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Supplement to my Paper "The Theory of Construction of Finite Semigroups II"

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When a finite semilattice $T=\{\tau_0, \tau_1, \cdots, \tau_{n-1}\}$ and a system of finite semigroups S_τ $(\tau \in T)$ are arbitrarily given, there exists a composition S of S_τ by T. (See [1], 3 or § 8, p. 30) Let τ_{n-1} be a minimal element of T, and we may regard T as a composition of a semilattice $T_0=\{\tau_0, \cdots, \tau_{n-2}\}$ and $\{\tau_{n-1}\}$ i.e. $T=(T_0, \gamma, \tau_{n-1})$. (cf. [1] § 8, p. 28) We use the successive method in construction of S. More precisely, the composition of S_τ $(\tau \in T)$ by T is constructed as a composition $S_0=\sum_{\tau \in T_0} S_\tau$ and $S_{\tau_{n-1}}$. (cf. Theorem 24 in [1]) The following problem was proposed as an unsolved one in the previous paper [1], 3 of § 8.

Can we adopt as S_0 an arbitrary composition of S_τ $(\tau \in T_0)$ by T_0 when we construct S? This means a question whether there are Φ_0 $(\subset \Phi)$ and Ψ_0 $(\subset \Psi)$ fulfilling (8.4.1), (8.4.2) and (8.4.3) for any composition S_0 of S_τ by T_0.

In the present short note, this question is denied, giving a simple counter example in the following manner.

Let $T=(T_0, \gamma, \tau_5)=\{\tau_i\,;\ 0 \leqq i \leqq 5\}$ be a semilattice of order 6 with multiplication defined by the following diagram.

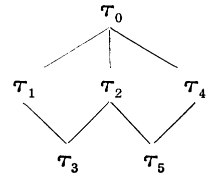


Fig. 1

where $T_0=\{\tau_i\,;\ 0 \leqq i \leqq 4\}$ and $\gamma=\begin{pmatrix}\tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4\\ \tau_0 & \tau_0 & \tau_2 & \tau_2 & \tau_4\end{pmatrix}$.

S_{τ_i} $(0 \leqq i \leqq 5)$ are defined as

$$S_{\tau_0}=\{a,\ b\} \quad \text{with} \quad a^2=ba=a\,, \quad ba=b^2=b\,,$$

$$S_{\tau_1}=\{c\}\,, \quad S_{\tau_2}=\{d\}\,, \quad S_{\tau_3}=\{e\}\,, \quad S_{\tau_4}=\{f\}\,, \quad S_{\tau_5}=\{g\}\,.$$

Let σ be the mapping $\begin{pmatrix} a & b & c & d & e & f & g \\ \tau_0 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \end{pmatrix}$. Consider a composition S_0 of S_{τ_i} $(0 \leqq i \leqq 4)$ by T_0 with the following multiplication.

	a	b	c	d	e	f
a	a	b	b	b	a	a
b	a	b	b	b	b	a
c	a	b	c	b	c	a
d	a	b	b	d	d	a
e	a	b	c	d	e	a
f	a	b	b	b	a	f

Since $S_0' = \{a, b, c, d, e\}$ was obtained as 1079_5 in [2] and S_0 is a composition of S_0' and $\{f\}$, S_0 is seen to be a semigroup by testing the conditions of §1 in [1].

In order to get a composition S of S_0 and $\{g\}$ such that S is a composition of S_{τ_i} $(0 \leqq i \leqq 5)$ by T, we must find a right translation φ of S_0 such that $\sigma\varphi = \gamma\sigma$ i.e.

$$\varphi(a),\ \varphi(b),\ \text{and}\ \varphi(c)\ \text{are}\ a\ \text{or}\ b,$$
$$\varphi(d) = \varphi(e) = d, \qquad \varphi(f) = f.$$

To tell the truth, there is no such φ. Because, if there is a right translation φ, then

$$\varphi(a) = \varphi(af) = a\varphi(f) = af = a\,,$$

while

$$\varphi(a) = \varphi(ae) = a\varphi(e) = ad = b\,.$$

Consequently there is no composition S of S_0 and $\{g\}$ which is, at the same time, a composition of S_{τ_i} $(0 \leqq i \leqq 5)$ by T.

As seen in [2], any semilattice of order at most 5 is either a lattice or a semilattice having a minimal element τ_4 which is covered by only one element; and hence, for the minimal element τ_4, the question is affirmed. Furthermore, if T is a semilattice of order 6 except one having the tyne of Fig. 1, then the question is affirmed for a suitable minimal element. For, we obtain easily that a semilattice of order 6, whose every minimal element is covered by two elements at least, is nothing but a semilattice having the type of Fig. 1.

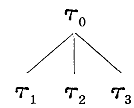
We notice, however, that there is a minimal element for which the question is denied even if T is of order 5.

For example, let S_0 be a semigroup with multiplication

	a	b	c	d	e
a	a	b	a	a	b
b	a	b	b	a	b
c	a	b	c	a	b
d	a	b	a	d	b
e	a	b	b	a	e

(1027_5 in [2])

which is a composition of $S_{\tau_0}=\{a, b\}$, $S_{\tau_1}=\{c\}$, $S_{\tau_2}=\{d\}$, and $S_{\tau_3}=\{e\}$ by the semilattice T_0:



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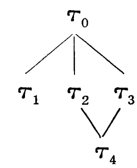
Let T be a semilattice with diagram



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There is no composition of S_0 and $S_{\tau_4}=\{f\}$ which is a composition of S_{τ_i} $(0 \leqq i \leqq 4)$ by T at the same time. Because we find no right translation φ of S_0 which satisfies

$$\varphi(c)=a \text{ or } b\,, \quad \varphi(d)=d\,, \quad \varphi(e)=e\,.$$

Generally it is suggested that the nature of the question depends not only on T but also on the structure of each S_τ, S_0 and S_1. We shall take up the more precise study of this problem again after the structure of finite s-indecomposable semigroups is clarified.

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References

[1] T. Tamura: The theory of construction of finite semigroups (Compositions of semigroups, and finite s-decomposable semigroups), Osaka Math. J. **9**, 1–42 (1957).

[2] ———: All semigroups of order at most 5, J. of Gakugei, Tokushima Univ. **6**, 19–39 (1955).