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*An Investigation on the Logical Structure of Mathematics (IX)**

Deductions in the Natural-Number Theory $T_1(N)$

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The consistency of the natural-number theory $\mathrm{T}_1(\mathrm{N})$ is proved in Section B, Part (VIII), and the natural-number theory $T_1(\mathrm{N})$ in a generalized sense is defined in § 2, Part (VIII), where $T_1(\mathrm{N})$ denotes the natural-number-theoretic extension of any arbitrary elementary natural-number theory, so that the consistency of $T_1(\mathrm{N})$ can be proved by the same method as that of $\mathrm{T}_1(\mathrm{N})$. Thus, if we have a series of elementary natural-number theories $\mathrm{T}_0(\mathrm{N})\subset\mathrm{T}_0'(\mathrm{N})\subset\mathrm{T}_0''(\mathrm{N})\subset\cdots$, then we have the series of consistent natural-number theories $\mathrm{T}_1(\mathrm{N})\subset\mathrm{T}_1'(\mathrm{N})\subset\mathrm{T}_1''(\mathrm{N})\subset\cdots$. By $T_1(\mathrm{N})$ we denote any one of these natural-number theories $\mathrm{T}_1(\mathrm{N})$, $\mathrm{T}_1'(\mathrm{N})$, $\mathrm{T}_1''(\mathrm{N})$, $\cdots$. But $T_1(\mathrm{N})$ does not denote representatively a subsystem of UL which belongs to a *formally* defined class of subsystems of UL; the notation $T_1(\mathrm{N})$ is a word belonging to the intuitive language. After fixing some natural-number theory $\mathrm{T}_1^{(i)}(\mathrm{N})$ within $T_1(\mathrm{N})$, the possibility of extending further this $\mathrm{T}_1^{(i)}(\mathrm{N})$ within $T_1(\mathrm{N})$ remains always open. A fixed $\mathrm{T}_1^{(i)}(\mathrm{N})$ has generally various directions of extension within $T_1(\mathrm{N})$, while some extension of $\mathrm{T}_1^{(i)}(\mathrm{N})$ may not remain within $T_1(\mathrm{N})$ (Beendigung) or, more strongly, may become inconsistent[1] (Hemmung).

This Part is divided into two Sections A and B. In Section A we treat addition and in Section B multiplication. In Section A addition is discussed in detail, while in Section B multiplication is discussed briefly to such an extent that we can know that the multiplication can be treated quite in a similar method as in Section A.

The formulas in Section A are numbered as $\mathrm{N}+k$ and those in Section B as $\mathrm{N}\times k$, where k is the number of a formula in each Section.

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* Continuation of Part (VIII), Nagoya Math. J. **14** (1959), 129–158. Other Parts referred to in this Part are as follows: Part (II), Hamburger Abh. forthcoming; Parts (III) and (IV), Nagoya Math. J. **13** (1958); Part (VII), ibid. **14** (1959).

1) Analogy is found in the definition of Brouwer's *spread* (*Menge*). See, for instance, A. Heyting: Intuitionism, Studies in Logic and Foundations of Mathematics, Amsterdam (1956), pp. 32–37, or L. E. J. Brouwer: Zur Begründung der intuitionistischen Mathematik I. Math. Ann. **93** (1925), pp. 244-5.

The deductions in this Part are not performed on the basis of Peano's axiom system but directly on the basis of the defining formula of N.

We shall give a sketch of Section A. This Section is divided into the following seven articles:

§1 Definition of Addition,
§2 Properties of $\mathrm{L}(c, \sigma)$,
§3 Preliminaries for Addition,
§4 Commutativity and Associativity of Addition,
§5 Regularity of Addition,
§6 Domain and Range of Addition,
§7 Characterization of Addition.

The definition of UL-constant Add (Addition) in §1 is as follows. First, we introduce the symbols $\mathrm{J}(xu\sigma)$ and $\mathrm{L}(z\sigma)$ for abbreviation by the formulas ($x' = \{x\}$)

$$\mathrm{J}(xy\sigma) \equiv. \ \langle xy\rangle \in \sigma \rightarrow \langle x'y'\rangle \in \sigma ,$$
$$\mathrm{L}(z\sigma) \equiv. \ \langle 0z\rangle \in \sigma \wedge \forall xy. \ \mathrm{J}(xy\sigma) .$$

The symbols $\mathrm{J}(xu\sigma)$ and $\mathrm{L}(z\sigma)$ are not symbols of UL. We define then the UL-constant L by the defining formula

$$\forall u . u \in \mathrm{L} \equiv \overset{\mathrm{Un}}{\exists} f . \mathrm{L}(uf)$$

The constant L is used only as set for induction in the proof of

N+4 $\qquad \mathrm{N} \subseteq \mathrm{L}, \qquad (\text{or} \quad c \in \mathrm{N} \rightarrow c \in \mathrm{L})$.

This formula means that for any natural number c there is a mapping f such that the image 0 by f is c and that if the image of x by f is y then the image of x' by f is y'. Therefore, it is reasonable to define the UL-constant Add by

$$\forall u: \ u \in \mathrm{Add} \equiv .$$
$$\exists fxyz. \ u = \langle xyz\rangle \wedge f \in \mathrm{Un} \wedge \mathrm{L}(yf) \wedge \langle xz\rangle \in f .$$

Namely, Add is the class of all the ordered triple $\langle abc\rangle$ such that there exists a mapping f (i.e. a variable f with $f \in \mathrm{Un}$) with the properties $\langle 0b\rangle \in f$, $\langle xy\rangle \in f \rightarrow \langle x'y'\rangle \in f$ for all x and y, and $\langle ac\rangle \in f$. It is shown in §5 and §6 that in a certain natural-number theory $T_1(\mathrm{N})$ $\langle abc\rangle \in \mathrm{Add}$ means $a+b=c$ in the "usual sense", provided that a and b, consequently c, are natural numbers.

Throughout this Part (IX), the constant Add is used only as concepts not as sets, namely in such a way that Add can be eliminated from any proof in this Part. From § 4 on, we use for simplicity the notation $a+b=c$ instead of $\langle abc\rangle \in \mathrm{Add}$. But $a+b=c$ stands exclusively for the abbreviation of $\langle abc\rangle \in \mathrm{Add}$, so that $a+b=c$ does not mean $\mathrm{Add}`\langle ab\rangle = c$. After extending the natural-number theory so as to include $\mathrm{Add}\restriction \mathrm{N}\times\mathrm{N}`\langle ab\rangle$ and other recursive functions $a\times b$, a^b, etc. as sets we can simplify the proofs of some assertions in this Part (IX). The object of this Part is, however, to show that the deductions of the fundamental part of the natural-number theory can be performed within the natural-number theory $T_1(\mathrm{N})$.

The formula N+4 mentioned above is a fundamental formula in deducing the properties of Add. In order to prove N+4, we need as sets, besides the set L for induction defined above, the identical mapping ι and the dependent variable κ_σ which are defined respectively by

$$\forall u.u\in\iota \equiv \exists x.u=\langle xx\rangle\,,$$
$$\forall u.u\in\kappa_\sigma \equiv \exists xy.u=\langle xy\rangle \wedge \exists z.y=z' \wedge \langle xz\rangle\in\sigma\,.$$

The other dependent variables used as sets in the proofs of N+1 – N+4 in § 1 are only some elementary sets generated by 0.

In order to extend $\mathrm{T}_1(\mathrm{N})$[2] consistently to a natural-number theory in which the formulas N+1 – N+4 become theorems, let, *first*, $\mathrm{T}_0'(\mathrm{N})/\mathrm{T}_0(\mathrm{N})$ be the extension of $\mathrm{T}_0(\mathrm{N})$ obtained by adjoining ι and the elementary sets generated by V, 0, N and ι to $\mathrm{T}_0(\mathrm{N})$ as sets. As in $\mathrm{T}_0(\mathrm{N})$, the negative constituent [NN] associated with the defining formula of N is not allowed to use in $\mathrm{T}_0'(\mathrm{N})$. In order to prove the consistency of $\mathrm{T}_0'(\mathrm{N})$, we use the intuitive knowledge I′ defined, as follows, by extending the intuitive knowledge I[3] concerning the constants of $\mathrm{T}_0(\mathrm{N})$. Namely the knowledge I′ consists of the facts:

(i) V, 0, N and ι are all different;

(ii) V, 0, N and ι are different from any elementary constant;

(iii) the criteria of the intuitive truth and falsehood of $m\in l$ for an elementary constant l and any constant m of $\mathrm{T}_0'(\mathrm{N})$, and of $m=l$ for elementary constants m and l of $\mathrm{T}_0'(\mathrm{N})$ are determined in the usual manner on the basis of (i) and (ii) above;

(iv) N consists only of 0, $\{0\}$, $\{\{0\}\}$, $\cdots$;

(v) $m\in\iota$ is true, exactly if m is intuitively of the from $\langle ll_1\rangle$ where l and l_1 are intuitively equal constants.

2) For the definitions $\mathrm{T}_0(N)$ and $\mathrm{T}_1(N)$, see Part (VIII), § 3 and § 8 respectively.

3) For the definition of I, see Part (VIII), pp. 130–131.

Using this knowledge I′, the consistency proof of $T_0'(N)$ proceeds similarly to that of $T_0(N)$ given in detail in §§ 5–7, Part (VIII).

Namely, the proof constituents we have to consider for the newly adjoined constant ι are:

$$[\iota A] \qquad \overline{m\notin\iota \qquad \exists x.m=\langle xx\rangle}\ ,$$

$$[\iota A^*] \qquad \overline{m=\langle ll\rangle}\ ,$$

$$[\iota N] \qquad \overline{m\in\iota \qquad \neg\exists x.m=\langle xx\rangle}\ ,$$

$$[\iota N^*] \qquad \overline{m\neq\langle ss\rangle}\ ,$$

where s is the eigen variable of $[\iota N^*]$. We, first, add the following procedures $b*4$ and $b*5$ to $b*1-b*3$ given in § 5, Part (VIII), of assigning constants to eigen variables. Namely,

$b*4$ If E is $[\iota N^*]$, and if m is not of the form $\langle ll_1\rangle$ where l and l_1 are intuitively equal constants, we assign any constant to the eigen variable s of $[\iota N^*]$.

$b*5$ If E is $[\iota N^*]$, and if m is of the form $\langle ll_1\rangle$ where l and l_1 are intuitively equal constants, we assign l (or l_1) to the eigen variable s of $[\iota N^*]$.

Next, we supplement suitably (i)–(iv) in $b*3$, § 5, Part (VIII), by taking into consideration the possibility that m and n in the same place may be ι.

After supplementing the procedures of assigning constants to eigen variables in this way, the consistency of $T_0'(N)$ is proved just in the same way as in §§ 6, 7, Part (VIII), again by supplementing c* and d* in § 6 suitably. Since, thus, $T_0'(N)$ is an elementary natural-number theory, the natural-number-theoretic extension $T_1'(N)/T_0'(N)$ is consistent by Theorem 2, § 2, Part (VIII).

By Theorem 3, § 2, Part (VIII), a formula contradicting the intuitive knowledge used in the consistency proof of $T_1'(N)$ is $T_1'(N)$-unprovable. For instance, $\iota\in\iota$ is $T_1'(N)$-unprovable, while the problem, whether $\iota\notin\iota$ is $T_1'(N)$-provable or not, remains open, unless we find a $T_1'(N)$-proof of $\iota\notin\iota$ or a $T_1'(N)$-unprovability proof for $\iota\notin\iota$.

Let, *second*, $T_0''(N)/T_0'(N)$ be the extension of $T_0'(N)$ obtained by adjoining as sets the dependent variable κ_σ with an independent variable σ and other dependent variables, necessary to make the species of sets of $T_0''(N)$ closed with respect to substitutions[4]. For the latter purpose, we have to adjoin, for instance, κ_ι, κ_V, κ_{κ_ι}, κ_{κ_σ}, etc., and the elementary sets generated by the dependent variables of $T_0'(N)$ and those thus adjoined. Because of the above definitions of the sets of $T_0''(N)$, it is easily seen that the species of all *constant sets* of $T_0''(N)$ consists of the sets recursively defined as follows:

(i) V, 0, N and ι are constant sets of $T_0''(N)$.

(ii) If $m_1, \cdots, m_k$ are constant sets of $T_0''(N)$, then $\{m_1, \cdots, m_k\}$ is a constant set of $T_0''(N)$.

(iii) If l is a constant set of $T_0''(N)$, then κ_l is a constant set of $T_0''(N)$. (Recursive definition finished.)

The intuitive knowledge I″ concerning the constant sets of $T_0''(N)$ consists of I′ and of the intuitive knowledge concerning the constants κ_l. The latter is recursively defined as follows.

(i) κ_0 and κ_N are 0.

(ii) Denoting κ_σ by $\kappa(\sigma)$, put $\kappa_0(V)=\kappa(V)$, $\kappa_{n+1}(V)=\kappa(\kappa_n(V))$ $(n=0, 1, \cdots)$ and $\kappa_0(\iota)=\kappa(\iota)$, $\kappa_{n+1}(\iota)=\kappa(\kappa_n(\iota))$ $(n=0, 1, \cdots)$. Then, $m \in \kappa_n(V)$ is true, exactly if m is of the form $\langle gk^{(n+1)}\rangle$, and $m \in \kappa_n(\iota)$ is true, exactly if m is of the form $\langle kk^{(n+1)}\rangle$, where g and k are any constant sets of $T_0''(N)$ and $k^{(n+1)}=\{\{\cdots\{k\}\cdots\}\}$ $(n+1$ pairs of brackets).

(iii) 0, V, N, ι, $\kappa_0(V)$, $\kappa_1(V)$, $\cdots$, $\kappa_0(\iota)$, $\kappa_1(\iota)$, $\cdots$ are different each other.

(iv) If l is an elementary constant of $T_0''(N)$, then κ_l is 0 or an elementary constant which, by using the defining formula of κ_σ, can be determined by a finite number of verifications. (Recrusive definition of I″ finished)

Thus we see by the knowledge I″ that the species of constant sets of $T_0''(N)$ consists of V, 0, N, ι, $\kappa_n(V)$, and $\kappa_n(\iota)$ $(n=0, 1, \cdots)$ and of the elementary constants generated by these constants. By using the intuitive knowledge I″ concerning the constant sets of $T_0''(N)$, we shall prove the consistency of $T_0''(N)$. The proof constituents we have to consider for the newly adjoined set κ_σ is as follows:

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4) In the deductions of this Part, κ_σ is used only with σ as an independent variable (see the proof of N+4). Therefore, if we wish to prove the consistency of the formulas deduced in this Part, we have only to adjoin κ_σ with σ as independent variable and a constant κ_m with any arbitray constant m, for instance $(m=)$ 0, to the species of sets of $T_0'(N)$. The system, say $T_0^*(N)$, is a subsystem of $T_0''(N)$, so that the consistency proof of $T_0^*(N)$ is simpler than that of $T_0''(N)$. We shall prove, however, the consistency of $T_0''(N)$ not for the practical reason but in order to show the method of proving the consistency of a subsystem of UL which has the closure property with respect to substitution of variables. See also the last paragraph (p. 42) of this Part.

[κ_σA] $$\overline{m \notin \kappa_\sigma \qquad \exists xy{:}m=\langle xy\rangle \land \exists z.y=z' \land \langle xz\rangle \in \sigma}$$

[κ_σA*] $$\overline{m=\langle gh\rangle \qquad h=k' \qquad \langle gk\rangle \in \sigma}$$

[κ_σN] $$\overline{m \in \kappa_\sigma \qquad \neg\exists xy{:}m=\langle xy\rangle \land \exists z.y=z' \land \langle xz\rangle \in \sigma}$$

[κ_σN*] $$\begin{array}{c}\overline{m \neq \langle rs\rangle} \\ s \neq t' \\ \langle rt\rangle \notin \sigma\end{array}$$

where r, s, and t are the eigen variables of [κ_σN*].

Assuming the m and σ in [κ_σN*] are constant sets of $T_0''(N)$, we adjoin to the previous procedures b*1–b*5 the following procedures b*6 and b*7 of assigning constants to eigen variables r, s, and t of [κ_σN*].

b*6 If $m \in \kappa_\sigma$ does not hold, we assign any constant sets of $T_0''(N)$ to r, s, and t.

b*7 If $m \in \kappa_\sigma$ holds, then m is of the form $\langle gh'\rangle$ with constants g and h ($h'=\{h\}$). We assign g to r, h' to s, and h to t. (Since $\langle gh\rangle \in \sigma$ holds by the assumption, the three formulas carried by [κ_σN*] become all false.)

We must also supplement the procedure in b*3, §5, Part (VIII), so as to assign constants to eigen variables of [$=$A] (see §3, Part (VIII)) when at least one of m and n in [$=$A] is ι, $\kappa_n(V)$, or $\kappa_n(\iota)$. There is no difficulty to decide this procedure appropriately.

After supplementing thus the procedures of assigning constants to eigen variables, the consistency of $T_0''(N)$ is proved[5] just in the same way as in §§6, 7, Part (VIII), by finding an intuitive string in an assumed $T_0''(N)$-proof of a contradiction. Since in this way $T_0''(N)$ is proved to be an elementary natural-number theory, the natural-number-theoretic extension $T_1''(N)/T_0''(N)$ is consistent by Theorem 2, §2, Part (VIII).

Now, not only the deductions of the formulas N+1 – N+4 in §1 but also all the deductions in §§2–5 are performed within the consistent natural-number theory $T_1''(N)$, so that the proved formulas N+1 – N+29 are all $T_1''(N)$-theorems. In fact, the constants Un and Add in these deductions are only used as "concepts"[6] so that these two constants are removable from the proofs of the formulas N+1 – N+29, and all the

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5) Let $\sum$ be the species of sets of $T_0''(N)$ and $\sum_0$ the species of constant sets, defined above, of $T_0''(N)$. Then T ($=T_0''(N)$) is an elementary extension of T_{Σ_0}. The consistency proof, briefly sketched above, is, strictly speaking, the consistency proof of T_{Σ_0}. Since T_Σ is an elementary extension of T_{Σ_0} ($=T_0''(N)$), the consistency proof of $T_0''(N)$ follows from Theorem 4, §2, Part (VIII).

6) See Part (X), forthcoming elsewhere.

other dependent variables, other than the sets for induction, belong to the species of sets of $\mathrm{T}_0''(\mathrm{N})$.

The formulas N+30—N+33 proved in §6 are also theorems of $\mathrm{T}_1''(\mathrm{N})$. Herein, it is to be remarked that the dependent variables Un, $\mathrm{N}\times\mathrm{V}$, $\mathrm{N}\times\mathrm{N}$, $\mathrm{Add}\restriction\mathrm{N}\times\mathrm{V}$, $\mathrm{D}_{\mathrm{Add}}$, and $\mathrm{W}_{\mathrm{Add}\restriction\mathrm{N}\times\mathrm{N}}$, occurring in these formulas, can be looked upon as concepts[6].

The only set used in §7, other than the set P for induction in the proof of N+34, is the set $\tau_{b,\sigma}$ in the proof of N+35. Adding $\tau_{b,\sigma}$ to the species of sets of $\mathrm{T}_0''(\mathrm{N})$ and taking the closure with respect to the substitution of variables, we obtain an elementary natural-number theory $\mathrm{T}_0'''(\mathrm{N})/\mathrm{T}_0''(\mathrm{N})$, of which the consistency is proved in a similar way as before. Thus, the formulas N+34—N+36 in §7 are theorems of the consistent natural-number-theoretic extension $\mathrm{T}_1'''(\mathrm{N})/\mathrm{T}_0'''(\mathrm{N})$.

The way of defining of, and deducing formulas concerning, multiplication is explained in Section B.

Thus, the method and extent of formulating consistently the natural-number theory within the weakest natural-number theory $T_1(\mathrm{N})$ are shown in this Part (IX). It is left for further continuation of our investigation to formulate the consistent extention $T_2(\mathrm{N})/T_1(\mathrm{N})$ in which the primitive recursive functions and the image of a natural number by these functions are added as sets and in which the sets for induction are defined by definiens which contain these sets.

Remarks on the Deductions in Section A

1. The use of the premise [I] of the extensionality is throughout this Part (IX) ordinary[7]. No indication is given about the place of use of [I].
2. The premises of the assertions and of the proofs are omitted, since they are easily known by the proofs given.
3. The cut formulas indicated as N*k are proved in Part (VII). The other cut formulas used are proved in Part (III).
4. The formulas, in the froofs of which there is no Spf[8] (superfluous formula) and the cut formulas are the formulas with strongly irreducible[9] proofs, are strongly irreducible. All the proofs in this Part are irreducible.
5. Even when the weakly irreducible formula[9] *N*1 is used in a proof P in the leftmost part of a mathematical induction [NN], the

7) The definition is in Part (IV), §4.
8) The definition is in Part (II), §12.
9) The definition is in Part (II), §20.

proof P is irreducible, since the superfluous formulas in the proof of $*$N$*$1 are effectively used in P concerning the mathematical induction.

6. In order to keep the degree of irreducibility of proofs as strong as possible, particular attention is required in determining whether the range of a variable be left universal or be restricted to the totality N of natural numbers. E.g. the element variable u of the set P for induction in the proof of N+11 is restricted to N, while the bound variable x in the defining formula of the same P is left universal. See also the defining formula of P in the proof of N+22. Bound variables and free variables occurring in the formulas to be proved are sometimes left universal and sometimes restricted to N.

7. If a mathematical induction is applied with a set for induction in the definiens of which occurs N, then the mathematiclal induction is called *impredicative*; otherwise *predicative*. General definition of the predicative and impredicative inferences will be defined in Part (X).

8. See Part (IV) for further details of the conventions and usages in deductions.

Section A Addition

§1 Definition of Addition

N+1 $\sigma \in \mathrm{Un} \rightarrow \kappa_\sigma \in \mathrm{Un}$

$$
\begin{array}{ll}
- & \mathrm{N{+}1} \\
\hline
1 & \neg \forall xyz.\ \langle xy\rangle \in \sigma \wedge \langle xz\rangle \in \sigma \rightarrow y=z \\
- & \forall xyz.\ \langle xy\rangle \in \kappa_\sigma \wedge \langle xz\rangle \in \kappa_\sigma \rightarrow y=z \\
\hline
{\scriptstyle (r,s,t)} & \\
- & \langle rs\rangle \notin \kappa_\sigma \\
- & \langle rt\rangle \notin \kappa_\sigma \\
2 & s=t \\
\hline
- & \neg \exists xy.\ \langle rs\rangle = \langle xy\rangle \wedge \exists z.\ y=z' \wedge \langle xz\rangle \in \sigma \\
{\scriptstyle (r,s,a)}\ - & \neg \exists xy.\ \langle rt\rangle = \langle xy\rangle \wedge \exists z.\ y=z' \wedge \langle xz\rangle \in \sigma \\
\hline
{\scriptstyle (r,t,b)} & \\
3 & s \neq a' \qquad 4\quad \langle ra\rangle \notin \sigma \\
5 & t \neq b' \qquad 6\quad \langle rb\rangle \notin \sigma \\
\hline
{\scriptstyle (1)} & \\
- & \neg.\ \langle ra\rangle \in \sigma \wedge \langle rb\rangle \in \sigma \rightarrow a=b \\
\hline
 & \underset{(4)}{\langle ra\rangle \in \sigma} \quad \underset{(6)}{\langle rb\rangle \in \sigma} \quad 7\ \ a \neq b \quad \text{Cut N}*3 \\
\hline
 & -\ \ \neg.\ a=b \rightarrow a'=b' \\
\hline
 & \underset{(7)}{a=b} \quad \underset{(2,3,5,=)}{a' \neq b'}
\end{array}
$$

N+2 $\mathrm{L}(0, \iota)$

\- N+2

\- $\langle 00\rangle \in \iota \wedge \forall xy.\ \mathrm{J}(x, y, \iota)$

$\langle 00\rangle \in \iota$

$\iota * 1$

(r, s) - $\forall xy.\ \mathrm{J}(x, y, \iota)$

\- $\langle rs\rangle \notin \iota$

\- $\langle r's'\rangle \in \iota$

1 $r \neq s$

2 $r' = s'$ Cut N*3

\- $\neg.\ r = s \to r' = s'$

$\underset{(1)}{r = s}$ $\underset{(2)}{r' \neq s'}$

N+3 $\mathrm{L}(a, \sigma) \to \mathrm{L}(a', \kappa_\sigma)$

\- N+3

\- $\neg.\ \langle 0a\rangle \in \sigma \wedge \forall xy.\ \mathrm{J}(x, y, \sigma)$

1 $\langle 0a'\rangle \in \kappa_\sigma \wedge \forall xy.\ \mathrm{J}(x, y, \kappa_\sigma)$

2 $\langle 0a\rangle \notin \sigma$

3 $\neg \forall xy.\ \mathrm{J}(x, y, \sigma)$

(1)

\- $\langle 0a'\rangle \in \kappa_\sigma$

\- $\exists z.\ a' = z' \wedge \langle 0z\rangle \in \sigma$

$\underset{(=)}{a' = a'}$ $\underset{(2)}{\langle 0a\rangle \in \sigma}$

(r, s) - $\forall xy.\ \mathrm{J}(x, y, \kappa_\sigma)$

\- $\mathrm{J}(r, s, \kappa_\sigma)$

\- $\langle rs\rangle \notin \kappa_\sigma$

4 $\langle r's'\rangle \in \kappa_\sigma$

(t) - $\neg \exists z.\ s = z' \wedge \langle rz\rangle \in \sigma$

5 $s \neq t'$

6 $\langle rt\rangle \notin \sigma$

(4)

$[t']$ - $\exists z.\ s' = z' \wedge \langle r'z\rangle \in \sigma$

\- $s' = t'' \wedge \langle r't'\rangle \in \sigma$

7 $s' = t''$ Cut N*3

\- $\neg.\ s = t' \to s' = t''$

$\underset{(5)}{s = t'}$ $\underset{(7)}{s' \neq t''}$

7 $\langle r't'\rangle \in \sigma$

(3)

\- $\neg \mathrm{J}(r, t, \sigma)$

\- $\neg.\ \langle rt\rangle \in \sigma \to \langle r't'\rangle \in \sigma$

$\underset{(6)}{\langle rt\rangle \in \sigma}$ $\underset{(7)}{\langle r't'\rangle \notin \sigma}$

By using L as set for induction, we now prove

N+4 $\quad c\in \mathrm{N}\to \overset{\mathrm{Un}}{\exists} f.\ \mathrm{L}(c, f)$ [or $c\in\mathrm{N}\to c\in\mathrm{L}$]

- N+4

- $c\notin \mathrm{N}$
1 $c\in \mathrm{L}$

- $\neg\forall x\colon 0\in x\wedge\forall y.\ y\in x\to y'\in x.\to c\in x$

[L]

- $0\in\mathrm{L}$

[ι]

- $\exists f.\ f\in\mathrm{Un}\wedge\mathrm{L}(0, f)$

$\iota\in\mathrm{Un}$ $\quad$ $\mathrm{L}(0, \iota)$
$\iota*3$ $\quad$ N+2

- $\forall y.\ y\in\mathrm{L}\to y'\in\mathrm{L}$ $\quad$ $\underset{(1)}{c\notin\mathrm{L}}$

(s)

- $s\notin\mathrm{L}$
- $s'\in\mathrm{L}$

- $\neg\exists f.\ f\in\mathrm{Un}\wedge\mathrm{L}(s, f)$
(σ)
2 $\exists f.\ f\in\mathrm{Un}\wedge\mathrm{L}(s', f)$

- $\neg.\ \sigma\in\mathrm{Un}\wedge\mathrm{L}(s, \sigma)$

---

3 $\sigma\notin\mathrm{Un}$
4 $\neg\mathrm{L}(s, \sigma)$

[κ_σ](2)

- $\kappa_\sigma\in\mathrm{Un}\wedge\mathrm{L}(s', \kappa_\sigma)$

$\underset{(3)}{\kappa_\sigma\in\mathrm{Un}}$
N+1

5 $\mathrm{L}(s', \kappa_\sigma))$ $\quad$ Cut N+3

- $\neg.\ \mathrm{L}(s, \sigma)\to\mathrm{L}(s', \kappa_\sigma)$

$\underset{(4)}{\mathrm{L}(s, \sigma)}$ $\quad$ $\underset{(5)}{\neg\mathrm{L}(s', \kappa_\sigma)}$

§2 Properties of $\mathrm{L}(c, \sigma)$

N+5 $\quad a\in\mathrm{N}\wedge\mathrm{L}(0, \sigma)\to\langle aa\rangle\in\sigma$

Define the set P for induction by

$$u\in\mathrm{P}\equiv\langle uu\rangle\in\sigma\ .$$

- N+5

---

1 $a\notin\mathrm{N}$
- $\neg\mathrm{L}(0, \sigma)$
2 $\langle aa\rangle\in\sigma$

---

3 $\langle 00\rangle\notin\sigma$
4 $\neg\forall xy.\ \langle xy\rangle\in\sigma\to\langle x'y'\rangle\in\sigma$

(s)[P](1)

$$
\dfrac{-\ 0\in \mathrm{P}}{\underset{(3)}{\langle 00\rangle}\in\sigma}
\qquad
\dfrac{\begin{array}{l}-\ s\notin \mathrm{P}\\ -\ s'\in \mathrm{P}\end{array}}{\begin{array}{l}5\ \ \langle ss\rangle\notin\sigma\\ 6\ \ \langle s's'\rangle\in\sigma\end{array}}
\qquad
\dfrac{-\ a\notin \mathrm{P}}{\underset{(2)}{\langle aa\rangle}\notin\sigma}
$$

(4)

$$
\underset{(5)}{\langle ss\rangle}\in\sigma \qquad \underset{(6)}{\langle s's'\rangle}\notin\sigma
$$

N+6 $\mathrm{L}(c,\sigma)\wedge a\in \mathrm{N}\rightarrow \exists x.\ \langle ax\rangle\in\sigma$

Define the set P for induction by

$$u\in \mathrm{P}\equiv\exists x.\ \langle ux\rangle\in\sigma .$$

$$
\dfrac{-\ \ \mathrm{N{+}6}}{\begin{array}{l}1\ \ \neg \mathrm{L}(c,\sigma)\qquad 2\ \ a\notin \mathrm{N}\\ 3\ \ \exists x.\ \langle ax\rangle\in\sigma\end{array}}
$$

(1)

$$
\begin{array}{l}4\ \ \langle 0c\rangle\notin\sigma\\ 5\ \ \neg\forall xy.\ \langle xy\rangle\in\sigma\rightarrow\langle x'y'\rangle\in\sigma\end{array}
$$

(r)[P](2)

$$
\dfrac{\dfrac{-\ 0\in \mathrm{P}}{\exists x.\ \langle 0x\rangle\in\sigma}}{\underset{(4)}{\langle 0c\rangle}\in\sigma}
\qquad
\dfrac{\begin{array}{l}-\ r\notin \mathrm{P}\\ -\ r'\in \mathrm{P}\end{array}}{\begin{array}{l}-\ \neg\exists x.\ \langle rx\rangle\in\sigma\\ \overset{(s)}{6}\ \ \exists x.\ \langle r'x\rangle\in\sigma\end{array}}
\qquad
\dfrac{-\ a\notin \mathrm{P}}{\neg\underset{(3)}{\exists x.}\ \langle ax\rangle\in\sigma}
$$

$$
7\ \ \langle rs\rangle\notin\sigma
$$

(6)

$$
8\ \ \langle r's'\rangle\in\sigma
$$

(5)

$$
\dfrac{-\ \ \neg .\ \langle rs\rangle\in\sigma\rightarrow\langle r's'\rangle\in\sigma}{\underset{(7)}{\langle rs\rangle}\in\sigma\qquad \underset{(8)}{\langle r's'\rangle}\notin\sigma}
$$

N+7 $\sigma\in \mathrm{Un}\wedge \mathrm{L}(c,\sigma)\wedge a\in \mathrm{N}\wedge\langle a'b\rangle\in\sigma\rightarrow\exists x.\ b=x'$

Define the set P for induction by

$$u\in \mathrm{P}\equiv\exists x.\ \langle u'x'\rangle\in\sigma .$$

$$
\dfrac{-\ \ \mathrm{N{+}7}}{\begin{array}{l}1\ \ \sigma\notin \mathrm{Un}\qquad 2\ \ \neg \mathrm{L}(c,\sigma)\qquad 3\ \ a\notin \mathrm{N}\\ 4\ \ \langle a'b\rangle\notin\sigma\qquad 5\ \ \exists x.\ b=x'\end{array}}
$$

(2)

$$
\begin{array}{l}6\ \ \langle 0c\rangle\notin\sigma\\ 7\ \ \neg\forall xy.\ \langle xy\rangle\in\sigma\rightarrow\langle x'y'\rangle\in\sigma\end{array}
$$

$(s)[\mathrm{P}](3)$

$- \quad 0 \in \mathrm{P}$ $\qquad - \quad s \notin \mathrm{P}$, $- \quad s' \in \mathrm{P}$ $\qquad - \quad a \notin \mathrm{P}$

$- \quad \exists x.\ \langle 0'x' \rangle \in \sigma$

$8 \quad \langle 0'c' \rangle \in \sigma$

(7)

$- \quad \neg.\ \langle 0c \rangle \in \sigma \to \langle 0'c' \rangle \in \sigma$

$\underset{(6)}{\langle 0c \rangle \in \sigma} \qquad \underset{(8)}{\langle 0'c' \rangle \notin \sigma}$

$- \quad \neg \exists x.\ \langle s'x' \rangle \in \sigma$

(t)

$8 \quad \exists x.\ \langle s''x' \rangle \in \sigma$

$9 \quad \langle s't' \rangle \notin \sigma$

(8)

$10 \quad \langle s''t'' \rangle \in \sigma$

(7)

$- \quad \neg.\ \langle s't' \rangle \in \sigma \to \langle s''t'' \rangle \in \sigma$

$\underset{(9)}{\langle s't' \rangle \in \sigma} \qquad \underset{(10)}{\langle s''t'' \rangle \notin \sigma}$

$- \quad \neg \exists x.\ \langle a'x' \rangle \in \sigma$

(s)

$8 \quad \langle a's' \rangle \notin \sigma$

(1)

$- \quad \neg.\ \langle a'b \rangle \in \sigma \wedge \langle a's' \rangle \in \sigma \to b =$

$\underset{(4)}{\langle a'b \rangle \in \sigma} \qquad \underset{(8)}{\langle a's' \rangle \in \sigma} \qquad 9 \quad b \neq$

(5)

$\underset{(9)}{b =}$

N+8 $\qquad \sigma \in \mathrm{Un} \wedge \mathrm{L}(c, \sigma) \wedge a \in \mathrm{N} \wedge \langle a'b' \rangle \in \sigma \to \langle ab \rangle \in \sigma$

$- \quad$ N+8

1 $\sigma \notin \mathrm{Un}$	2 $\neg \mathrm{L}(c, \sigma)$	3 $a \notin \mathrm{N}$
4 $\langle a'b' \rangle \notin \sigma$	5 $\langle ab \rangle \in \sigma$	Cut N+6

$- \quad \neg.\ \mathrm{L}(c, \sigma) \wedge a \in \mathrm{N} \to \exists x.\ \langle ax \rangle \in \sigma$

$\underset{(2)}{\mathrm{L}(c, \sigma)} \qquad \underset{(3)}{a \in \mathrm{N}} \qquad - \quad \neg \exists x.\ \langle ax \rangle \in \sigma$

(r)

$6 \quad \langle ar \rangle \notin \sigma$

(2)

$- \quad \neg \forall xy.\ \langle xy \rangle \in \sigma \to \langle x'y' \rangle \in \sigma$

$\underset{(6)}{\langle ar \rangle \in \sigma} \qquad 7 \quad \langle a'r' \rangle \notin \sigma$

(1)

$- \quad \neg \forall xyz.\ \langle xy \rangle \in \sigma \wedge \langle xz \rangle \in \sigma \to y = z$

$\underset{(4)}{\langle a'b' \rangle \in \sigma} \qquad \underset{(7)}{\langle a'r' \rangle \in \sigma} \qquad 8 \quad b' \neq r' \qquad$ Cut N*3

$- \quad \neg.\ b' = r' \to b = r$

$\underset{(8)}{b' = r'} \qquad \underset{(5, 6, =)}{b \neq r}$

N+9 $\qquad \sigma \in \mathrm{Un} \wedge c \in \mathrm{N} \wedge \mathrm{L}(c, \sigma) \wedge a \in \mathrm{N} \wedge \langle ab \rangle \in \sigma \to b \in \mathrm{N}$

Define the set P for induction by

$$u \in \mathrm{P} \equiv u \in \mathrm{N} \wedge \forall x.\ \langle ux \rangle \in \sigma \to x \in \mathrm{N}$$

$- \quad$ N+9

1 $\sigma \notin \mathrm{Un}$	2 $c \notin \mathrm{N}$	3 $\neg \mathrm{L}(c, \sigma)$
4 $a \notin \mathrm{N}$	5 $\langle ab \rangle \notin \sigma$	6 $b \in \mathrm{N}$

(3)

7 $\quad \langle 0c \rangle \notin \sigma \quad$ (3 is used here partly)

(4)

- $\neg\forall x{:}\ 0\in x\wedge\forall y.\ y\in x\rightarrow y'\in x.\rightarrow a\in x$

(s)

$0\in \mathrm{P}$ (*) $\qquad s\notin \mathrm{P}$, $s'\in \mathrm{P}$ (**) $\qquad a\notin \mathrm{P}$ (***)

- (*)

$0\in \mathrm{N}$ $\quad$ $*\mathrm{N}*1$

(s) - $\forall x.\ \langle 0x\rangle\in\sigma\rightarrow x\in \mathrm{N}$

8 $\langle 0s\rangle\notin\sigma$

9 $s\in \mathrm{N}$

(1)

- $\neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\rightarrow y=z$

$\langle 0s\rangle\in\sigma$ (8) $\qquad \langle 0c\rangle\in\sigma$ (7) $\qquad s\neq c$ (2, 9, =)

- (***)

- $\neg.\ a\in \mathrm{N}\wedge\forall x.\ \langle ax\rangle\in\sigma\rightarrow x\in \mathrm{N}$

Spf $\qquad a\notin \mathrm{N}$

- $\neg\forall x.\ \langle ax\rangle\in\sigma\rightarrow x\in \mathrm{N}$

$\langle ab\rangle\in\sigma$ (5) $\qquad b\notin \mathrm{N}$ (6)

- (**)

- $\neg.\ s\in \mathrm{N}\wedge\forall x.\ \langle sx\rangle\in\sigma\rightarrow x\in \mathrm{N}$

8 $s'\in \mathrm{N}\wedge\forall x.\ \langle s'x\rangle\in\sigma\rightarrow x\in \mathrm{N}$

9 $s\notin \mathrm{N}$

10 $\neg\forall x.\ \langle sx\rangle\in\sigma\rightarrow x\in \mathrm{N}$

(8)

$s'\in \mathrm{N}$ (9) $\quad$ Cut $\mathrm{N}*2$

(t) - $\forall x.\ \langle s'x\rangle\in\sigma\rightarrow x\in \mathrm{N}$

11 $\langle s't\rangle\notin\sigma$

12 $t\in \mathrm{N}$ $\qquad$ Cut $\mathrm{N}+7$

- $\neg.\ \sigma\in \mathrm{Un}\wedge \mathrm{L}(c,\sigma)\wedge s\in \mathrm{N}\wedge\langle s't\rangle\in\sigma\rightarrow\exists x.\ t=x'$

$\sigma\in \mathrm{Un}$ (1) $\quad \mathrm{L}(c,\sigma)$ (3) $\quad s\in \mathrm{N}$ (9) $\quad \langle s't\rangle\in\sigma$ (11) $\quad$ (r) - $\neg\exists x.\ t=x'$

13 $t\neq r'$

(10)

- $\neg.\ \langle sr\rangle\in\sigma\rightarrow r\in \mathrm{N}$

14 $\langle sr\rangle\in\sigma$ $\qquad$ Cut $\mathrm{N}+8$ $\qquad r\notin \mathrm{N}$ (12, 13, =) Cut $\mathrm{N}*2$

- $\neg.\ \sigma\in \mathrm{Un}\wedge \mathrm{L}(c,\sigma)\wedge s\in \mathrm{N}\wedge\langle s'r'\rangle\in\sigma\rightarrow\langle sr\rangle\in\sigma$

$\sigma\in \mathrm{Un}$ (1) $\quad \mathrm{L}(c,\sigma)$ (3) $\quad s\in \mathrm{N}$ (9) $\quad \langle s'r'\rangle\in\sigma$ (11, 13, =) $\quad \langle sr\rangle\notin\sigma$ (14)

N+10 $\quad \sigma\in\mathrm{Un}\wedge \mathrm{L}(c,\sigma)\wedge \mathrm{L}(c,\tau)\wedge a\in\mathrm{N}\wedge\langle ab\rangle\in\sigma\rightarrow\langle ab\rangle\in\tau$

Define the set P for induction by

$$u\in\mathrm{P}\equiv u\in\mathrm{N}\wedge\forall x.\ \langle ux\rangle\in\sigma\rightarrow\langle ux\rangle\in\tau.$$

- N+10

---

1 $\sigma\notin\mathrm{Un}$ 2 $\neg\mathrm{L}(c,\sigma)$ 3 $\neg\mathrm{L}(c,\tau)$

4 $a\notin\mathrm{N}$ 5 $\langle ab\rangle\notin\sigma$ 6 $\langle ab\rangle\in\tau$

(3) ---

7 $\langle 0c\rangle\notin\tau$ 8 $\neg\forall xy.\ \langle xy\rangle\in\tau\rightarrow\langle x'y'\rangle\in\tau$

(2) ---

9 $\langle 0c\rangle\notin\sigma$ (2 is used here partly)

(4) ---

- $\neg\forall x.\ 0\in x\wedge\forall y.\ y\in x\rightarrow y'\in x.\ \rightarrow a\in x$

(r)[P] ---

$0\in\mathrm{P}$ (*) $r\notin\mathrm{P}$, $r'\in\mathrm{P}$ (**) $a\notin\mathrm{P}$ (***)

- (*)

- $0\in\mathrm{N}\wedge\forall x.\ \langle 0x\rangle\in\sigma\rightarrow\langle 0x\rangle\in\tau$

$0\in\mathrm{N}$ - $\forall x.\ \langle 0x\rangle\in\sigma\rightarrow\langle 0x\rangle\in\tau$

*N*1   (s) ---

10 $\langle 0s\rangle\notin\sigma$

11 $\langle 0s\rangle\in\tau$

(1) ---

- $\neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\rightarrow y=z$

$\langle 0s\rangle\in\sigma$ (10) $\langle 0c\rangle\in\sigma$ (9) $s\neq c$ (7, 11, =)

- (***)

- $\neg.\ a\in\mathrm{N}\wedge\forall x.\ \langle ax\rangle\in\sigma\rightarrow\langle ax\rangle\in\tau$

Spf $a\notin\mathrm{N}$

- $\neg\forall x.\ \langle ax\rangle\in\sigma\rightarrow\langle ax\rangle\in\tau$

- $\neg.\ \langle ab\rangle\in\sigma\rightarrow\langle ab\rangle\in\tau$

$\langle ab\rangle\in\sigma$ (5) $\langle ab\rangle\notin\tau$ (6)

- (**)

- $\neg.\ r\in N\wedge\forall x.\ \langle rx\rangle\in\sigma\to\langle rx\rangle\in\tau$

10 $r'\in N\wedge\forall x.\ \langle r'x\rangle\in\sigma\to\langle r'x\rangle\in\tau$

---

11 $r\notin N$

12 $\neg\forall x.\ \langle rx\rangle\in\sigma\to\langle rx\rangle\in\tau$

(10) ---

$r'\in N$ (11) Cut N*2 | - (s) $\forall x.\ \langle r'x\rangle\in\sigma\to\langle r'x\rangle\in\tau$

---

13 $\langle r's\rangle\notin\sigma$

14 $\langle r's\rangle\in\tau$ Cut N+7

- $\neg.\ \sigma\in Un\wedge L(c,\sigma)\wedge r\in N\wedge\langle r's\rangle\in\sigma\to\exists x.\ s=x'$

$\sigma\in Un$ (1) $L(c,\sigma)$ (2) $r\in N$ (11) $\langle r's\rangle\in\sigma$ (13) (t) - $\neg\exists x.\ s=x'$

---

15 $s\neq t'$

(12) ---

- $\neg.\ \langle rt\rangle\in\sigma\to\langle rt\rangle\in\tau$

---

16 $\langle rt\rangle\in\sigma$ Cut N+8 | 16 $\langle rt\rangle\notin\tau$

--- (8) ---

- $\neg.\ \sigma\in Un\wedge L(c,\sigma)\wedge r\in N\wedge\langle r't'\rangle\in\sigma\to\langle rt\rangle\in\sigma$ | - $\neg.\ \langle rt\rangle\in\tau\to\langle r't'\rangle\in\tau$

$\sigma\in Un$ (1) $L(c,\sigma)$ (2) $r\in N$ (11) $\langle r't'\rangle\in\sigma$ (13, 15, =) $\langle rt\rangle\notin\sigma$ (16) | $\langle rt\rangle\in\tau$ (16) $\langle r't'\rangle\notin\tau$ (14, 15, =)

N+11 $\sigma\in Un\wedge L(b,\sigma)\wedge L(b',\tau)\wedge a\in N\wedge\langle ac\rangle\in\sigma\to\langle ac'\rangle\in\tau$

Define the set P for induction by

$$u\in P\equiv u\in N\wedge\forall x.\ \langle ux\rangle\in\sigma\to\langle ux'\rangle\in\tau\,.$$

- N+11

---

1 $\sigma\notin Un$ 2 $\neg L(b,\sigma)$ 3 $\neg L(b',\tau)$

4 $a\notin N$ 5 $\langle ac\rangle\notin\sigma$ 6 $\langle ac'\rangle\in\tau$

(2, 3) ---

7 $\langle 0b\rangle\notin\sigma$ 8 $\langle 0b'\rangle\notin\tau$ (2 is used here partly)

9 $\neg\forall xy.\ \langle xy\rangle\in\tau\to\langle x'y'\rangle\in\tau$

(r)[P](4) ---

$0\in P$ (*) | $r\notin P$, $r'\in P$ (**) | $a\notin P$ (***)

$-$ (*)

$0 \in \mathrm{N}$ $\quad - \quad \forall x.\ \langle 0x \rangle \in \sigma \rightarrow \langle 0x' \rangle \in \tau$

$*\mathrm{N}*1$ $\quad (s, 8)$

10 $\langle 0s \rangle \notin \sigma$

11 $\langle 0s' \rangle \in \tau$

(1)

$\underset{(7)}{\langle 0b \rangle \in \tau} \qquad \underset{(10)}{\langle 0s \rangle \in \sigma} \qquad \underset{(8, 11, =)}{b \neq s}$

$-$ (***)

Spf $\qquad a \notin \mathrm{N}$

$- \quad \neg \forall x.\ \langle ax \rangle \in \sigma \rightarrow \langle ax' \rangle \in \tau$

$- \quad \neg.\ \langle ac \rangle \in \sigma \rightarrow \langle ac' \rangle \in \tau$

$\underset{(5)}{\langle ac \rangle \in \sigma} \qquad \underset{(6)}{\langle ac' \rangle \notin \tau}$

$-$ (**)

$- \quad \neg :\ r \in \mathrm{N} \wedge \forall x.\ \langle rx \rangle \in \sigma \rightarrow \langle rx' \rangle \in \tau$

10 $\quad r' \in \mathrm{N} \wedge \forall x.\ \langle r'x \rangle \in \sigma \rightarrow \langle r'x' \rangle \in \tau$

---

11 $\quad r \notin \mathrm{N}$

12 $\quad \neg \forall x.\ \langle rx \rangle \in \sigma \rightarrow \langle rx' \rangle \in \tau$

(10)

$\underset{(11)}{r' \in \mathrm{N}}$ $\quad (s) \quad - \quad \forall x.\ \langle r'x \rangle \in \sigma \rightarrow \langle r'x' \rangle \in \tau$

N$*$2

13 $\quad \langle r's \rangle \notin \sigma$

14 $\quad \langle r's' \rangle \in \tau$ $\qquad$ Cut N+7

$- \quad \neg.\ \sigma \in \mathrm{Un} \wedge \mathrm{L}(b, \sigma) \wedge r \in \mathrm{N} \in \langle r's \rangle \in \sigma \rightarrow \exists x.\ s = x'$

$\underset{(1)}{\sigma \in \mathrm{Un}} \quad \underset{(2)}{\mathrm{L}(b, \sigma)} \quad \underset{(11)}{r \in \mathrm{N}} \quad \underset{(13)}{\langle r's \rangle \in \sigma}$ $\quad (t) \quad - \quad \neg \exists x.\ s = x'$

15 $\quad s \neq t'$ $\qquad$ Cut N+8

$- \quad \neg.\ \sigma \in \mathrm{Un} \wedge \mathrm{L}(b, \sigma) \wedge r \in \mathrm{N} \wedge \langle r't' \rangle \in \sigma \rightarrow \langle rt \rangle \in \sigma$

$\underset{(1)}{\sigma \in \mathrm{Un}} \quad \underset{(2)}{\mathrm{L}(b, \sigma)} \quad \underset{(11)}{r \in \mathrm{N}} \quad \underset{(13, 15, =)}{\langle r't' \rangle \in \sigma}$

16 $\quad \langle rt \rangle \notin \sigma$

(12)

$\underset{(16)}{\langle rt \rangle \in \sigma}$ $\qquad$ 17 $\quad \langle rt' \rangle \notin \tau$

(9)

$\underset{(15, 17, =)}{\langle rs \rangle \notin \tau} \qquad \underset{(14)}{\langle r's' \rangle \notin \tau}$

§ 3 Preliminaries for Addition

N+12 $a \in \mathrm{N} \to \langle 0aa \rangle \in \mathrm{Add}$

\- N+12

1 $a \notin \mathrm{N}$

\- $\langle 0aa \rangle \in \mathrm{Add}$

2 $\exists fxyz.\ \langle 0aa \rangle = \langle xyz \rangle \wedge f \in \mathrm{Un} \wedge \mathrm{L}(y, f) \wedge \langle xz \rangle \in f$ Cut N+4

\- $\neg.\ a \in \mathrm{N} \to \exists f.\ f \in \mathrm{Un} \wedge \mathrm{L}(a, f)$

$\underset{(1)}{a \in \mathrm{N}}$ (σ) - $\neg \exists f.\ f \in \mathrm{Un} \wedge \mathrm{L}(a, f)$

3 $\sigma \notin \mathrm{Un}$

4 $\neg \mathrm{L}(a, \sigma)$

(2)

$\underset{(=)}{\langle 0aa \rangle = \langle 0aa \rangle}$ $\underset{(3)}{\sigma \in \mathrm{Un}}$ $\underset{(4)}{\mathrm{L}(a, \sigma)}$ 5 $\langle 0a \rangle \in \sigma$

(4)

$\langle 0a \rangle \notin \sigma$

Spf $\underset{(5)}{\cdots}$

N+13 $a \in \mathrm{N} \to \langle a0a \rangle \in \mathrm{Add}$

\- N+13

1 $a \notin \mathrm{N}$

2 $\langle a0a \rangle \in \mathrm{Add}$ Cut N+4

\- $\neg.\ 0 \in \mathrm{N} \to \exists f.\ f \in \mathrm{Un} \wedge \mathrm{L}(0, f)$

$0 \in \mathrm{N}$ - $\neg \exists f.\ f \in \mathrm{Un} \wedge \mathrm{L}(0, f)$

$*\mathrm{N}*1$ (σ)

3 $\sigma \notin \mathrm{Un}$

4 $\neg \mathrm{L}(0, \sigma)$

(2)

\- $\exists fxyz.\ \langle a0a \rangle = \langle xyz \rangle \wedge f \in \mathrm{Un} \wedge \mathrm{L}(0, f) \wedge \langle xz \rangle \in f$

$\underset{(3)}{\sigma \in \mathrm{Un}}$ $\underset{(4)}{\mathrm{L}(0, \sigma)}$ 5 $\langle aa \rangle \in \sigma$ Cut N+5

\- $\neg.\ a \in \mathrm{N} \wedge \mathrm{L}(0, \sigma) \to \langle aa \rangle \in \sigma$

$\underset{(1)}{a \in \mathrm{N}}$ $\underset{(4)}{\mathrm{L}(0, \sigma)}$ $\underset{(3)}{\langle aa \rangle \notin \sigma}$

REMARK: Although in the proof of N+13 a cut by a weakly irreducible formula (§ 20, Part (II)) $*\mathrm{N}*1$ is used, the proof of N+13 is irreducible, since the superfluous formulas in the proof of $*\mathrm{N}*1$ are effectively used in the proofs of the other cut formulas N+4 and N+5.

N+14 $a\in N\wedge b\in N\wedge\langle abc\rangle\in \mathrm{Add}\rightarrow c\in N$

- N+14

---

1 $a\notin N$ 2 $b\notin N$ 3 $c\in N$

- $\langle abc\rangle\notin \mathrm{Add}$

- $\neg\exists fxyz.\ \langle abc\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge L(y,f)\wedge\langle xz\rangle\in f$

(a,b,c,σ)

---

4 $\sigma\notin \mathrm{Un}$ 5 $\neg L(b,\sigma)$ 6 $\langle ac\rangle\notin\sigma$ Cut N+9

- $\neg.\ \sigma\in \mathrm{Un}\wedge b\in N\wedge L(b,\sigma)\wedge a\in N\wedge\langle ac\rangle\in\sigma\rightarrow c\in N$

$\underset{(4)}{\sigma\in \mathrm{Un}}$ $\underset{(2)}{b\in N}$ $\underset{(5)}{L(b,\sigma)}$ $\underset{(1)}{a\in N}$ $\underset{(6)}{\langle ac\rangle\in\sigma}$ $\underset{(3)}{c\notin N}$

N+15 $a\in N\wedge\langle abc\rangle\in \mathrm{Add}\wedge\langle abd\rangle\in \mathrm{Add}\rightarrow c=d$

- N+15

---

1 $a\notin N$ 2 $\langle abc\rangle\notin \mathrm{Add}$

3 $\langle abd\rangle\notin \mathrm{Add}$ 4 $c=d$

$(2,3)$

- $\neg\exists fxyz.\ \langle abc\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge L(y,f)\wedge\langle xz\rangle\in f$

(σ,a,b,c)

- $\neg\exists fxyz.\ \langle abd\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge L(y,f)\wedge\langle xz\rangle\in f$

(τ,a,b,d)

---

5 $\sigma\notin \mathrm{Un}$ 6 $\neg L(b,\sigma)$ 7 $\langle ac\rangle\notin\sigma$

Spf $\tau\notin \mathrm{Un}$ 8 $\neg L(b,\tau)$ 9 $\langle ad\rangle\notin\tau$ Cut N+10

- $\neg.\ \sigma\in \mathrm{Un}\wedge L(b,\sigma)\wedge L(b,\tau)\wedge a\in N\wedge\langle ac\rangle\in\sigma\rightarrow\langle ac\rangle\in\tau$

$\underset{(5)}{\sigma\in \mathrm{Un}}$ $\underset{(6)}{L(b,\sigma)}$ $\underset{(8)}{L(b,\tau)}$ $\underset{(1)}{a\in N}$ $\underset{(7)}{\langle ac\rangle\in\sigma}$ $\underset{(3,4,9)}{\langle ac\rangle\notin\tau}$

N+16 $a\in N\wedge\langle a'bc\rangle\in \mathrm{Add}\rightarrow\exists x.\ c=x'$

- N+16

---

1 $a\notin N$

- $\langle a'bc\rangle\notin \mathrm{Add}$

2 $\exists x.\ c=x'$

- $\exists fxyz.\ \langle a'bc\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge L(y,f)\wedge\langle xz\rangle\in f$

(σ,r,s,t)

---

3 $\langle a'bc\rangle\neq\langle rst\rangle$

4 $\sigma\notin \mathrm{Un}$

5 $\neg L(s,\sigma)$

6 $\langle rt\rangle\notin\sigma$ Cut N+7

- $\neg.\ \sigma\in \mathrm{Un}\wedge L(b,\sigma)\wedge a\in N\wedge\langle a'c\rangle\in\sigma\rightarrow\exists x.\ c=x'$

$\underset{(4)}{\sigma\in \mathrm{Un}}$ $\underset{(5,3,=)}{L(b,\sigma)}$ $\underset{(1)}{a\in N}$ $\underset{(6,3,=)}{\langle a'c\rangle\in\sigma}$ $\underset{(2)}{\neg\exists x.\ c=x'}$

N+17 $a \in \mathrm{N} \wedge \langle ab0 \rangle \in \mathrm{Add} \rightarrow a = 0$

$$- \quad \mathrm{N}{+}17$$

1 $a \notin \mathrm{N}$ 2 $\langle ab0 \rangle \notin \mathrm{Add}$

3 $a = 0$ Cut N∗10

$$\overset{\mathrm{N}}{-} \quad \neg.\ a \in \mathrm{N} \wedge a \neq 0 \rightarrow \exists x.\ a = x'$$

$\underset{(1)}{a \in \mathrm{N}}$ $\underset{(3)}{a \neq 0}$ (p) $\overset{\mathrm{N}}{-} \quad \neg \exists x.\ a = x'$

4 $p \notin \mathrm{N}$

5 $a \neq p'$ Cut N+16

$$- \quad \neg.\ p \in \mathrm{N} \wedge \langle p'b0 \rangle \in \mathrm{Add} \rightarrow \exists x.\ 0 = x'$$

$\underset{(4)}{p \in \mathrm{N}}$ $\underset{(2,5,=)}{\langle p'b0 \rangle \in \mathrm{Add}}$ (q) $- \quad \neg \exists x.\ 0 = x'$

$0 \neq q'$

N∗4

N+18 $\langle abc \rangle \in \mathrm{Add} \rightarrow \langle a'bc' \rangle \in \mathrm{Add}$

$$- \quad \mathrm{N}{+}18$$

$- \quad \langle abc \rangle \notin \mathrm{Add}$

$- \quad \langle a'bc' \rangle \in \mathrm{Add}$

(σ, a, b, c) $- \quad \neg \exists fxyz.\ \langle abc \rangle = \langle xyz \rangle \wedge f \in \mathrm{Un} \wedge \mathrm{L}(y, f) \wedge \langle xz \rangle \in f$

1 $\exists fxyz.\ \langle a'bc' \rangle = \langle xyz \rangle \wedge f \in \mathrm{Un} \wedge \mathrm{L}(y, f) \wedge \langle xz \rangle \in f$

2 $\sigma \notin \mathrm{Un}$

3 $\neg \mathrm{L}(b, \sigma)$

4 $\langle ac \rangle \notin \sigma$

(1)

$\underset{(2)}{\sigma \in \mathrm{Un}}$ $\underset{(3)}{\mathrm{L}(b, \sigma)}$ 5 $\langle a'c' \rangle \in \sigma$

(3)

Spf $\langle 0b \rangle \notin \sigma$

$- \quad \neg \forall xy.\ \langle xy \rangle \in \sigma \rightarrow \langle x'y' \rangle \in \sigma$

$\underset{(4)}{\langle ac \rangle \in \sigma}$ $\underset{(5)}{\langle a'c' \rangle \notin \sigma}$

N+19 $a \in \mathrm{N} \wedge b \in \mathrm{N} \rightarrow \langle abc \rangle \in \mathrm{Add} \rightarrow \langle ab'c' \rangle \in \mathrm{Add}$

$$- \quad \mathrm{N}{+}19$$

1 $a \notin \mathrm{N}$ 2 $b \notin \mathrm{N}$

$- \quad \langle abc \rangle \notin \mathrm{Add}$

$- \quad \langle ab'c' \rangle \in \mathrm{Add}$

$$
\begin{array}{lll}
 & - & \neg\exists fxyz.\ \langle abc\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge \mathrm{L}(y,f)\wedge\langle xz\rangle\in f \\
(\sigma,a,b,c) & 3 & \exists fxyz.\ \langle ab'c'\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge \mathrm{L}(y,f)\wedge\langle xz\rangle\in f \\
\hline
 & 4 & \sigma\notin \mathrm{Un} \\
 & 5 & \neg \mathrm{L}(b,\sigma) \\
 & 6 & \langle ac\rangle\notin\sigma \qquad \text{Cut N+4} \\
\hline
 & - & \neg.\ b'\in \mathrm{N}\to\exists f.\ f\in \mathrm{Un}\wedge \mathrm{L}(b',f)
\end{array}
$$

$$
\begin{array}{ll}
\underset{(2)}{b'\notin \mathrm{N}} & (\tau)\ -\ \neg\exists f.\ f\in \mathrm{Un}\wedge \mathrm{L}(b',f) \\
\text{N*2} & 7\quad \tau\notin \mathrm{Un} \\
 & 8\quad \neg \mathrm{L}(b',\tau)
\end{array}
$$

(3)

$$
\underset{(7)}{\tau\in \mathrm{Un}}\quad \underset{(8)}{\mathrm{L}(b',\tau)}\quad 9\quad \langle ac'\rangle\in\tau \qquad \text{Cut N+11}
$$

$$
-\ \neg.\ \sigma\in \mathrm{Un}\wedge \mathrm{L}(b,\sigma)\wedge \mathrm{L}(b',\tau)\wedge a\in \mathrm{N}\wedge\langle ac\rangle\in\sigma\to\langle ac'\rangle\in\tau
$$

$$
\underset{(1)}{\sigma\in \mathrm{Un}}\quad \underset{(5)}{\mathrm{L}(b,\sigma)}\quad \underset{(8)}{\mathrm{L}(b',\tau)}\quad \underset{(1)}{a\in \mathrm{N}}\quad \underset{(6)}{\langle ac\rangle\in\sigma}\quad \underset{(9)}{\langle ac'\rangle\notin\tau}
$$

N+20 $\quad a\in \mathrm{N}\wedge\langle a'bc'\rangle\in \mathrm{Add}\to\langle abc\rangle\in \mathrm{Add}$

$$
\begin{array}{lll}
 & - & \text{N+20} \\
\hline
 & 1 & a\notin \mathrm{N} \\
 & - & \langle a'bc'\rangle\notin \mathrm{Add} \\
 & - & \langle abc\rangle\in \mathrm{Add} \\
\hline
 & - & \neg\exists fxyz.\ \langle a'bc'\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge \mathrm{L}(y,f)\wedge\langle xz\rangle\in f \\
(\sigma,r,s,t) & 2 & \exists fxyz.\ \langle abc\rangle=\langle xyz\rangle\wedge f\in \mathrm{Un}\wedge \mathrm{L}(y,f)\wedge\langle xz\rangle\in f \\
\hline
 & 3 & \langle a'bc'\rangle\neq\langle rst\rangle \\
 & 4 & \sigma\notin \mathrm{Un} \\
 & 5 & \neg \mathrm{L}(s,\sigma) \\
 & 6 & \langle rt\rangle\notin\sigma
\end{array}
$$

(2)

$$
\underset{(1)}{\sigma\in \mathrm{Un}}\quad \underset{(5,3,=)}{\mathrm{L}(b,\sigma)}\quad 7\quad \langle ac\rangle\in\sigma
$$

Cut N+8

$$
-\ \neg.\ \sigma\in \mathrm{Un}\wedge \mathrm{L}(b,\sigma)\wedge a\in \mathrm{N}\wedge\langle a'c'\rangle\in\sigma\to\langle ac\rangle\in\sigma
$$

$$
\underset{(4)}{\sigma\in \mathrm{Un}}\quad \underset{(5,3,=)}{\mathrm{L}(b,\sigma)}\quad \underset{(1)}{a\in \mathrm{N}}\quad \underset{(6,3,=)}{\langle a'c'\rangle\in\sigma}\quad \underset{(7)}{\langle ac\rangle\notin\sigma}
$$

N+21 $a\in N \wedge b\in N \wedge \langle a'bc\rangle \in \mathrm{Add} \to \langle ab'c\rangle \in \mathrm{Add}$

$-$ N+21

1	$a\notin N$	3	$\langle a'bc\rangle \notin \mathrm{Add}$	
2	$b\notin N$	4	$\langle ab'c\rangle \in \mathrm{Add}$	Cut N+16

$-$ $\neg.\ a\in N \wedge \langle a'bc\rangle \in \mathrm{Add} \to \exists x.\ c=x'$

$\underset{(1)}{a\in N}$ $\quad \underset{(3)}{\langle a'bc\rangle \in \mathrm{Add}}$ $\quad {}_{(p)}-$ $\neg \exists x.\ c=x'$

5 $c\neq p'$ Cut N+20

$-$ $\neg.\ a\in N \wedge \langle a'bp'\rangle \in \mathrm{Add} \to \langle abp\rangle \in \mathrm{Add}$

$\underset{(1)}{a\in N}$ $\quad \underset{(3,5,=)}{\langle a'bp'\rangle \in \mathrm{Add}}$ $\quad$ 6 $\langle abp\rangle \notin \mathrm{Add}$

Cut N+19

$-$ $\neg.\ a\in N \wedge b\in N \wedge \langle abp\rangle \in \mathrm{Add} \to \langle ab'p'\rangle \in \mathrm{Add}$

$\underset{(1)}{a\in N}$ $\quad \underset{(2)}{b\in N}$ $\quad \underset{(6)}{\langle abp\rangle \in \mathrm{Add}}$ $\quad \underset{(4,5,=)}{\langle ab'p'\rangle \notin \mathrm{Add}}$

§4 Commutativity and Associativity of Addition

N+22 $a, b\in N \wedge a+b=c \to b+a=c$

Define the set P for induction by

$$u\in P \equiv u\in N \wedge \overset{N}{\forall x \forall y}.\ Q(u, x, y)$$

where

$$Q(a, b, c) \equiv.\ a+b=c \to b+a=c\ .$$

$-$ N+22

1	$a\notin N$	3	$a+b\neq c$
2	$b\notin N$	4	$b+a=c$

$(r)[P](1)$

$\underset{(*)}{0\in P}$ $\quad \dfrac{r\notin P}{\underset{(**)}{r'\notin P}}$ $\quad \underset{(***)}{a\notin P}$

$-$ $(*)$

$0\in N$ $\quad {}_{(p,q)}-$ $\overset{N}{\forall x \forall y}.\ Q(0, x, y)$

$*N*1$

5	$p\notin N$	
6	$0+p\neq q$	
7	$p+0=q$	Cut N+15

$- \quad \neg.\ 0\in \mathrm{N} \wedge 0+p=p \wedge 0+p=q \rightarrow p=q$

$0\in\mathrm{N} \quad \underset{(5)}{0+p=p} \quad \underset{(6)}{0+p=q} \quad \underset{(5,7,=)}{p\neq q}$

$*\mathrm{N}*1 \quad \mathrm{N}{+}12 \quad \text{Cut}\ \mathrm{N}{+}13$

$- \quad (***)$

Spf $\quad a\notin\mathrm{N}$

$- \quad \neg \overset{\mathrm{N}}{\forall x \forall y}.\ \mathrm{Q}(a, x, y)$

$\underset{(2)}{b\in\mathrm{N}} \quad \underset{(3)}{a+b=c} \quad \underset{(4)}{b+a\neq c}$

$- \quad (**)$

$- \quad \neg.\ r\in\mathrm{N} \wedge \overset{\mathrm{N}}{\forall x \forall y}.\ \mathrm{Q}(r, x, y)$

5 $\quad r'\in\mathrm{N} \wedge \overset{\mathrm{N}}{\forall x \forall y}.\ \mathrm{Q}(r', x, y)$

---

6 $\quad r\notin\mathrm{N}$

7 $\quad \neg \overset{\mathrm{N}}{\forall x \forall y}.\ \mathrm{Q}(r, x, y)$

(5) ---

$\underset{(6)}{r'\in\mathrm{N}} \quad {\scriptstyle(s,t)} - \quad \overset{\mathrm{N}}{\forall x \forall y}.\ \mathrm{Q}(r', x, y)$

$\mathrm{N}*2$

---

8 $\quad s\notin\mathrm{N}$

$- \quad \mathrm{Q}(r', s, t)$

---

9 $\quad r'+s\neq t$

Cut $\mathrm{N}{+}16$ $\quad$ 10 $\quad s+r'=t$

$- \quad \neg.\ r\in\mathrm{N} \wedge r'+s=t \rightarrow \exists x.\ t=x'$

$\underset{(6)}{r\in\mathrm{N}} \quad \underset{(9)}{r'+s=t} \quad {\scriptstyle(p)} - \quad \neg \exists x.\ t=x'$

---

11 $\quad t\neq p'$ $\quad$ Cut $\mathrm{N}{+}20$

$- \quad \neg.\ r\in\mathrm{N} \wedge r'+s=p' \rightarrow r+s=p$

$\underset{(6)}{r\in\mathrm{N}} \quad \underset{(9,11,=)}{r'+s=p'} \quad$ 12 $\quad r+s\neq p$

(7) ---

$\underset{(8)}{s\in\mathrm{N}} \quad \underset{(12)}{r+s=p} \quad$ 13 $\quad s+r\neq p$

Cut $\mathrm{N}{+}19$

$- \quad \neg.\ s\in\mathrm{N} \wedge r\in\mathrm{N} \wedge s+r=p \rightarrow s+r'=p'$

$\underset{(8)}{s\in\mathrm{N}} \quad \underset{(6)}{r\in\mathrm{N}} \quad \underset{(13)}{s+r=p} \quad \underset{(10,11,=)}{s+r'\neq p'}$

N+23 $\quad a, b, c\in \mathrm{N}\wedge a+b=k\wedge b+c=l\wedge a+l=m\rightarrow k+c=m$

Define the set P for induction by

$$u\in \mathrm{P}\equiv u\in \mathrm{N}\wedge \overset{\mathrm{N}}{\forall bc\forall klm}.\ \mathrm{Q}(u, b, c, k, l, m)$$

where $\quad \mathrm{Q}(a, b, c, k, l, m)\equiv .\ a+b=k\wedge b+c=l\wedge a+l=m\rightarrow k+c=m$.

$$\begin{array}{l}
- \qquad \mathrm{N}{+}23 \\
\hline
1\ a\notin \mathrm{N} \quad 2\ b\notin \mathrm{N} \quad 3\ c\notin \mathrm{N} \\
4\ a+b\neq k \quad 5\ b+c\neq l \quad 6\ a+l\neq m \\
7\ k+c=m \\
\hline_{(r)[\mathrm{P}](1)}
\underset{(*)}{0\in \mathrm{P}} \qquad \underset{(**)}{r\notin \mathrm{P} \quad r'\in \mathrm{P}} \qquad \underset{(***)}{a\notin \mathrm{P}}
\end{array}$$

$$- \qquad (*)$$

$$\begin{array}{l}
\hline
\underset{*\mathrm{N}*1}{0\in \mathrm{N}} \qquad - \quad \overset{\mathrm{N}}{\forall bc\forall klm}.\ \mathrm{Q}(0, b, c, k, l, m) \\
\hline_{(p,q,r,s,t)}
8\ p\notin \mathrm{N} \quad 9\ q\notin \mathrm{N} \\
10\ 0+p\neq r \quad 11\ p+q\neq s \\
12\ 0+s\neq t \quad 13\ r+q=t \qquad \text{Cut N+12} \\
\hline
- \quad \neg.\ p\in \mathrm{N}\rightarrow 0+p=p \\
\hline
\underset{(8)}{p\in \mathrm{N}} \qquad 14\ \ 0+p\neq p \qquad \text{Cut N+15} \\
\hline
- \quad \neg.\ 0\in \mathrm{N}\wedge 0+p=r\wedge 0+p=p\rightarrow r=p \\
\hline
\underset{*\mathrm{N}*1}{0\in \mathrm{N}} \quad \underset{(10)}{0+p=r} \quad \underset{(14)}{0+p=p} \qquad 15\ \ r\neq p \qquad \text{Cut N+14} \\
\hline
- \quad \neg.\ p, q\in \mathrm{N}\wedge p+q=s\rightarrow s\in \mathrm{N} \\
\hline
\underset{(8)}{p\in \mathrm{N}} \quad \underset{(9)}{q\in \mathrm{N}} \quad \underset{(11)}{p+q=s} \qquad 16\ \ s\notin \mathrm{N} \qquad \text{Cut N+12} \\
\hline
- \quad \neg.\ s\in \mathrm{N}\rightarrow 0+s=s \\
\hline
\underset{(16)}{s\in \mathrm{N}} \qquad 17\ \ 0+s\neq s \qquad \text{Cut N+15} \\
\hline
- \quad \neg.\ 0\in \mathrm{N}\wedge 0+s=t\wedge 0+s=s\rightarrow t=s \\
\hline
\underset{*\mathrm{N}*1}{0\in \mathrm{N}} \quad \underset{(12)}{0+s=t} \quad \underset{(17)}{0+s=s} \quad \underset{(11,13,15,=)}{t\neq s}
\end{array}$$

– $(***)$

Spf $a \notin N$

– $\neg \overset{N}{\forall} bc \forall klm.\ Q(a, b, c, k, l, m)$

$\underset{(2)}{b \in N}$ $\underset{(3)}{c \in N}$ $\underset{(4)}{a+b=k}$ $\underset{(5)}{b+c=l}$ $\underset{(6)}{a+l=m}$ $\underset{(7)}{k+c \neq m}$

– $(**)$

– $\neg .\ r \in N \wedge \overset{N}{\forall} bc \forall klm.\ Q(r, b, c, k, l, m)$

8 $r' \in N \wedge \overset{N}{\forall} bc \forall klm.\ Q(r', b, c, k, l, m)$

---

9 $r \notin N$

10 $\neg \overset{N}{\forall} bc \forall klm.\ Q(r, b, c, k, l, m)$

(8) ---

$\underset{(9)}{r' \in N}$ $\quad$ $\underset{(b, c, k, l, m)}{–}$ $\overset{N}{\forall} bc \forall klm.\ Q(r', b, c, k, l, m)$

N*2

---

11 $b \notin N$ $\qquad$ 12 $c \notin N$

13 $r'+b \neq k$ $\qquad$ 14 $b+c \neq l$

Cut N+16 $\qquad$ 15 $r'+l \neq m$ $\qquad$ 16 $k+c=m$

– $\neg .\ r \in N \wedge r'+b=k \rightarrow \exists x.\ k=x'$

– $\neg .\ r \in N \wedge r'+l=m \rightarrow \exists x.\ m=x'$

$\underset{(9)}{r \in N}$ $\underset{(13)}{r'+b=k}$ $\underset{(15)}{r'+l=m}$ $\quad$ $\underset{(s)}{–}$ $\neg \exists x.\ k=x'$

$\underset{(t)}{–}$ $\neg \exists x.\ m=x'$

---

17 $k \neq s'$

Cut N+18 $\qquad$ 18 $m \neq t'$

– $\neg .\ s+c=t \rightarrow s'+c=t'$

---

19 $s+c=t$ $\qquad$ Cut N+20 $\qquad$ $\underset{(16, 17, 18, =)}{s'+c \neq t'}$

– $\neg .\ r \in N \wedge r'+b=s' \rightarrow r+b=s$

– $\neg .\ r \in N \wedge r'+l=t' \rightarrow r+l=t$

$\underset{(9)}{r \in N}$ $\underset{(13, 17, =)}{r'+b=s'}$ $\underset{(15, 18, =)}{r'+l=t'}$ $\qquad$ 20 $r+b \neq s$

21 $r+l \neq t$

(10) ---

$\underset{(11)}{b \in N}$ $\underset{(12)}{c \in N}$ $\underset{(20)}{r+b=s}$ $\underset{(14)}{b+c=l}$ $\underset{(21)}{r+l=t}$ $\underset{(19)}{s+c \neq t}$

N+24 $a, b, c \in \mathrm{N} \wedge a+b=k \wedge b+c=l \wedge k+c=m \rightarrow a+l=m$

This is proved by using cuts by N+14, N+22 and N+23.
From N+23 and N+24 follows

N+25 $a, b, c \in \mathrm{N} \wedge a+b=k \wedge b+c=l \rightarrow .\ a+l=m \equiv k+c=m$

§5 Regularity of Addition

N+26 $a, b, c \in \mathrm{N} \wedge a+b=m \wedge a+c=m \rightarrow b=c$

Define the set P for induction by

$$u \in \mathrm{P} \equiv u \in \mathrm{N} \wedge \overset{\mathrm{N}}{\forall} xy \forall z.\ \mathrm{Q}(u, x, y, z)$$

where $\mathrm{Q}(a, b, c, m) \equiv a+b=m \wedge a+c=m \rightarrow b=c$.

– N+26

1 $r \notin \mathrm{N}$	2 $b \notin \mathrm{N}$	3 $c \notin \mathrm{N}$
4 $a+b \neq m$	5 $a+c \neq m$	6 $b=c$

(r)[P](1)

$0 \in \mathrm{P}$ (*)	$r \notin \mathrm{P}$, $r' \in \mathrm{P}$ (**)	$a \notin \mathrm{P}$ (***)

– (*)

$0 \in \mathrm{N}$ *N*1   – $\overset{\mathrm{N}}{\forall} x \forall yz.\ \mathrm{Q}(0, x, y, z)$

(r, s, t)

7 $r \notin \mathrm{N}$	9 $0+r \neq t$	11 $r=s$
8 $s \notin \mathrm{N}$	10 $0+s \neq t$	Cut N+12

– $\neg .\ r \in \mathrm{N} \rightarrow 0+r=r$

– $\neg .\ s \in \mathrm{N} \rightarrow 0+s=s$

$r \in \mathrm{N}$ (7)	$s \in \mathrm{N}$ (8)	12 $0+r \neq r$	
		13 $0+s \neq s$	Cut N+15

– $\neg .\ 0 \in \mathrm{N} \wedge 0+r=t \wedge 0+r=r \rightarrow t=r$

– $\neg .\ 0 \in \mathrm{N} \wedge 0+s=t \wedge 0+s=s \rightarrow t=s$

$0 \in \mathrm{N}$ *N*1	$0+r=t$ (9)	$0+r=r$ (12)	$0+s=t$ (10)	$0+s=s$ (13)	$t \neq r$, $t \neq s$ (11, =)

– (***)

Spf $a \notin \mathrm{N}$

– $\neg \overset{\mathrm{N}}{\forall} xy \forall z.\ \mathrm{Q}(a, x, y, z)$

$b \in \mathrm{N}$ (2)	$c \in \mathrm{N}$ (3)	$a+b=m$ (4)	$a+c=m$ (5)	$b \neq c$ (6)

$$
\begin{array}{ll}
- & (**) \\
\hline
- & \neg.\ r\in \mathrm{N}\wedge \overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{Q}(r, x, y, z) \\
7 & r'\in \mathrm{N}\wedge \overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{Q}(r', x, y, z) \\
\hline
8 & r\notin \mathrm{N} \\
9 & \neg \overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{Q}(r, x, y, z)
\end{array}
$$

(7)

$$
\begin{array}{l|l}
\begin{array}{c} r'\in \mathrm{N} \\ {\scriptstyle (8)} \\ \mathrm{N}{*}2 \end{array} &
\begin{array}{l}
- \quad \overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{Q}(r', x, y, z) \\
\hline_{(s,t,p)} \\
10\ \ s\notin \mathrm{N} \qquad 12\ \ r'+s\neq p \qquad 14\ \ s=t \\
11\ \ t\notin \mathrm{N} \qquad 13\ \ r'+t\neq p \qquad \text{Cut N+16}
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\hline
- & \neg.\ r\in \mathrm{N}\wedge r'+s=p\rightarrow \exists x.\ p=x' \\
\hline
\end{array}
$$

$$
\begin{array}{lll}
\underset{(8)}{r\in \mathrm{N}} & \underset{(12)}{r'+s=p} &
\begin{array}{l}
- \quad \neg\exists x.\ p=x' \\
\hline_{(q)} \\
15 \quad p\neq q' \qquad \text{Cut N+20}
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\hline
- & \neg.\ r\in \mathrm{N}\wedge r'+s=q'\rightarrow r+s=q \\
- & \neg.\ r\in \mathrm{N}\wedge r'+t=q'\rightarrow r+t=q \\
\hline
\end{array}
$$

$$
\begin{array}{llll}
\underset{(8)}{r\in \mathrm{N}} & \underset{(12,15,=)}{r'+s=q'} & \underset{(13,15,=)}{r'+t=q'} & \begin{array}{l} 16\ \ r+s\neq q \\ 17\ \ r+t\neq q \end{array}
\end{array}
$$

(9)

$$
\begin{array}{lllll}
\underset{(10)}{s\in \mathrm{N}} & \underset{(11)}{t\in \mathrm{N}} & \underset{(16)}{r+s=q} & \underset{(17)}{r+t=q} & \underset{(14)}{s\neq t}
\end{array}
$$

N+27 $a, b, c\in \mathrm{N}\wedge b+a=m\wedge c+a=m\rightarrow b=c$ (From N+22 and N+26.)

N+28 $a, b\in \mathrm{N}\wedge a+b=b\rightarrow a=0$ (From N+12 and N+27.)

N+29 $a, b\in \mathrm{N}\wedge b+a=b\rightarrow a=0$ (From N+13 and N+26.)

§6 Domain and Range of Addition

N+30 $\mathrm{Add}\upharpoonright \mathrm{N}\times \mathrm{V}\in \mathrm{Un}$

$$
\begin{array}{ll}
- & \text{N+30} \\
\hline_{(w,s,t)} & \\
- & \langle ws\rangle\notin \mathrm{Add}\upharpoonright \mathrm{N}\times \mathrm{V} \\
- & \langle wt\rangle\notin \mathrm{Add}\upharpoonright \mathrm{N}\times \mathrm{V} \\
1 & s=t
\end{array}
$$

- $\neg.\ \langle ws\rangle\in\mathrm{Add}\wedge w\in\mathrm{N}\times\mathrm{V}$
- $\neg.\ \langle wt\rangle\in\mathrm{Add}\wedge w\in\mathrm{N}\times\mathrm{V}$

---

2 $\langle ws\rangle\notin\mathrm{Add}$
3 $\langle wt\rangle\notin\mathrm{Add}$
Spf $w\notin\mathrm{N}\times\mathrm{V}$
- $w\notin\mathrm{N}\times\mathrm{V}$

- $\neg\exists xy.\ w=\langle xy\rangle\wedge x\in\mathrm{N}$

(a, b) ---

4 $w\neq\langle ab\rangle$
5 $a\notin\mathrm{N}$ Cut N+15

- $\neg.\ a\in\mathrm{N}\wedge\langle abs\rangle\in\mathrm{Add}\wedge\langle abt\rangle\in\mathrm{Add}\rightarrow s=t$

$\underset{(5)}{a\in\mathrm{N}}$ $\underset{(2,4,=)}{\langle abs\rangle\in\mathrm{Add}}$ $\underset{(3,4,=)}{\langle abt\rangle\in\mathrm{Add}}$ $\underset{(1)}{s\neq t}$

REMARK: The Spf-formula $w\notin\mathrm{N}\times\mathrm{V}$ and the same formula directly under it are both derivatives of the defining formula of $\mathrm{Add}\upharpoonright\mathrm{N}\times\mathrm{V}$ and the lower $w\notin\mathrm{N}\times\mathrm{V}$ is used effectively in the proof, so that the proof of N+30 is irreducible. The specialized definition of $\mathrm{N}\times\mathrm{V}$ is used under the formula $w\notin\mathrm{N}\times\mathrm{V}$. The analytically provable formula $\langle\langle ab\rangle c\rangle=\langle abc\rangle$ is used at the bottom of the two middle strings of the proof.

N+31 $a, b\in\mathrm{N}\rightarrow\exists x.\ \langle abx\rangle\in\mathrm{Add}$

Define the set P for induction by

$$u\in\mathrm{P}\equiv\exists x.\ \langle ubx\rangle\in\mathrm{Add}\,.$$

- N+31

---

1 $a\notin\mathrm{N}$ 2 $b\notin\mathrm{N}$
3 $\exists x.\ \langle abx\rangle\in\mathrm{Add}$

(s)[P](1) ---

- $0\in\mathrm{P}$

- $\exists x.\ \langle 0bx\rangle\in\mathrm{Add}$

$\underset{(2)}{\langle 0bb\rangle\in\mathrm{Add}}$
N+12

- $s\notin\mathrm{P}$
- $s'\in\mathrm{P}$

- $\neg\exists x.\ \langle sbx\rangle\in\mathrm{Add}$
(t) 4 $\exists x.\ \langle s'bx\rangle\in\mathrm{Add}$
---
5 $\langle sbt\rangle\notin\mathrm{Add}$
(4) ---
6 $\langle s'bt'\rangle\in\mathrm{Add}$ Cut N+18

- $\neg.\ \langle sbt\rangle\in\mathrm{Add}\rightarrow\langle s'bt'\rangle\in\mathrm{Add}$

$\underset{(5)}{\langle sbt\rangle\in\mathrm{Add}}$ $\underset{(6)}{\langle s'bt'\rangle\notin\mathrm{Add}}$

- $a\notin\mathrm{P}$

$\neg\exists x.\ \underset{(3)}{\langle abx\rangle}\in\mathrm{Add}$

N+32 $N \times N \subseteq D_{Add}$

$$
\begin{array}{ll}
- & \text{N+32} \\
\hline
{\scriptstyle (w)} & \\
- & w \notin N \times N \\
- & w \in D_{Add} \\
\hline
{\scriptstyle (r,s)} & \\
- & \neg \exists xy.\ w = \langle xy \rangle \wedge x \in N \wedge y \in N \\
1 & \exists x.\ \langle wx \rangle \in Add \\
\hline
2 & w \neq \langle rs \rangle \\
3 & r \notin N \\
4 & s \in N \qquad \text{Cut N+31} \\
\hline
- & \neg .\ r, s \in N \rightarrow \exists x.\ \langle rsx \rangle \in Add \\
\hline
\end{array}
$$

$$
\underset{(3)}{r \in N} \qquad \underset{(4)}{s \in N} \qquad
\begin{array}{ll}
- & \neg \exists x.\ \langle rsx \rangle \in Add \\
\hline
{\scriptstyle (t)} & \\
5 & \langle rst \rangle \notin Add \\
\hline
{\scriptstyle (1)} & \\
 & \underset{(5,2,=)}{\langle wt \rangle \in Add}
\end{array}
$$

N+33 $W_{Add \restriction N \times N} \subseteq N$

$$
\begin{array}{ll}
- & \text{N+33} \\
\hline
{\scriptstyle (c)} & \\
- & c \notin W_{Add \restriction N \times N} \\
1 & c \in N \\
\hline
- & \neg \exists x.\ \langle xc \rangle \in Add \restriction N \times N \\
\hline
{\scriptstyle (w)} & \\
- & \langle wc \rangle \notin Add \restriction N \times N \\
\hline
2 & \langle wc \rangle \notin Add \\
- & w \notin N \times N \\
\hline
- & \neg \exists xy.\ w = \langle xy \rangle \wedge x \in N \wedge y \in N \\
\hline
{\scriptstyle (a,b)} & \\
3 & w \neq \langle ab \rangle \\
4 & a \notin N \\
5 & b \notin N \qquad \text{Cut N+14} \\
\hline
- & \neg .\ a, b \in N \wedge \langle abc \rangle \in Add \rightarrow c \in N \\
\hline
\end{array}
$$

$$
\underset{(4)}{a \in N} \qquad \underset{(5)}{b \in N} \qquad \underset{(2,3,=)}{\langle abc \rangle \in Add} \qquad \underset{(1)}{c \notin N}
$$

§7 Characterization of Addition

N+34 $\langle 0bb\rangle\in\sigma\wedge\forall xy.\langle xby\rangle\in\sigma\to\langle x'by'\rangle\in\sigma.\wedge a\in\mathrm{N}\wedge\langle abc\rangle\in\mathrm{Add}\to\langle abc\rangle\in\sigma$

Define the set P for induction by

$$u\in\mathrm{P}\equiv u\in\mathrm{N}\wedge\forall x.\ \mathrm{K}(u,x)$$

where
$$\mathrm{K}(u,x)\equiv.\ \langle ux\rangle\in\tau\to\langle ubx\rangle\in\sigma\,.$$

- N+34

---

1 $\langle 0bb\rangle\notin\sigma$ 2 $\neg\forall xy.\ \langle xby\rangle\in\sigma\to\langle x'by'\rangle\in\sigma$

3 $a\notin\mathrm{N}$ 4 $\langle abc\rangle\notin\mathrm{Add}$ 5 $\langle abc\rangle\in\sigma$

(4)

- $\neg\exists fxyz.\ \langle abc\rangle=\langle xyz\rangle\wedge f\in\mathrm{Un}\wedge\mathrm{L}(yf)\wedge\langle xz\rangle\in f$

(τ,a,b,c)

6 $\tau\notin\mathrm{Un}$

7 $\neg\mathrm{L}(b,\tau)$

8 $\langle ac\rangle\notin\tau$

(7)

9 $\langle 0b\rangle\notin\tau$ (7 is used here partly)

$(r)[\mathrm{P}](3)$

$0\in\mathrm{P}$ (*) | $r\notin\mathrm{P}$, $r'\in\mathrm{P}$ (**) | $a\notin\mathrm{P}$ (***)

- (*)

- $0\in\mathrm{N}\wedge\forall x.\ \mathrm{K}(0,x)$

$0\in\mathrm{N}$ *N*1

- $\forall x.\ \mathrm{K}(0,x)$

(r)

- $\mathrm{K}(0,r)$

10 $\langle 0r\rangle\notin\tau$

11 $\langle 0br\rangle\in\sigma$

(6)

$\langle 0r\rangle\in\tau$ (10) | $\langle 0b\rangle\in\tau$ (9) | $r\neq b$ (1, 11, =)

- (***)

- $\neg.\ a\in\mathrm{N}\wedge\forall x.\ \mathrm{K}(a,x)$

Spf $a\notin\mathrm{N}$

- $\neg\forall x.\ \mathrm{K}(a,x)$

- $\neg\mathrm{K}(a,c)$

$\langle ac\rangle\in\tau$ (8) | $\langle abc\rangle\notin\sigma$ (5)

- (**)

- $\neg.\ r\in \mathrm{N}\wedge \forall x.\ \mathrm{K}(r, x)$

10 $r'\in \mathrm{N}\wedge \forall x.\ \mathrm{K}(r', x)$

---

11 $r\notin \mathrm{N}$

12 $\neg \forall x.\ \mathrm{K}(r, x)$

(10)

$r'\in \mathrm{N}$ (11) N*2 | (s) - $\forall x.\ \mathrm{K}(r', x)$

- $\mathrm{K}(r', s)$

13 $\langle r's\rangle\notin\tau$

14 $\langle r'bs\rangle\in\sigma$ Cut N+7

- $\neg.\ \tau\in \mathrm{Un}\wedge \mathrm{L}(b, \tau)\wedge r\in \mathrm{N}\wedge\langle r's\rangle\in\tau\rightarrow\exists x.\ s=x'$

$\tau\in\mathrm{Un}$ (6) $\quad \mathrm{L}(b, \tau)$ (7) $\quad r\in \mathrm{N}$ (11) $\quad \langle r's\rangle\in\tau$ (13) $\quad$ (t) - $\neg\exists x.\ s=x'$

15 $s\neq t'$

(2)

- $\neg.\ \langle rbt\rangle\in\sigma\rightarrow\langle r'bt'\rangle\in\sigma$

---

16 $\langle rbt\rangle\in\sigma$ $\quad \langle r'bt'\rangle\notin\sigma$ (14, 15, =)

(12)

- $\neg.\ \langle rt\rangle\in\tau\rightarrow\langle rbt\rangle\in\sigma$

---

17 $\langle rt\rangle\in\tau$ $\quad \langle rbt\rangle\notin\sigma$ (16)

Cut N+8

- $\neg.\ \tau\in \mathrm{Un}\wedge \mathrm{L}(b, \tau)\wedge r\in \mathrm{N}\wedge\langle r't'\rangle\in\tau\rightarrow\langle rt\rangle\ni\tau$

$\tau\in\mathrm{Un}$ (6) $\quad \mathrm{L}(b, \tau)$ (7) $\quad r\in \mathrm{N}$ (11) $\quad \langle r't'\rangle\in\tau$ (13, 15, =) $\quad \langle rt\rangle\notin\tau$ (17)

N+35 $\quad \sigma\in\mathrm{Un}\wedge\langle 0bb\rangle\in\sigma\wedge\forall xy.\ \langle xby\rangle\in\sigma\rightarrow\langle x'by'\rangle\in\sigma.\ \wedge\langle abc\rangle\in\sigma\rightarrow\langle abc\rangle\in\mathrm{Add}$

Define the set $\tau_b(=\tau_{b,\sigma})$ by $u\in\tau_b\equiv\exists xy.\ u=\langle xy\rangle\wedge\langle xby\rangle\in\sigma$.

- N+35

---

1 $\sigma\notin\mathrm{Un}$ $\quad$ 2 $\langle 0bb\rangle\notin\sigma$ $\quad$ 3 $\neg\forall xy.\ \langle xby\rangle\in\sigma\rightarrow\langle x'by'\rangle\in\sigma$

4 $\langle abc\rangle\notin\sigma$ $\quad$ 5 $\langle abc\rangle\in\mathrm{Add}$

(5)

- $\exists fxyz.\ \langle abc\rangle=\langle xyz\rangle\wedge f\in\mathrm{Un}\wedge \mathrm{L}(y, f)\wedge\langle xz\rangle\in f$

$[\tau_b]$

$\tau_b\in\mathrm{Un}$ (*) $\quad \mathrm{L}(b, \tau_b)$ (**) $\quad$ - $\langle ac\rangle\in\tau_b$

$\langle abc\rangle\in\sigma$ (4)

$$
\begin{array}{ll}
- & (*) \\
\hline_{(r,s,t)} & \\
- & \langle rs\rangle \notin \tau_b \\
{\scriptstyle (r,s)}\ - & \langle rt\rangle \notin \tau_b \\
{\scriptstyle (r,t)}\ 6 & s=t \\
\hline
7 & \langle rbs\rangle \notin \sigma \\
8 & \langle rbt\rangle \notin \sigma \\
\hline_{(1)} & \\
\end{array}
$$

$$
\underset{(7)}{\langle rbs\rangle \in \sigma} \qquad \underset{(8)}{\langle rbt\rangle \in \sigma} \qquad \underset{(6)}{s \neq t}
$$

$$
\begin{array}{l|l}
\multicolumn{2}{c}{- \qquad (**)} \\
\hline
- \ \langle 0b\rangle \in \tau_b & {\scriptstyle (r,t)}\ - \ \forall xy.\ \langle xy\rangle \in \tau_b \rightarrow \langle x'y'\rangle \in \tau_b \\
\hline
\underset{(2)}{\langle 0bb\rangle \in \sigma} & - \ \langle rt\rangle \notin \tau_b \\
 & {\scriptstyle (r,t)}\ - \ \langle r't'\rangle \in \tau_b \\
\cline{2-2}
 & 6 \ \langle rbt\rangle \notin \sigma \\
 & 7 \ \langle r'bt'\rangle \in \sigma \\
\hline_{(3)} & \\
\underset{(6)}{\langle rbt\rangle \in \sigma} & \underset{(7)}{\langle r'bt'\rangle \notin \sigma}
\end{array}
$$

N+36 $\sigma \in \mathrm{Un} \wedge \langle 0bb\rangle \in \sigma \wedge \forall xy.\ \langle xby\rangle \in \sigma \rightarrow \langle x'by'\rangle \in \sigma.\ \wedge a \in \mathrm{N}:$
$\rightarrow. \langle abc\rangle \in \sigma \equiv \langle abc\rangle \in \mathrm{Add}$

From N+34 and N+35.

N+37 $\sigma \in \mathrm{Un} \wedge \overset{\mathrm{N}}{\forall} x.\ \langle 0xx\rangle \in \sigma.\ \wedge \forall xyz.\ \langle xyz\rangle \in \sigma \rightarrow \langle x'yz'\rangle \in \sigma:$
$\rightarrow \sigma \upharpoonright \mathrm{N}^2 = \mathrm{Add} \upharpoonright \mathrm{N}^2$

$$
\begin{array}{ll}
- & \text{N+37} \\
\hline
1 & \sigma \notin \mathrm{Un} \qquad 2 \ \neg \overset{\mathrm{N}}{\forall} x.\ \langle 0xx\rangle \in \sigma \\
3 & \neg \forall xyz.\ \langle xyz\rangle \in \sigma \rightarrow \langle x'yz'\rangle \in \sigma \\
- & \sigma \upharpoonright \mathrm{N}^2 = \mathrm{Add} \upharpoonright \mathrm{N}^2 \\
\hline_{(w)} & \\
\end{array}
$$

$$
\begin{array}{l|l}
- \ w \in \sigma \upharpoonright \mathrm{N}^2 & - \ w \in \mathrm{Add} \upharpoonright \mathrm{N}^2 \\
- \ w \notin \mathrm{Add} \upharpoonright \mathrm{N}^2 & - \ w \notin \sigma \upharpoonright \mathrm{N}^2. \\
\hline
(*) & (**)
\end{array}
$$

(*)

---

4 $\exists xy.\ w=\langle xy\rangle \wedge w\in\sigma \wedge x\in \mathrm{N}^2$

$-$ $\neg\exists xy.\ w=\langle xy\rangle \wedge w\in \mathrm{Add} \wedge x\in \mathrm{N}^2$

(r, c) ---

5 $w\neq\langle rc\rangle$

6 $w\notin \mathrm{Add}$

7 $r\notin \mathrm{N}^2$

(7) ---

$-$ $\neg\exists xy.\ r=\langle xy\rangle \wedge x\in\mathrm{N} \wedge y\in\mathrm{N}$

(a, b) ---

8 $r\neq\langle ab\rangle$

9 $a\notin\mathrm{N}$

10 $b\notin\mathrm{N}$

(4) ---

$\underset{(5)}{w=\langle rc\rangle}$ 11 $w\in\sigma$ $\underset{(7)}{r\in\mathrm{N}^2}$

(2) ---

$\underset{(10)}{b\in\mathrm{N}}$ 12 $\langle 0bb\rangle\in\sigma$

(3) ---

13 $\neg\forall xz.\ \langle xbz\rangle\in\sigma\rightarrow\langle x'bz'\rangle\in\sigma$ Cut N+34

$\underset{(12)}{\langle 0bb\rangle\in\sigma}$ $\underset{(13)}{\forall xz.\ \langle xbz\rangle\in\sigma\rightarrow\langle x'bz'\rangle\in\sigma}$ $\underset{(9)}{a\in\mathrm{N}}$ $\underset{(5,8,6,=)}{\langle abc\rangle\in\mathrm{Add}}$ $\underset{(5,8,11,=)}{\langle abc\rangle\notin\sigma}$

(**)

---

4 $\exists xy.\ w=\langle xy\rangle \wedge w\in \mathrm{Add} \wedge x\in \mathrm{N}^2$

$-$ $\neg\exists xy.\ w=\langle xy\rangle \wedge w\in\sigma \wedge x\in \mathrm{N}^2$

(r, c) ---

5 $w\neq\langle rc\rangle$

6 $w\notin\sigma$

7 $r\notin \mathrm{N}^2$

(7) ---

$-$ $\neg\exists xy.\ r=\langle xy\rangle \wedge x\in\mathrm{N} \wedge y\in\mathrm{N}$

(a, b) ---

8 $r\neq\langle ab\rangle$

Spf $a\notin\mathrm{N}$

9 $b\notin\mathrm{N}$

(4) ---

$\underset{(5)}{w=\langle rc\rangle}$ 10 $w\in\mathrm{Add}$ $\underset{(7)}{r\in\mathrm{N}^2}$

(2) ---

$\underset{(9)}{b\in\mathrm{N}}$ 11 $\langle 0bb\rangle\notin\sigma$

(3) ---

12 $\neg\forall xz.\ \langle xbz\rangle\in\sigma\rightarrow\langle x'bz'\rangle\in\sigma$ Cut N+35

$\underset{(1)}{\sigma\in\mathrm{Un}}$ $\underset{(11)}{\langle 0bb\rangle\in\sigma}$ $\underset{(12)}{\forall xy.\ \langle xby\rangle\in\sigma\rightarrow\langle x'by'\rangle\in\sigma}$ $\underset{(5,8,6,=)}{\langle abc\rangle\in\sigma}$ $\underset{(5,8,10,=)}{\langle abc\rangle\notin\mathrm{Add}}$

Section B Multiplication

In order to define the constant Mlt (multiplication), we introduce, as in the case of addition, the abbreviations:

$$\mathrm{K}(xyzu\sigma) \equiv \langle xy\rangle \in \sigma \wedge y+u \underset{\mathrm{N}}{=} z \rightarrow \langle x'z\rangle \in \sigma,$$

$$\mathrm{M}(u\sigma) \equiv \langle 00\rangle \in \sigma \wedge \forall xy \forall z.\ \mathrm{K}(xyzu\sigma),$$

and the sets ν, λ_σ, and M by the defining formulas:

$$u \in \nu \equiv \exists xy.\ u=\langle xy\rangle \wedge y \underset{\mathrm{N}}{=} 0,$$

$$u \in \lambda_\sigma \equiv \exists xy.\ u \underset{\mathrm{Un}}{=} \langle xy\rangle \wedge \exists z.\ \langle xz\rangle \in \sigma \wedge x+z=y,$$

$$u \in \mathrm{M} \equiv u \in \mathrm{N} \wedge \exists f.\ \mathrm{M}(u,f),$$

of which the last set M is used as set for induction in the proof of N×5. We prove first N×1, *N×2, N×3, and N×4 as lemmas for N×5.

N×1 $\nu \in \mathrm{Un}$

- N×1
(r,s,t) ————————
- $\langle rs\rangle \notin \nu$ (r,s)
- $\langle rt\rangle \notin \nu$ (r,t)
1 $s=t$
- - - - - - - - - - -
$s \neq 0$
$t \neq 0$ (1)

*N×2 $\mathrm{M}(0, \nu)$

- N×2
————————————————
- $\langle 00\rangle \in \nu$ | - $\overset{\mathrm{N}}{\forall xy \forall z}.\ \mathrm{K}(xyz0\nu)$ (r,s,t)
———— | ————————
$0=0$ (=) | *Spf $r \notin \mathrm{N}$ 1 $s \notin \mathrm{N}$
| - $\langle rs\rangle \notin \nu$ (r,s)
| 2 $s+0 \neq t$
| - $\langle r't\rangle \in \nu$
| ————————
| 3 $s \neq 0$
| 4 $t=0$ Cut N+15
————————————————
- $\neg.\ s \in \mathrm{N} \wedge s+0=t \wedge s+0=0 \rightarrow t=0$
————————————————
$s \in \mathrm{N}$ (1) $s+0=t$ (2) $s+0=0$ (1, 3, =) $t \neq 0$ (4)
Cut N+13

N×3 $\sigma\in\mathrm{Un}\rightarrow\lambda_{\sigma}\in\mathrm{Un}$

$-$ N×3

---

1 $\neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\rightarrow y=z$

$-$ $\forall xyz.\ \langle xy\rangle\in\lambda_{\sigma}\wedge\langle xz\rangle\in\lambda_{\sigma}\rightarrow y=z$

(r,s,t)

$-$ $\langle rs\rangle\notin\lambda_{\sigma}$

(r,s)

$-$ $\langle rt\rangle\notin\lambda_{\sigma}$

(r,t)

2 $s=t$

$-$ $\neg\overset{\mathrm{N}}{\exists} z.\ \langle rz\rangle\in\sigma\wedge r+z=s$

(a)

$-$ $\neg\overset{\mathrm{N}}{\exists} z.\ \langle rz\rangle\in\sigma\wedge r+z=t$

(b)

---

3 $a\notin\mathrm{N}$ 4 $\langle ra\rangle\notin\sigma$ 5 $r+a\neq s$

Spf $b\notin\mathrm{N}$ 6 $\langle rb\rangle\notin\sigma$ 7 $r+b\neq t$

(1)

$\underset{(4)}{\langle ra\rangle\in\sigma}$ $\underset{(6)}{\langle rb\rangle\in\sigma}$ 8 $a\neq b$ Cut N+15

$-$ $\neg.\ a\in\mathrm{N}\wedge r+a=s\wedge r+a=t\rightarrow s=t$

$\underset{(3)}{a\in\mathrm{N}}$ $\underset{(5)}{r+a=s}$ $\underset{(7,8,=)}{r+a=t}$ $\underset{(2)}{s\neq t}$

N×4 $a\in\mathrm{N}\wedge\mathrm{M}(a,\sigma)\rightarrow\mathrm{M}(a',\lambda_{\sigma})$

$-$ N×4

---

1 $a\notin\mathrm{N}$

$-$ $\neg.\ \langle 00\rangle\in\sigma\wedge\overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{K}(xyza\sigma)$

2 $\langle 00\rangle\in\lambda_{\sigma}\wedge\overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{K}(xyza'\lambda_{\sigma})$

---

3 $\langle 00\rangle\notin\sigma$

4 $\neg\overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{K}(xyza\sigma)$

(2)

$-$ $\langle 00\rangle\in\lambda_{\sigma}$

$-$ $\overset{\mathrm{N}}{\exists} z.\ \langle 0z\rangle\in\sigma\wedge 0+z=0$

$\underset{*\mathrm{N}*1}{0\in\mathrm{N}}$ $\underset{(3)}{\langle 00\rangle\in\sigma}$ $\underset{\text{Cut N+12, }*\mathrm{N}*1}{0+0=0}$

$-$ $\overset{\mathrm{N}}{\forall} xy\forall z.\ \mathrm{K}(xyza'\lambda_{\sigma})$

(r,s,t)

---

5 $r\notin\mathrm{N}$

6 $s\notin\mathrm{N}$

$-$ $\mathrm{K}(rsta'\lambda_{\sigma})$

$-$ $\langle rs\rangle\notin\lambda_{\sigma}$

(r,s)

7 $s+a'\neq t$

$-$ $\langle r't\rangle\in\lambda_{\sigma}$

- (p) $\overset{\mathrm{N}}{\neg\exists z}.\ \langle rz\rangle\in\sigma\wedge r+z=s$

8 $\overset{\mathrm{N}}{\exists z}.\ \langle r'z\rangle\in\sigma\wedge r'+z=t$

---

9 $p\notin\mathrm{N}$

10 $\langle rp\rangle\notin\sigma$

11 $r+p\neq s$ Cut N+31

- $\neg.\ p, a\in\mathrm{N}\rightarrow\exists x.\ p+a=x$

$\underset{(9)}{p\in\mathrm{N}}$ $\underset{(1)}{a\in\mathrm{N}}$ (q) - $\neg\exists x.\ p+a=x$

12 $p+a\neq q$

(8)

$\underset{(9,1,12)}{q\in\mathrm{N}}$ 13 $\underset{(*)}{\langle r'q\rangle\in\sigma}$ 13 $\underset{(**)}{r'+q=t}$

Cut N+14

- (*)

(4)

- $\neg.\ r\in\mathrm{N}\wedge p\in\mathrm{N}\wedge\mathrm{K}(rpqa\sigma)$

$\underset{(5)}{r\in\mathrm{N}}$ $\underset{(9)}{p\in\mathrm{N}}$ $\underset{(10)}{\langle rp\rangle\in\sigma}$ $\underset{(12)}{p+a=q}$ $\underset{(13)}{\langle r'q\rangle\notin\sigma}$

- (**) Cut N+24

- $\neg.\ r', p, a\in\mathrm{N}\wedge r'+p=s'\wedge p+a=q\wedge s'+a=t\rightarrow r'+q=t$

$\underset{(5)}{r'\in\mathrm{N}}$ $\underset{(9)}{p\in\mathrm{N}}$ $\underset{(1)}{a\in\mathrm{N}}$ $\underset{(11)}{r'+p=s'}$ $\underset{(12)}{p+a=q}$ $\underset{(7)}{s'+a=t}$ $\underset{(13)}{r'+q\neq t}$

N*2 N+18 (1, 6, N*2) N+21 N+22

N×5 $a\in\mathrm{N}\rightarrow\overset{\mathrm{Un}}{\exists f}.\ \mathrm{M}(a,f)$ (or $a\in\mathrm{N}\rightarrow a\in\mathrm{M}$)

- N×5

- $a\notin\mathrm{N}$

1 $\overset{\mathrm{Un}}{\exists f}.\ \mathrm{M}(a,f)$

(r)[M]

- $0\in\mathrm{M}$

$0\in\mathrm{N}$ - $\overset{\mathrm{Un}}{\exists f}.\ \mathrm{M}(0,f)$

*N*1

$\nu\in\mathrm{Un}$ $\mathrm{M}(0,\nu)$

N×1 *N×2

- $r\notin\mathrm{M}$

2 $r'\in\mathrm{M}$

---

3 $r\notin\mathrm{N}$

- $\neg\overset{\mathrm{Un}}{\exists f}.\ \mathrm{M}(r,f)$

(σ)

---

4 $\sigma\notin\mathrm{Un}$

5 $\neg\mathrm{M}(r,\sigma)$

- $a\notin\mathrm{M}$

Spf $a\notin\mathrm{N}$

- $\neg\overset{\mathrm{Un}}{\exists f}.\ \underset{(1)}{\mathrm{M}(a,f)}$

(2) ————————————————————————————

$$\begin{array}{lcc} & & \overset{\mathrm{Un}}{\exists} f.\ \mathrm{M}(r',f) \\ \underset{\text{N*2}}{\overset{(3)}{r' \in \mathrm{N}}} & [\lambda_\sigma] & \overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\ & & \underset{\mathrm{N}\times 3}{\overset{(4)}{\lambda_\sigma \in \mathrm{Un}}} \qquad \underset{\mathrm{N}\times 4}{\overset{(3,5)}{\mathrm{M}(r', \lambda_\sigma)}} \end{array}$$

Now we define the constant Mlt by

$$u \in \mathrm{Mlt} \equiv \exists fxyz.\ u = \langle xyz \rangle \wedge f \in \mathrm{Un} \wedge \mathrm{M}(yf) \wedge \langle xz \rangle \in f\,.$$

The formula N×5 i.e. $\mathrm{N} \subseteq \mathrm{M}$ is the fundamental formula in deducing the properties of Mlt, just in the same way as the formula N+4 is so in Section A concerning Add.

First, we deduce the properties of $\mathrm{M}(c, \sigma)$, corresponding to the deduction of the properties of $\mathrm{L}(c, \sigma)$ in § 2, Section A. For example, the formula

$$a,\ c \in \mathrm{N} \wedge \sigma \in \mathrm{Un} \wedge \mathrm{M}(c\sigma) \wedge \mathrm{M}(c\tau) \wedge \langle ab \rangle \in \sigma \rightarrow \langle ab \rangle \in \tau$$

is proved by using the set P for induction defined by

$$u \in \mathrm{P} \equiv u \in \mathrm{N} \wedge \forall x.\ \langle ux \rangle \in \sigma \rightarrow \langle ux \rangle \in \tau\,.$$

Next, the deductions of fundamental properties of Mlt are performed in a similar way as in §§ 3–6, Section A. In these deductions some formulas proved in Section A are used as cut formulas.

For the characterization of Mlt, i.e. for proving the formula

$$\begin{aligned} &\sigma \in \mathrm{Un} \wedge \overset{\mathrm{N}}{\forall} x.\ \langle 0x0 \rangle \in \sigma \\ &\wedge \overset{\mathrm{N}}{\forall} uxy \forall z.\ \langle xuy \rangle \in \sigma \wedge y + u = z \rightarrow \langle x'uz \rangle \in \sigma: \\ &\rightarrow \sigma \restriction \mathrm{N} \times \mathrm{N} = \mathrm{Mlt} \restriction \mathrm{N} \times \mathrm{N}\,, \end{aligned}$$

we need again the same set $\tau_{b,\sigma}$ used in the proof of N+35, Section A.

In order to make simpler the consistency proof of these deductions, we have to abondon the closure property with respect to substitutions of variables in restricting the substitution of variables to the extent which is required in these deductions. Such a restriction is categorically necessary for further consistent formulation of a developed stage of mathematics in order to avoid contradictions.

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