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Alexander Polynomials as Isotopy Invariants, II

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Introduction

In this paper we shall consider the Alexander polynomials of linear graphs and closed surfaces, which may not be connected, in the 3-sphere S^3. The former have been already studied in [2] and in §1 the fact of §5 in [2] will be generalized. This result will be used in §§2-3. In §2 we shall define the Alexander polynomial, more explicitly a system of the Alexander polynomials, of a closed surface in S^3. This Alexander polynomial contains some arbitrary constants, and the number of it will be discussed in §3.

§1.

Let L be a linear graph with integral coefficients in S^3. Suppose further that $\partial L=0$. Let α_0 and α_1 be the number of vertices and edges of $|L|$ respectively. Then we have

$$\alpha_0-\alpha_1 = \mu-p_1\,, \tag{1}$$

where μ is the number of components and p_1 is the 1-dimensional Betti number of $|L|$ respectively.

Now let p be a normal projection of $|L|$ in a suitably chosen plane E^2. Further let s be the number of crossing points of $p(|L|)$ and r the number of regions of E^2 divided by $p(|L|)$. Then we have

$$(\alpha_0-s)-\alpha_1+r = 2\,. \tag{2}$$

From (1) and (2) it follows that

$$1+p_1-\mu = r-(s+1)\,. \tag{3}$$

The Alexander polynomial of L is calculated from the matrix

$$\left(\frac{\partial R_i}{\partial x_j}\right)^{\psi\varphi},$$

where R_i is a defining relation and x_j is a generator of $F(S^3-|L|)$[1].

1) See [2].

From a normal projection of $|L|$ given above, we can obtain the generators and defining relations of $F(S^3-|L|)$. Actually they are seen to consist of r generators and $s+1$ defining relations by the method of [2]. Then from (3) it follows that

$$\Delta^{(d)}(t_1, \cdots, t_\mu) \quad \text{and} \quad \Delta^{(d)}(t)$$

are equal to 0, if $0 \leqq d < r-(s+1) = 1+p_1-\mu$. Thus we have the following

Theorem 1. *Let L be a linear graph with integral coefficients in S^3. Further suppose that $\partial L = 0$. Let μ be the number of components of $|L|$ and p_1 the 1-dimensional Betti number of $|L|$. Then if $0 \leqq d < 1+p_1-\mu$, $\Delta^{(d)}(t_1, \cdots, t_\mu)$ and $\Delta^{(d)}(t)$ are all equal to 0.*

Hence it is natural to say that $\Delta^{(1+p_1-\mu)}(t_1, \cdots, t_\mu)$ and $\Delta^{(1+p_1-\mu)}(t)$ are Alexander polynomials of L. From now on we shall consider only Alexander polynomials of the type $\Delta^{(d)}(t)$.

§ 2.

New let M be a closed surface in S^3 which may not connected. Further let $M_1, M_2, \cdots, M_\mu$ be components of M and g_i the genus of $M_i (i=1, 2, \cdots, \mu)$.

Put $g(M) = \sum_{i=1}^{\mu} g_i$. Then M divides S^3 into $\mu+1$ regions $C_0, C_1, \cdots, C_\mu$. For each C_i we can define the Alexander polynomial as follws: Suppose that the boundary of C_i consists of $M_{i_1}, \cdots, M_{i_{\nu_i}}$ and that $g_{i_1}, \cdots, g_{i\nu_i}$ are genera of them respectively. Put $g^i = \sum_{j=1}^{\nu_i} g_{i_j}$. Then clearly $p_1(C_i) = g^i$. Now we consider $F(C_i)$. If φ is a homomorphism of $F(C_i)/[F(C_i), F(C_i)]$ into the infinite cyclic group Z, then we have a sequence of homomorphisms

$$X \longrightarrow F(C_i) \longrightarrow F(C_i)/[F(C_i), F(C_i)] \xrightarrow{\varphi} Z.$$

From this we can define by the usual way the Alexander polynomial $\Delta^{(1+g^i-\nu_i)}(t_i)$. Since φ is arbitrary, we have actually a family of Alexander polynomials $\Delta_{C_i}^{(1+g^i-\nu_i)}(t)$. If i moves from 0 to μ, then we have a system of Alexander polynomials

$$\{\Delta_{C_i}^{(1+g^i-\nu_i)}(t)\}. \tag{4}$$

From now on we shall say that (4) is the Alexander polynomial of M.

REMARK. This definition of the Alexander polynomial of M can be naturally extended to the case, where an n-dimensional manifold lies in the $(n+1)$-dimensional sphere S^{n+1}.

It is proved by R. H. Fox [1] that each C_i is homeomorphic to a complementary region of a suitably chosen linear graph $|L_i|$. The 1-dimensional homology group of $S^3-|L_i|$ is a free abelian group with $p_1(|L_i|)$ generators. From this it is easy to see that *the Alexander polynomial* $\Delta_{C_i}^{(1+g^i-\nu_i)}(t)$ *of* C_i *is a polynominal with at most* $p_i(|L_i|)=g^i$ *arbitrary constants.*[2] Thus the Alexander polynomial (4) of the closed surface M has at most $2g(M)$ arbitrary constants. These illustrate also the way to calculate the Alexander polynomial of a given closed surface.

§ 3.

Using the notation of § 2, we shall now prove the following

Theorem 2. *Let M be a closed surface which may not be connected. Then the number of arbitrary constants of the system of Alexander polynomials of M is at most $2g(M)-1$ for every $g(M)\geqq 1$.*

Proof. It is proved by R. H. Fox [1] that a closed surface M in S^3 can be deformed to a system of 2-spheres by a sequence of suitably chosen cuts, which are done along the disk D whose interior $\operatorname{int} D$ is disjoint from N[3] and whose boundary $\operatorname{bd} D$ lies on a component, say N_1, of positive genus and is not homotopic to 0 on N_1. Our proof will be done by induction on the minimal number $n(M)$ of these cuts used for this purpose.

If $n=1$, then our theorem is trivial. Now we assume that our theorem is true for $n\leqq k-1$. Suppose $n(M)=k$. Then M can be deformed to a closed surface N by a cut along a disk D described above, where $n(N)=k-1$. It occurs two cases.

The first case is that $\operatorname{bd} D$ is homologous to 0 on M. In this case $g(M)=g(N)$. Suppose that $\operatorname{bd} D$ lies on M_1 and that M_1 is the boundary of C_0 and C_1. Further suppose that $\operatorname{int} D$ lies in C_0. Then $\operatorname{int} D$ divides C_0 into two regions C_{00} and C_{01}. Now let $\Delta_{C_{00}}^{(1+g^{00}-\mu_{01})}(t)$, $\Delta_{C_{01}}^{(1+g^{01}-\mu_{01})}(t)$ and $\Delta_{C_0}^{(1+g^0-\mu_0)}(t)$ be Alexander polynomials of C_{00}, C_{01} and C_0 respectively. Then it is easy to see that $g^{00}+g^{01}=g^0$ and $\mu_{00}+\mu_{01}=\mu_0+1$. Furthermore it follows from the construction of M and N that

$$\Delta_{C_{00}}^{(1+g^{00}-\mu_{00})}(t)\cdot\Delta_{C_{01}}^{(1+g^{01}-\mu_{01})}(t)\equiv\Delta_{C_0}^{(1+g^0-\mu_0)}(t)\,.$$

Thus the number of arbitrary constants of $\Delta_{C_0}^{(1+g^0-\mu_0)}(t)$ is equal to the sum of that of $\Delta_{C_{00}}^{(1+g^{00}-\mu_{00})}(t)$ and $\Delta_{C_{01}}^{(1+g^{01}-\mu_{01})}(t)$.

Now we shall consider C_1. Let E_1 be a region of S^3-N which contains C_1. Let $\Delta_{C_1}^{(1+g^1-\mu_1)}(t)$ and $\Delta_{E_1}^{(1+h^1-\nu_1)}(t)$ be Alexander polynomials of C_1

2) Arbitrary constants are integers.

3) N is a closed surface which appears while M is deformed to a system of 2-spheres.

and E_1 respectively, where $g^1=h^1$ and $\mu_1=\nu_1-1$. From the construction of M and N it is easy to see that

$$\Delta_{E_1}^{(1+h^1-\nu_1)}(t)\equiv f(t)\cdot\Delta_{C_1}^{(1+g^1-\mu_1)}(t)\,,$$

where $f(t)$ is a polynomial. Then the number of arbitrary constants of $\Delta_{C_1}^{(1+g^1-\mu_1)}(t)$ is equal to or smaller than that of $\Delta_{E_1}^{(1+g^1-\nu_1)}(t)$. Thus our proof of the first case is complete.

The second case is now that bd D is not homologous to 0 on M. In this case $g(M)=g(N)+1$. Suppose that bd D lies on M_1 and int D lies in C_0. Then C_0 is homeomorphic to a complementary region of a linear graph which is the join[4)] of a circle and another linear graph whose complementary region is homeomorphic to C_0-D. Then we can see directly that the number of arbitrary constants of the Alexander polynomial of C_0 is at most g^0-1. Therefore the number of arbitrary constants of the Alexander polynomial of M is at most $2g(M)-1$. Thus our proof is complete.

As an application of Theorem 2 we have the following fact. Let M and N be for instance two connected closed surfaces with the same genus g in S_1^3 and S_2^3 respectively, and let C be a complementary regions of M and E that of N respectively. Further suppose that Alexander polynomials of C and E have g arbitrary constants respectively. Then from our theorem 2 it follows that *C and E do not make a 3-sphere by any identification of M and N.*

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References

[1] R. H. Fox: On the imbedding of polyhedron in 3-space, Ann. Math. **49**, 462-470 (1948).

[2] S. Kinoshita: Alexander polynomials as isotopy invariants, I, Osaka Math. J. **10**, 263-271 (1958).

4) Suppose that A is a point on S^2 which lies in S^3. Let $|L_1|$ and $|L_2|$ be two linear graphs such that $|L_1|\frown S^2=A$ and $|L_2|\frown S^2=A$. Further let $|L_1|-A$ and $|L_2|-A$ be contained in the different components of S^3-S^2. Then $|L_1|\smile|L_2|$ is said to be a *join* of $|L_1|$ and $|L_2|$.