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Gentzen Method in Modal Calculi, II[1)]

To Professor Zyoiti SUETUNA on his 60th birthday

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Various decision procedures for modal sentential calculus $S5$ have been given by W. T. Parry [8], R. Carnap [10], M. Itoh [11], M. Ohnishi-K. Matsumoto [7] and S. Kanger [5]. Among them Gentzen-type procedure is only that of S. Kanger. The object of this note is to give an alternative Gentzen-type decision procedure for $S5$.

I

Our formulation of Lewis's modal sentential calculus $S5$[2)] is based upon "Sequenzenkalkül LK", which was constructed by G. Gentzen [3]. Namely:

- logical symbols:
  - $\cdot$ (and), $\sim$ (not), $\vee$ (or)
- rules of inference:
  - structural rules
    - weakening, contraction, exchange and cut.
  - logical rules
    - $(\rightarrow\cdot)$, $(\cdot\rightarrow)$; $(\rightarrow\sim)$, $(\sim\rightarrow)$; $(\rightarrow\vee)$, $(\vee\rightarrow)$.

Next we add to LK a new logical symbol $\Box$ (necessary), and we define as follows: if α is a formula, then $\Box\alpha$ is also a formula.

New rules for modality are

$$\frac{\alpha,\ \Gamma\rightarrow\Theta}{\Box\alpha,\ \Gamma\rightarrow\Theta}\ (\Box\rightarrow), \qquad \frac{\Box\Gamma\rightarrow\Box\Theta,\ \alpha}{\Box\Gamma\rightarrow\Box\Theta,\ \Box\alpha}\ (\rightarrow\Box).$$

By Γ, Θ we mean a series of formulas as in LK. $\Box\Gamma$ (or $\sim\Gamma$) means a series of formulas which is formed by prefixing $\Box$ (or $\sim$) in front of each formula of Γ.

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1) This is a continuation to M. Ohnishi and K. Matsumoto [7].

2) C. I. Lewis and C. H. Langford [6].

Thus established sentential calculus which contains LK is $S5$[3].

M. Wajsberg [9] gave a decision procedure for monadic functional calculus, and W. T. Parry [8] remarked that for an arbitrary formula γ in $S5$ there exists a γ^* equivalent to γ and of degree at most 1[4], and using this fact he showed the equivalence of $S5$ and monadic functional calculus, hence gave a decision procedure for $S5$.

In the following γ^* always means a formula of degree at most 1 and equivalent to γ.

When all sequent-formulas of each rule in $S5$ are of degree at most 1, we denote this system by $S5^*$.

Theorem 1. *$\to\gamma$ is provable in S5, if and only if $\to\gamma^*$ is provable in $S5^*$.*

Lemma 1. *Let $\to\gamma$ be an S5-provable sequent. Then there exists an S5-proof-figure for $\to\gamma$ which does not include any mix other than $\Box$-mix*[5].

The proof of Lemma 1 is carried out by the induction on the rank and grade of mix-formula.

We define a modalized formula (abbr. mf) as follows: (1–3)

1. $\Box\alpha$ is an mf.
2. If α is an mf, then so is $\sim\alpha$.
3. If α and β are mfs, then so are $\alpha\vee\beta$ and $\alpha\cdot\beta$.

Clearly we have

Lemma 2[6]. *In the formulation of S5 we can replace $(\to\Box)$ by*

$$\frac{\Gamma'\to\Theta',\ \alpha}{\Gamma'\to\Theta',\ \Box\alpha}\ (\to\Box)',$$

where Γ' and Θ' mean series of mfs.

Lemma 3. *Assuming that all mixes appearing in an S5-proof-figure of $\to\gamma$ are $\Box$-mixes, there exists a proof-figure of $\to\gamma$, where the degrees of mix-formulas are all at most 1.*

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3) The following example shows that Gentzen's Hauptsatz does not hold in our $S5$:

$$\dfrac{\dfrac{\dfrac{\Box\alpha\to\Box\alpha}{\to\Box\alpha,\ \sim\Box\alpha}\,(\to\sim)}{\to\Box\alpha,\ \Box\sim\Box\alpha}\,(\to\Box)\qquad\dfrac{\alpha\to\alpha}{\Box\alpha\to\alpha}\,(\Box\to)}{\to\Box\sim\Box\alpha,\ \alpha}\,(\Box\alpha)$$

4) For the definition of "a formula of degree n", see A. R. Anderson [1], p. 203.

5) $\Box$-mix means a mix, the outermost symbol of whose mix-formula is $\Box$.

6) This Lemma is a formal generalization of R. Feys's formulation [2].

The proof of Lemma 3 can be carried out by the induction on n, where n is the maximal degree of $\square$-mix-formula appearing in the proof-figure of $\rightarrow\gamma$. Let $\square\xi$ be an upper-most (in the proof-figure) mix-formula of degree n (≥ 2). By the aid of the following Wajsberg's recurring equivalences[7]: $\square\square\alpha=\square\alpha$, $\square\sim\square\alpha=\sim\square\alpha$, $\square(\square\alpha\vee\beta)=\square\alpha\vee\square\beta$, $\square(\sim\square\alpha\vee\beta)=\sim\square\alpha\vee\square\beta$ and $\square(\alpha\cdot\beta)=\square\alpha\cdot\square\beta$, clearly we can define $\xi^{\dagger}$ with the following properties: 1. $\xi^{\dagger}$ is an mf equivalent to $\square\xi$, 2. $\xi^{\dagger}$ is of degree at most $n-1$, and 3. in the upper part of the $\square$-mix of $\square\xi$ we can replace $\square\xi$ by $\xi^{\dagger}$ without any $\square$-mix of degree n.

Now, Theorem 1 can be proved by Lemmas 1, 2 and 3.

Theorem 2. *The Hauptsatz does hold in S5*.*

In order to prove Theorem 2 we need the following

Lemma 4. *In S5*, if* $\square\Gamma\rightarrow\square\Theta, \alpha$ *is provable without any mix, then either* $\square\Gamma\rightarrow\square\Theta$ *or* $\square\Gamma\rightarrow\alpha$ *is provable without any mix, where all formulas of* Γ, Θ *and* α *are of degree 0 (i.e. LK-formulas).*

The proof of Lemma 4 can be easily given by the idea of elimination of formula-bundle of each formula of $\square\Theta$ in the proof-figure of $\square\Gamma\rightarrow\square\Theta, \alpha$.

Now we have only to consider the following cases: 1. When $\rho=2$ and the mix is a $\square$-mix, i.e.

$$\dfrac{\dfrac{\square\Gamma\rightarrow\square\Theta,\ \alpha}{\square\Gamma\rightarrow\square\Theta,\ \square\alpha}\,(\rightarrow\square) \qquad \dfrac{\alpha,\ \Sigma\rightarrow\Pi}{\square\alpha,\ \Sigma\rightarrow\Pi}\,(\square\rightarrow)}{\square\Gamma,\ \Sigma\rightarrow\square\Theta,\ \Pi}\,(\square\alpha),$$

we transform this into:

$$\dfrac{\dfrac{\square\Gamma\rightarrow\square\Theta,\ \alpha \qquad \alpha,\ \Sigma\rightarrow\Pi}{\square\Gamma,\ \Sigma^{*}\rightarrow(\square\Theta)^{*},\ \Pi}\,(\alpha).}{\square\Gamma,\ \Sigma\rightarrow\square\Theta,\ \Pi}$$

2. When $\rho>2$ and the left rank >1,

2.1
$$\dfrac{\dfrac{\alpha,\ \Gamma\rightarrow\Theta}{\square\alpha,\ \Gamma\rightarrow\Theta}\,(\square\rightarrow) \qquad \Sigma\rightarrow\Pi}{\square\alpha,\ \Gamma,\ \Sigma^{*}\rightarrow\Theta^{*},\ \Pi}\,(\mathfrak{M}).$$

This case is trivial.

2.2
$$\dfrac{\dfrac{\square\Gamma\rightarrow\square\Theta,\ \alpha}{\square\Gamma\rightarrow\square\Theta,\ \square\alpha}\,(\rightarrow\square) \qquad \Sigma\rightarrow\Pi}{\square\Gamma,\ \Sigma^{*}\rightarrow(\square\Theta)^{*},\ (\square\alpha)^{*},\ \Pi}\,(\square\mathfrak{M}).$$

7) See M. Wajsberg [9].

According to Lemma 4 either $\Box\Gamma\to\Box\Theta$ or $\Box\Gamma\to\alpha$ is provable. In the former case 2.21 we transform this into:

$$\dfrac{\dfrac{\Box\Gamma\to\Box\Theta \qquad \Sigma\to\Pi}{\Box\Gamma,\ \Sigma^*\to(\Box\Theta)^*,\ \Pi}\ (\Box\mathfrak{M})}{\Box\Gamma,\ \Sigma^*\to(\Box\Theta)^*,\ (\Box\alpha)^*,\ \Pi}\ \text{(weakening if necessary)}.$$

In the latter case 2.22 we distinguish the following two cases:

2.221. when $\mathfrak{M}=\alpha$,

$$\dfrac{\dfrac{\dfrac{\Box\Gamma\to\alpha}{\Box\Gamma\to\Box\alpha}\ (\to\Box) \qquad \Sigma\to\Pi}{\Box\Gamma,\ \Sigma^*\to\Pi}\ (\Box\alpha),}{\Box\Gamma,\ \Sigma^*\to(\Box\Theta)^*,\ \Pi}$$

2.222. when $\mathfrak{M}\neq\alpha$,

$$\dfrac{\dfrac{\dfrac{\Box\Gamma\to\alpha}{\Box\Gamma\to\Box\alpha}\ (\to\Box)}{\Box\Gamma\to\Box\Theta,\ \Box\alpha} \qquad \Sigma\to\Pi}{\Box\Gamma,\ \Sigma^*\to(\Box\Theta)^*,\ \Box\alpha,\ \Pi}\ (\Box\mathfrak{M}).$$

3. When $\rho>2$, the left rank$=1$ and the right rank>1,

3.1

$$\dfrac{\Sigma\to\Pi \qquad \dfrac{\alpha,\ \Gamma\to\Theta}{\Box\alpha,\ \Gamma\to\Theta}\ (\Box\to)}{\Sigma,\ (\Box\alpha)^*,\ \Gamma^*\to\Pi^*,\ \Theta}\ (\mathfrak{M}),$$

we transform this into: 3.11. in case $\mathfrak{M}=\Box\alpha$,

$$\dfrac{\dfrac{\Sigma\to\Pi \qquad \dfrac{\dfrac{\Sigma\to\Pi \qquad \alpha,\ \Gamma\to\Theta}{\Sigma,\ \alpha,\ \Gamma^*\to\Pi^*,\ \Theta}\ (\Box\alpha)}{\Sigma,\ \Box\alpha,\ \Gamma^*\to\Pi^*,\ \Theta}\ (\Box\to)}{\Sigma,\ \Sigma^*,\ \Gamma^*\to\Pi^*,\ \Pi^*,\ \Theta}\ (\Box\alpha)}{\Sigma,\ \Gamma^*\to\Pi^*,\ \Theta}$$

3.12. in case $\mathfrak{M}\neq\Box\alpha$,

$$\dfrac{\dfrac{\dfrac{\Sigma\to\Pi \qquad \alpha,\ \Gamma\to\Theta}{\Sigma,\ \alpha^*,\ \Gamma^*\to\Pi^*,\ \Theta}\ (\mathfrak{M})}{\Sigma,\ \alpha\ ,\ \Gamma^*\to\Pi^*,\ \Theta}\ \text{(weakening if necessary)}}{\Sigma,\ \Box\alpha,\ \Gamma^*\to\Pi^*,\ \Theta}\ (\Box\to)$$

3.2

$$\dfrac{\Sigma\to\Pi \qquad \dfrac{\Box\Gamma\to\Box\Theta,\ \alpha}{\Box\Gamma\to\Box\Theta,\ \Box\alpha}\ (\to\Box)}{\Sigma,\ (\Box\Gamma)^*\to\Pi^*,\ \Box\Theta,\ \Box\alpha}\ (\Box\mathfrak{M}),$$

where Π contains $\Box\mathfrak{M}$, rnd the left rank$=1$. Therefore the non-trivial case is

$$\cfrac{\cfrac{\Box\Sigma \rightarrow \Box\Pi,\ \mathfrak{M}}{\Box\Sigma \rightarrow \Box\Pi,\ \Box\mathfrak{M}}(\rightarrow\Box) \qquad \cfrac{\Box\Gamma \rightarrow \Box\Theta,\ \alpha}{\Box\Gamma \rightarrow \Box\Theta,\ \Box\alpha}(\rightarrow\Box)}{\Box\Sigma,\ (\Box\Gamma)^* \rightarrow \Box\Pi,\ \Box\Theta,\ \Box\alpha}(\Box\mathfrak{M}).$$

e transform this into:

$$\cfrac{\cfrac{\Box\Sigma \rightarrow \Box\Pi,\ \Box\mathfrak{M} \qquad \Box\Gamma \rightarrow \Box\Theta,\ \alpha}{\Box\Sigma,\ (\Box\Gamma)^* \rightarrow \Box\Pi,\ \Box\Theta,\ \alpha}(\Box\mathfrak{M})}{\Box\Sigma,\ (\Box\Gamma)^* \rightarrow \Box\Pi,\ \Box\Theta,\ \Box\alpha}(\rightarrow\Box).$$

This completes the proof of Theorem 2.
Theorems 1 and 2 yield a new decision procedure for $S5$.

II

W. T. Parry [8] showed that if a formula γ of degree 1 is provable in $S5$, then γ is also provable in $S3$. S. Halldén [4] remarked that if a formula γ of degree 1 is provable in any one system of $S2$, $S3$, $S4$ and $S5$, then γ is also provable in each of other systems. That is,

Theorem 3. *$\rightarrow\gamma^*$ is provable in S5, if and only if $\rightarrow\gamma^*$ is provable in S2, where γ^* is of degree 1.*

In our former paper [7], we have already obtained the following result: "*$\rightarrow\gamma$ is provable in S2 if and only if $p\prec p\rightarrow\gamma$*[8] *is provable in Q2, where p is a sentence-variable*".

This result leads to an alternative proof of Theorem 3. That is, we have only to show that $p\prec p\rightarrow\gamma^*$ is provable in $Q2$ if $\rightarrow\gamma^*$ is provable in $S5^*$ (see Theorem 1). The essential part of this proof is to show the $Q2$-admissibility of

$$\frac{p\prec p,\ \Box\Gamma \rightarrow \Box\Theta,\ \alpha}{p\prec p,\ \Box\Gamma \rightarrow \Box\Theta,\ \Box\alpha},$$

where Γ, Θ and α are all LK-formulas. According to an analogous form of Lemma 4 if $p\prec p,\ \Box\Gamma\rightarrow\Box\Theta,\ \alpha$ is provable in $Q2$, then either $p\prec p,\ \Box\Gamma\rightarrow\Box\Theta$ or $p\prec p,\ \Box\Gamma\rightarrow\alpha$ is provable in $Q2$. The former case is clear. In the latter case eliminating all $\Box$'s in the proof of $p\prec p,\ \Box\Gamma\rightarrow\alpha$, we obtain a proof of $p\supset p,\ \Gamma\rightarrow\alpha$, hence of $p\prec p,\ \Box\Gamma\rightarrow\Box\alpha$.

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8) $\alpha\prec\beta$ is the abbreviation of $\Box(\sim\alpha\vee\beta)$.

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