subset N in $C(X)$ is called a *ring-ideal*[7] if it satisfies the following conditions:

(i) if $f, g \in N$, then $f + g \in N$,

(ii) if $f \in N$ and if $h \in C(X)$, then $hf \in N$.

A ring-ideal N is said to be *m-closed* if N is closed in the m-topology $C(X)$. Any neighborhood of $f \in C(X)$ in the m-topology is the set $\{g \mid g \in C(X), |g - f| < \pi\}$ for some everywhere positive function $\pi \in C(X)$ according to Hewitt [9]. Shirota [15], and Gillman, Henrikson, and Jerison [7] proved that any m-closed ring-ideal is an intersection of some maximal ring-ideals. We shall show that any m-closed ring-ideal is an α-ideal and it satisfies (A) (Cf. Theorem 5).

The following lemma is proved by [16] in the case where X is compact.

Lemma 3. *Let N be an α-ideal and let it have the property such that if $f \in N$ then $|f|^{1/2} \in N$. Then N satisfies the property (A).*

Proof. Suppose that a positive functional T on N satisfies the property such that $T(K) = 0$ and $T(f) = 1$ for some positive $f \in N$ (Cf. Theorem 1). We put $g_n = (nf - f^{1/2}) \vee 0$. Then $n^2 f \geqq \varphi_{P(g_n)}$ and $g_n \in K$. $0 = T(g_n) \geqq T(nf - f^{1/2})$, or $T(f^{1/2}) \geqq nT(f) = n$ for any n. This contradiction proves the lemma.

We can easily prove the following lemmas.

---

7) We use the word "ring-ideal" to avoid the confusion.

Lemma 4. *If N is a maximal ideal ($=l$-ideal), then it satisfies the property* (A).

Proof. By Lemma 3, it is sufficient prove that (i) for any positive f in N, $f^{1/2} \in N$ and (ii) N is an α-ideal.

(i) Suppose that $f \in N$ and $f^{1/2} \notin N$. Since N is maximal, the set $\{h \mid h \in C(X),\ \lambda f^{1/2} + g \geqq |h|$ for some positive $g \in N$ and for some $\lambda > 0\}$ is identical to $C(X)$. Therefore $\lambda f^{1/2} + g \geqq f^{1/4}$ for some positive $g \in N$ and for some $\lambda > 0$, or $g \geqq f^{1/4} - \lambda f^{1/2} = f^{1/4}(1 - \lambda f^{1/4})$. For any x in X with $f(x) \leqslant (2\lambda)^{-4}$, $g(x) \geqslant 1/2 f^{1/4}(x)$, or $2g(x) \geqslant f^{1/4}(x)$. For any x in X with $f(x) > (2\lambda)^{-4}$, $(2\lambda)^3 f(x) - f^{1/4}(x) = f^{1/4}(x)((2\lambda)^3 f^{3/4}(x) - 1) \geqslant 0$, or $(2\lambda)^3 f(x) \geqslant f^{1/4}(x)$. Therefore $2g \vee (2\lambda)^3 f \geqslant f^{1/4}$, and so $f^{1/4} \in N$. By Lemma 2, we have $f^{1/2} \in N$. This contradication proves (i).

(ii) Let f be a positive function in $C(X)$ such that for any n $f \wedge n \in N$ and $f \notin N$. Since N is a maximal ideal, the set $\{h \mid \lambda f + g \geqq |h|$ for some positive $g \in N$ and for some $\lambda > 0\}$ is identical to $C(X)$. Therefore $\lambda f + g \geqq f^2$ for some positive $g \in N$ and $\lambda > 0$, or $g \geqslant f^2 - \lambda f = f(f - \lambda)$. For $x \in X$ with $f(x) \geqslant 1 + \lambda$, we have $g(x) \geqslant f(x)$. For $x \in X$ with $f(x) < 1 + \lambda$, we can select a natural number n such that $n \geqslant 1 + \lambda$. If we put $f_n = f \wedge n$, then $(1+\lambda)^{1/2} f_n^{1/2}(x) \geqslant f(x)$. Therefore $g \vee (1+\lambda)^{1/2} f_n^{1/2} \geqslant f$. Since $f_n^{1/2} \in N$ by (i), $f \in N$.

Lemma 5. *A maximal ring-ideal is a maximal ideal.*

Proof. Let M be a maximal ring-ideal. Then we must first prove that it is an ideal. We put $M_0 = \{f \mid f \in C(X),\ |f| \leqslant \alpha g$ for some positive $g \in M$ and some $\alpha > 0\}$. Then M_0 is a proper ring-ideal (for, $M_0 \not\ni 1$ since $M \not\ni 1$), and $M \subset M_0$, so $M = M_0$, i.e. M is an ideal. To prove the lemma, it is sufficient to show that if N is a maximal ideal, then it is a proper ring-ideal. We put $N_0 = \{f \mid f \in C(X),\ |f| \leqslant hg$ for some positive $h \in C(X)$ and some $g \in N\}$. Then N_0 is an ideal and $N \subset N_0$. Therefore it is sufficient to prove that N_0 is proper. Suppose that $N_0 = C(X)$. Then there exist $h \in C(X)$ and $g \in N$ such that $hg \geqslant 1$, so g is everywhere positive. If we put $N' = \{fg^{-1}; f \in N\}$, then N' is a maximal ideal and $N' \ni 1$. By the proof of Lemma 4, N' is an α-ideal, so $N' = C(X)$ and $N = C(X)$. This is a contradiction.

Now we can prove the following theorem.

Theorem 5. *Any m-closed ring-ideal is an α-ideal and it satisfies the property* (A).

Proof. Let N be an m-closed ring-ideal. Then N is an intersection of some maximal ideals M_α ([15] or [7]). Any M_α is a maximal ideal

(Lemma 5) and by the proof of Lemma 4, any M_α is an α-ideal and has the property such that for any positive $f \in M_\alpha$, $f^{1/2} \in M_\alpha$. Therefore N is an α-ideal and has the property such that for any positive $f \in N$, $f^{1/2} \in N$. By Lemma 3, N satisfies (A).

REMARK. If X is a P-space (Cf. Gillman and Henriksen [6]), then any ring-ideal in $C(X)$ satisfies (A) since any ring-ideal is m-closed ([6]. p. 345).

EXAMPLE. An m-closed ideal (not a ring-ideal) does not always satisfy the property (A). Such an example is the following: Let X be the semi-line $[0, \infty)$ and let $N=\{f \mid f \in C(X), |f(x)| \leqslant \alpha x$ for some $\alpha > 0$ and for $x \geqq 1\}$. Then we easily see that N is an m-closed ideal but it does not satisfy (A) since N is ∞-principal (Cf. Theorem 2. (2)).

§ 4. Property (B)

Let X be a locally compact space and let N be the set of all continuous functions whose whose carriers are compact on X. McShane [14] proved that N has the propety (B). We can regard N as an ideal in $C(X_0)$, where X_0 is the one-point compactification of X. We here consider ideals in $C(X)$, where X is a Q-space. Q-spaces are considered in [9]. Any separable metric space or any locally compact Hausdorff space which is sum of countable compact subsets is always a Q-space [9]. We here show that an α-ideal satisfies the property (B) if it satisfies (A) in the case X is a normal Q-space.

We first prove the following

Lemma 6. *Let X be a normal Q-space and let F be a closed subset in X. Let Y be the decomposition space*[8] *consisting of F and all elements in $X-F$. Then Y is also a Q-space.*

Proof. Let φ be the mapping such that $\varphi^{-1}(y_0)=F$ and $\varphi^{-1}(y)$ is a set consisting of only one point for any $y \in Y$, $y \neq y_0$. Then φ is continus. Now suppose that Y is not a Q-space. Then there exists a family

---

8) Let X be a topological space and let $\{F_\alpha\}$ be a division by closed sets of X, i.e. $X=\bigcup F_\alpha$, any F_α is closed and elements of $\{F_\alpha\}$ are mutually disjoint. We can consider new space Y whose points are $\{F_\alpha\}$. This space is called the decomposition space of X if for any open set $U \supset F_\alpha$, there exists an open set $V \supset F_\alpha$ such that $F_\beta \cap V \neq 0$ implies that $F_\alpha \subset U$. For anypoint $y_0=F_\alpha$ in the decomposition space Y, any neighborhood of y_0 is the set $\{y \mid y=F_\beta, F_\beta \subset U\}$ for some open set U in X (Cf. [1]). Then there exists a continuous mapping from X onto Y.

$\mathfrak{F} \subset \mathfrak{Z}(Y)$[9] such that (i) $\mathfrak{F}$ is Z-maximal, (ii) any countable family of $\mathfrak{F}$ has a non-void intersection and (iii) $\bigcap_{B \in \mathfrak{F}} B = 0$ (Cf. [9]). Let $\mathfrak{H} = \{A \mid A \in \mathfrak{Z}(X),\ A \supset \varphi^{-1}B$ for some $B \in \mathfrak{F}\}$. Then we shall first prove that $\mathfrak{H}$ is Z-maximal. If $\mathfrak{H}$ is not Z-maximal, then there exists an $A_0 \notin \mathfrak{H}$ $(A_0 \in \mathfrak{Z}(X))$ such that $\varphi^{-1}B \cap A_0 \neq 0$ for any $B \in \mathfrak{F}$. Since $A_0 \notin \mathfrak{H}$, $A_0 \not\supset \varphi^{-1}B$ for any $B \in \mathfrak{F}$, so $\varphi A_0 \not\supset B$ for any $B \in \mathfrak{F}$. For, if $\varphi A_0 \supset B$ for some $B \in \mathfrak{F}$, by (iii), there exists a $B_1 \in \mathfrak{F}$ such that $\varphi A_0 \supset B_1$ and $B_1 \not\ni y_0$. Therefore $A_0 \supset \varphi^{-1} B_1$. This is a contradiction, so $\varphi A_0 \not\supset B$ for any $B \in \mathfrak{F}$. Let $A_0 = Z(f)$ $(f \in C^*(C))$ and let $V(y_0)$ be a neighborhood of y_0 in Y. Then $f\varphi^{-1}$ is continous on $Y - V(y_0)$. Let g be an extended continous function of $f\varphi^{-1} \mid (Y - V(y_0))$ on Y (Y is a normal space). Then we have $Z(g) \subset \varphi A_0 \cup V(y_0)$. We take a $B_2 \in \mathfrak{F}$ $B_2 \not\ni y_0$ and $V(y_0)$ such that $V(y_0) \cap B_2 = 0$. To prove that $Z(g) \notin \mathfrak{F}$, we suppose that $Z(g) \in \mathfrak{F}$. Then $\varphi A_0 \supset \varphi A_0 \cap B_2 = (\varphi A_0 \cup V(y_0)) \cap B_2 \supset Z(g) \cap B_2$ and $Z(g) \cap B_2 \in \mathfrak{F}$, this is a contradiction. Since $\mathfrak{F}$ is Z-maximal, there exists a $B \in \mathfrak{F}$ such that $B \cap Z(g) = 0$. We can assume that $B \not\ni y_0$ and $B \cap V(y_0) = 0$. $0 = \varphi^{-1}B \cap \varphi^{-1}(Z(g)) \supset \varphi^{-1}B \cap \varphi^{-1}(\varphi A_0 - V(y_0)) = \varphi^{-1}[B \cap (\varphi A_0 - V(y_0))] = \varphi^{-1}(B \cap \varphi A_0) = \varphi^{-1}B \cap A_0$. But $\varphi^{-1}B \cap A_0 \neq 0$. This contradication shows that $\mathfrak{H}$ is Z-maximal. We easily see that any countable family of $\mathfrak{H}$ has non-empty intersection and $\bigcap_{A \in \mathfrak{H}} A = 0$. Therefore X is not a Q-space. This shows that Y is a Q-space.

By this lemma, we have

Theorem 6. *Let X be a normal Q-space and let N be an α-ideal. Then N satisfies the property* (B) *if and only if N satisfies the property* (A).

Proof. It is sufficient to prove that if N satisfies (A), then it satisfies (B). Suppose that N satisfied (A). Let T be a positive liner functional on N and let f be a positive function N. Then we define $T'(f) = \inf \lim_{\alpha} T(f_\alpha)$, where any f_α of a directed set $\{f_\alpha\}$ is non-negative and $f_\alpha \uparrow f$, and the infinimum is taken for all directed sets $\{f_\alpha\}$ such that $f_\alpha \uparrow f$, $f_\alpha \geqq 0$ and $f_\alpha \in N$. Then we have that for any $f, g (\geqq 0) \in N$, $T'(f+g) = T'(f) + T'(g)$ and $T'(tf) = tT'(f)$ for any $t \geqq 0$. For any $f \in N$, we put $T'(f) = T'(f^+) - T'(f^-)$. Then T' is linear functional on N. If we put $T'' = T - T'$, then $T'' \geqq 0$. To prove that $T = T'$, it is sufficient to show that $T''(K) = 0$ (Theorem 1). Therefore we have only to show that for any positive function $f \in K$, $T \mid K_f$ satisfies the condition (MA), where $K_f = \{g \mid g \in K,\ P(|g|) \subset P(f)\}$. Since N is an α-ideal, $K_f = \{g \mid g \in C(X),\ P(|g|) \subset P(f)\}$. If we put $F = \overline{P(f)}$[10], we can regard K_f as the set of

9) For any topological space X, we denote by $\mathfrak{Z}(X)$ the family $\{Z(f) \mid f \in C(X)\}$.

10) For any subset A, $\bar{A}$ denotes the closure of A.

all functions in $C(F)$ vanishing of $F-P(f)$. Therefore to prove the Theorem, we have only to show that if M is the set of all functions in $C(F)$ (F is a Q-space) vanishing on a fixed closed subset A in F, then M satisfies (B).

(i) If A is the empty set, then $M=C(F)$. If T is a positive linear functional on M, then there are a Baire measure γ on F and a compact set $C\subset F$ with $T(f)=\int_C f(x)d\gamma^*$ $(f\in M)$ ([10]. Theorem 18). Therefore M fulfills (B)[11].

(ii) Let A be a set consisting of one point and let $A=(p)$. Let T be a positive linear functional. Then T is continuous, i.e. $\|T\|=\sup_{\substack{0\leq f\leq 1\\ f\in M}} T(f)<+\infty$. We can assume that $\|T\|=1$. We put for any $f\in C(F)$, $T^*(f)=T(f-f(p))+f(p)$. We easily see that T^* is a positive linear functional on $C(F)$. By (i) T^* satisfies (MA) and so does T.

(iii) If A is an arbitrary closed subset in F, let Y be the decomposition space consisting of A and $\{x\}_{x\in F-A}$. Then by Lemma 6 Y is a Q-space. Let φ be the mapping such that $\varphi^{-1}(y_0)=A$ and $\varphi^{-1}(y)$ is a set consisting of only one point for any $y\in Y, y\neq y_0$. For any $f\in M$ we put $f'(y)=f(\varphi^{-1}y)$, then f' is continous on Y. $M^*=\{f'|f\in M\}$ is the set of all continuous functions vanishing at y_0. Let T be a positive linear functional on M and let $T_1(f')=T(f)$ for any $f'\in M^*$. Then T_1 is positive on M^*. By (ii) T_1 satisfies (MA) and so does T.

Let N be an ideal and let T be a non-negative linear functional on N. T is said to has the *property* (D) if $T(f)=T(g)$ for any $f, g\in N$ where $f=g$ on some open set $U\supset Z(N)$.

In the case where X is compact, we have

Theorem 7. *Let X be a compact space and let N be an ideal. Then N satisfies the property (B) (or (A)) if and only if any non-negative linear functional on N which has the property (D) is identically zero.*

Proof. By Theoreme 1 and 6, we have only to show that T has the property (D) if and only if $T(K)=0$. Since X is compact, it is clear.

REMARK. Any non-negative linear functional T on N which has the property (D) is of the following form. For any s-function $f\in N$ we put $M=\{g|g\in N, Z(g)\supset Z(f)$ and $\lim_{f\to 0} g/f$ exists$\}$. Then we have that for any $g\in M$

$$T(g)=c\lim_{f\to 0} g/f, \text{ where } c\geqq 0$$

For, if we put $\lim_{f\to 0} g/f=a$, then for any $\varepsilon>0$ there is an m such that

11) This fact is pointed out by [11].

$(a-\varepsilon)f(x)\leqslant g(x)\leqslant(a+\varepsilon)f(x)$ if $f(x)\leqslant m^{-1}$. $(a-\varepsilon)f\wedge g\leqslant g\leqslant(a+\varepsilon)f\vee g$, so $(a-\varepsilon)T(f)=T((a-\varepsilon)f\wedge g)\leqslant T(g)\leqslant T((a+\varepsilon)f\vee g)=(a+\varepsilon)T(f)$. Since ε is an arbitrary positive number, $T(g)=aT(f)=c\lim_{f\to 0} g/f$, where $c=T(f)$.

(Received September 17, 1959)

Bibliography

[1] P. Alexandroff and H. Hopf: Topologie, Berlin, 1935.

[2] G. Birkhoff: Lattice theory, New York, 1948.

[3] N. Bourbaki: Intégration, Livre VI, Paris, 1952.

[4] M. M. Day: Normed linear spaces, Berlin, 1958.

[5] I. Glucksberg: The representation of functionals by integrals, Duke Math. J. **19**, 253–261 (1952).

[6] L. Gillman and M. Henriksen: Concerning rings of continuous functions, Trans. Amer. Math. Soc. **77**, 340–362 (1954).

[7] L. Gillman, M. Henriksen, and M. Jerison: On a theorem of Gelfand and Kolmogroff concerning maximal ideals in rings of continuous functions, Proc. Amer. Math. Soc. **5**, 447–455 (1954).

[8] P. R. Halmos: Measure theory, New York, 1950.

[9] E. Hewitt: Rings of real-valued continuous functions I, Trans. Amer. Math. Soc. **64**, 45–99 (1948).

[10] ———: Linear functionals on spaces of continuous functions, Fund. Math. **37**, 161–189 (1950).

[11] T. Ishii: On the (MA)-condition of linear functionals on space of continuous functions, I. II. (in Japanese), Sûgaku, **8**, (1957).

[12] S. Kakutani: Concrete representation of abstract (M)-spaces, Ann. of Math. **42**, 994–1024 (1941).

[13] M. Katetov: Measures in fully normal spaces, Fund. Math. **38**, 73–84 (1951).

[14] E. J. McShane: Remark concerning integration, Proc. Nat. Acad. Sci. U.S.A. **35**, 46–49 (1949).

[15] T. Shirota: On ideals in rings of continuous functions, Proc. Japan Acad. **30**, 85–89 (1954).

[16] V. S. Varadarajan: On a theorem of F. Riesz concerning the form of linear functionals, Fund. Math. **46**, 209–220 (1958).