

## *Kernel Functions of Diffusion Equations (I)*

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Let  $D$  be an open bounded set in a  $d$ -dimensional Euclidean space  $\mathcal{E}$ .

By  $\Delta$  we understand the Laplacian with respect to given coordinates. Consider the diffusion equation

$$(1) \quad \frac{\partial U}{\partial t} = \Delta U$$

on  $D$ . By a *kernel function*  $K(x, y; t)$  we understand a function on  $\mathcal{E} \times \mathcal{E} \times [0, \infty)$  satisfying following properties:

(i)  $K(x, y; t) = 0$  when either  $x$  or  $y$  is on the boundary of  $D$ , if  $K(x, y; t)$  is continuous on boundary for a fixed  $t$ , or the boundary  $\partial D$  is smooth.

(ii) For a fixed  $y$

$$(2) \quad \frac{\partial}{\partial t} K(x, y; t) = \Delta_x K(x, y; t)$$

where  $\Delta_x$  is understood as the Laplacian on the variable  $x$ .

The purpose of this paper is to give a new way of constructing the kernel function on  $D$  which coincides with the Green's function when  $\partial D$ , the boundary of  $D$  is smooth.

In preparations we shall define some notations. Coordinates of points  $x, y, \dots$  on  $\mathcal{E}$  will be written  $x^i; 1 \leq i \leq d, y^j; 1 \leq j \leq d$ , etc. The euclidean distance between two points  $x$  and  $y$  is denoted by

$$(3) \quad |x - y| = \left( \sum_i^d (x^i - y^i)^2 \right)^{1/2}.$$

Let

$$(4) \quad E_t(x, y) = (2\sqrt{\pi t})^{-d} \exp(- (4t)^{-1} |x - y|^2).$$

**Lemma 1.** *Take a point  $x$  on  $D$ . Let  $S(h)$  be a solid sphere around*

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(1), see (15) and (8)

(2), see (8)

$x$  with its radius  $(4h)^{1/6}$  where  $h$  is sufficiently small that  $S(h)$  is within  $D$ . Then

$$(5) \quad 1 \geq \int_{S(h)} E_h(x, y) dy \geq 1 - 2^d \exp\left(-\frac{1}{4} (h)^{-2/3}\right).$$

Proof. Consider polar coordinates  $\rho, \theta^i, 1 \leq i \leq d-1$ , around  $x$  where  $\theta^i$ 's denote angular coordinates. Then

$$(6) \quad \int_{D-S(h)} E_h(x, y) dy \leq \int_{(4h)^{1/3}}^{\infty} d\rho \int_{\theta} \rho^{d-1} (2\sqrt{\pi h})^{-d} \exp(- (4h)^{-1} \rho^2) d\theta.$$

Notice that for any  $y$  outside of  $S(h)$

$$(7) \quad |x-y| \geq \frac{1}{2} (4h)^{1/6} + \frac{1}{2} |x-y|$$

and consequently

$$(8) \quad |x-y|^2 \geq \frac{1}{4} \left( (4h)^{1/3} + |x-y|^2 \right)$$

Hence

$$(9) \quad \begin{aligned} \int_{D-S(h)} E_h(x, y) dy &\leq (2\sqrt{\pi h})^{-d} \int_{\mathcal{E}-S(h)} \exp\left[-\frac{1}{4} (4h)^{-2/3} - \frac{1}{4} |x-y|^2/4h\right] dy \\ &\leq 2^d \exp\left(-\frac{1}{4} (4h)^{-2/3}\right) (2\sqrt{\pi 4h})^{-d} \int \exp(-|x-y|^2/16h) dy \\ &\leq 2^d \exp\left(-\frac{1}{4} (4h)^{-2/3}\right). \end{aligned}$$

However it is well-known that

$$(10) \quad \int_{S(h)} E_h(x, y) dy \leq 1,$$

for any  $h$ .

Hence we have

$$1 \geq \int E_h(x, y) dy \geq 1 - 2^d \exp\left(-\frac{1}{4} (h)^{-2/3}\right),$$

which proves the lemma.

Given two functions  $\phi$  and  $\psi$  of two variables, we define a convolution

$$(11) \quad (\phi * \psi)(x, y) = \int_D \phi(x, z) \psi(z, y) dz.$$

Then

**Lemma 2.**

$$(12) \quad \begin{aligned} (E_t * E_h)(x, y) &\leq E_{t+h}(x, y) \left[ \left( 2\sqrt{\pi} \sqrt{\frac{th}{t+h}} \right)^{-d} \int_D \exp\left(-\frac{t+h}{4th} \left| z - \frac{tx+hy}{t+h} \right|^2\right) dz \right] \end{aligned}$$

Proof. By direct computations

$$\begin{aligned}
 (13) \quad & E_t(x, z)E_h(z, y) \\
 &= (2\sqrt{\pi t})^{-d}(2\sqrt{\pi h})^{-d} \exp\left(-\frac{1}{4t}|x-z|^2-\frac{1}{4h}|z-y|^2\right) \\
 &= (2\sqrt{\pi(t+h)})^{-d}\left(2\sqrt{\pi}\sqrt{\frac{th}{t+h}}\right)^{-d} \exp\left(-|x-y|^2/4(t+h)-\frac{t+h}{4th}\left|z-\frac{tx+hy}{t+h}\right|^2\right).
 \end{aligned}$$

Hence we have (12) by integrating (13) with respect to  $z$  over  $D$ . By iterating these processes  $m$  times for  $E_{t/n}(x, y)$  we can define

$$(14) \quad \overbrace{(E_{t/n} * E_{t/n} * \dots * E_{t/n})}^m(x, y) = (E_{t/n} *)^m(x, y).$$

**Lemma 3.** *Suppose that  $x$  and  $y$  be on a compact convex set  $Q$  contained in  $D$ . Then for small  $t$ ,*

$$(15) \quad (E_{t/l(n)} *)^{2^n}(x, y) = E_t(x, y) (1 + o(\exp - t^{-1/2}))$$

where  $l(n) = 2^{-n}$ .

Proof. Let  $t$  be so small that around any  $x$  in  $Q$ ,  $S(t)$ , i.e. the solid sphere with its radius  $\sqrt[4]{4t}$  is contained in  $D$ .

It is easy to see that

$$\begin{aligned}
 (16) \quad & (E_{t/n} *)^2(x, y) \leq E_{2/tn}(x, y), \\
 & (E_{t/n} *)^3(x, y) \leq (E_{2/tn} * E_{t/n})(x, y), \\
 & \dots,
 \end{aligned}$$

By making convolutions successively

$$(16') \quad (E_{t/n} *)^n(x, y) \leq E_t(x, y).$$

From (16'), by replacing  $t$  by  $t/m$  we have

$$(E_{t/mn} *)^{mn}(x, y) \leq (E_{t/m} *)^m(x, y).$$

This means that if we introduce a partial order  $<$  into positive integers such that  $m < n$  means  $n$  is divided by  $m$ , then  $(E_{t/m} *)^m$  is decreasing when  $m$  is increasing. Hence there exists the limit in the sense of Moore Smith for integers  $2^n$  because these are linearly ordered.

Now let  $Q_k$  be the set of points whose distance from  $Q$  is less than

$$\sum_{j=1}^k \sqrt[4]{4tl(j)}$$

where  $l(j) = 2^{-j}$ .

Notice that the distance between the two sets  $Q_{k+1}$  and  $Q_k$  is equal to  $\sqrt[4]{4tl(k+1)}$ . Therefore for  $x$  and  $y$  in  $Q_k$

$$\begin{aligned}
 (17) \quad & \int_D E_{t l(k)}(x, z) E_{t l(k)}(x, y) dz \\
 & \geq \int_{Q_{k+1}} E_{t l(k)}(x, z) E_{t l(k)}(z, y) dz \\
 & \geq E_{2tl(k)}(x, y) \int_{Q_{k+1}} E_{t l(k)/2}\left(z, \frac{1}{2}(x+y)\right) dz \\
 & = E_{t l(k-1)}(x, y) \left(1 - 2^d \exp\left(-\frac{1}{4}(tl(k+1))\right)^{-2/3}\right).
 \end{aligned}$$

Repeating these processes

$$\begin{aligned}
 (18) \quad & E_{t l(n)} * )^{2^n}(x, y) \\
 & \geq (E_{t l(n-1)} * )^{2^{n-1}}(x, y) \left(1 - 2^d \exp\left(-\frac{1}{4}(tl(n+1))\right)^{-2/3}\right)^{l(n-1)} \\
 & \geq, \dots, E_t(x, y) \sum_{j=1}^n \left(1 - 2^d \exp\left(-\frac{1}{4}(tl(n-j+1))\right)^{-2/3}\right)^{l(j-1)} \\
 & \geq E_t(x, y) \left(1 - \sum_{j=1}^n 2^d 2^{n-j+1} \exp\left(-\frac{1}{4}(tl(n-j+1))\right)^{-2/3}\right) \\
 & \geq E_t(x, y) \left(1 - 2^d \int_1^\infty \exp\left(-\frac{1}{4}(t^{-2/3} \xi^{2/3})\right) d\xi (1 - 0(t))\right) \\
 & \geq E_t(x, y) (1 - 0(\exp(-t^{-1/2})))
 \end{aligned}$$

uniformly in  $n$  when  $t$  is small. This proves the lemma.

REMARK: Since  $E_t(x, y)$  is uniformly continuous in  $t$  over  $D$  and larger than a constant,  $1/n$  can be approximated with dyadic number  $pl(m)$  from below in such a way that

$$E_{t/n}(x, y) \geq E_{t pl(m)}(x, y) e^{-\delta/n}$$

for a preassigned  $\delta$ . Then

$$\begin{aligned}
 (E_{t/n} * )^n(x, y) & \geq (E_{t pl(m)} * )^n(x, y) e^{-\delta} \\
 & \geq E_{t pl(m)}(x, y) (1 - 0(\exp(-t^{-1/2}))).
 \end{aligned}$$

New let  $m$  go to infinity. Then  $\delta$  goes to zero and  $pl(m)$  to  $1/n$ . Hence we have

$$(19) \quad (E_{t/n} * )^n(x, y) \geq E_t(x, y) (1 - 0(\exp(-t^{-1/2}))).$$

These (16') and (19) prove the lemma.

**Lemma 4.** *When both  $x$  and  $y$  are in  $Q$ ,*

$$(20) \quad \lim_{n \rightarrow \infty} (E_{t/n} *)^n(x, y) = K(x, y; t) \text{ exists.}$$

Proof. Set  $t = h/m$  in (19). Then

$$\begin{aligned} (E_{hm/n} *)^{mn}(x, y) &\geq (E_{h/m} *)^m(x, y)(1 - 0(\exp(-h/m)^{-1/2}))^m \\ &= (E_{t/m} *)^m(x, y)(1 - 0(1/m)). \end{aligned}$$

Evidently, however

$$(21) \quad (E_{t/mn} *)^{mn}(x, y) \leq (E_{t/m} *)^m(x, y)$$

and  $(E_{t/m} *)^m(x, y)$  is uniformly bounded above by  $E_t(x, y)$ . Hence

$$(22) \quad |(E_{t/mn} *)^{mn}(x, y) - (E_{t/m} *)^m(x, y)| \leq 0(1/m)E_t(x, y).$$

If both  $m$  and  $n$  be larger than  $m_0$ , then from (23) it follows that

$$(23) \quad \begin{aligned} |(E_{t/m} *)^m(x, y) - (E_{t/n} *)^n(x, y)| &\leq |(E_{t/m} *)^m(x, y) - (E_{t/mn} *)^{mn}(x, y)| \\ &\quad + |(E_{t/mn} *)^{mn}(x, y) - (E_{t/n} *)^n(x, y)| \\ &\leq 0(1/m) + 0(1/n)E_t(x, y) = 0(1/m_0)E_t(x, y). \end{aligned}$$

This proves the existence of the limit (20).

**Lemma 5.** *The diffusion equation (1) on  $D$  has a unique solution up to the initial function under the condition of 0-boundary-value when  $t \neq 0$ .*

Proof. We have only to prove that if  $U$  be a solution of (1) with 0-boundary-value and  $U(x; t) = 0$  everywhere when  $t = 0$ , then  $U(x, t) = 0$  for any  $t$  and for any  $x$ .

Now

$$(24) \quad \frac{d}{dt} \int_D (U(x, t))^2 dx = 2 \int_D \frac{\partial}{\partial t} U(x, t) U(x, t) dx = \int_D (\Delta U) U dx,$$

and by the virtue of Stoke's theorem,

$$= -2 \int_D \sum_{i=1}^n \left( \frac{\partial}{\partial x^i} U(x, t) \right)^2 dx \leq 0.$$

Hence  $\int (U(x, 0))^2 dx = 0$  implies  $\int (U(x, t))^2 dx = 0$ , i.e.  $U(x, t) = 0$ .

This prove the lemma.

**Lemma 6.** *Let  $D'$  be an open convex domain, and  $p$  be a point at a distance less than  $ch^{1/2}$  from  $D'$ . Then there exists a constant  $\varepsilon > 0$*

depending only on  $c$  so that when  $h$  goes to zero

$$(25) \quad \int_{D'} E_h(p, y) dy \geq \varepsilon$$

holds uniformly in  $h$ .

Proof. Consider the transformation of coordinates  $x^i \rightarrow \xi^i = (2\sqrt{h})^{-1}(x^i - p^i)$  with its origin  $p$ . In  $\xi$ 's,  $D'$  is mapped to a convex open set  $D'_1(h)$  near to origin, and similar to  $D'$ . These  $D'_1(h)$  approaches a convex open domain  $D'_1$  when  $h$  goes to 0.

However, for a small positive  $\varepsilon$ ,

$$(26) \quad \int_{D'} E_h(p, y) dy = \int_{D'_1(h)} E_1(0, \eta) d\eta \geq \varepsilon.$$

where  $\eta^i = (2\sqrt{h})^{-1}(y^i - p^i)$ .

An open set  $O$  is called regularly open if it coincides with the open kernel of its closure.

**Lemma 7.** *Suppose that the boundary  $\partial D$  of a regularly open  $D$  be a rectilinear simplicial complex. If either  $x$  or  $y$  be on  $\partial D$ , then*

$$\lim_n (E_{t/n} *)^n(x, y) = 0.$$

Proof. Let  $x$  be on  $\partial D$ . It is easy to construct a convex open  $D'$  outside of  $D$  whose boundary contains  $x$ .

Set

$$m = [n/2].$$

Clearly

$$(E_{t/n} *)^m(x, y) \leq E_{mt/n}(x, y).$$

Take the  $c$  defined in Lemma 6 sufficiently large that for any  $h$  less than  $t$

$$(27) \quad \begin{aligned} & \int_{|y| \geq ct^{1/2}} E_h(0, y) dy \\ & \leq \int_{|\eta| \geq c/2} E_1(0, \eta) d\eta \leq \frac{1}{2} \varepsilon. \end{aligned}$$

By  $T_n$  we denote the set of points  $\{z; |x-z| \leq c(t/n)^{1/2}\}$ . The for  $y$  in  $T_n$

$$(28) \quad \begin{aligned} & (E_{t/n} *)^{m+1}(x, y) \leq \int_D E_{m/t_n}(z, y) dz \\ & = E_{(m+1)t/n}(x, y) \int_D E_{mt/(m+1)n} \left( z, \frac{x+my}{m+1} \right) dz \\ & \leq E_{(m+1)t/n}(x, y) \left( \int_{D \cap T_n} + \int_{\mathcal{E} - T_n} \right) E_{m/t(m+1)n} \left( z, \frac{x+my}{m+1} \right) dz \end{aligned}$$

$$\begin{aligned} &\leq E_{(m+1)t/n}(x, y) \left(1 - \varepsilon + \frac{1}{2} \varepsilon\right) \\ &= E_{(m+1)t/n}(x, y) \left(1 - \frac{1}{2} \varepsilon\right). \end{aligned}$$

because of (27) and of the previous lemma.

By iterating these processes, we have

$$(E_{t/n} *)^{m+i}(x, y) \leq E_{(m+i)t/n}(x, y) \left(1 - \frac{1}{2} \varepsilon\right)^i$$

and in particular

$$(29) \quad (E_{t/n} *)^{n-1}(x, y) \leq E_{(n-1)t/n} \left(1 - \frac{1}{2} \varepsilon\right)^{n-m-1}.$$

From this it follows that

$$\begin{aligned} (E_{t/n} *)^n(x, y) &\leq \int_{D \cap T_n} (E_{t/n} *)^{n-1}(x, z) E_{t/n}(z, y) dz \\ &\quad + \int_{D - T_n \cap D} E_{t/n}(x, z) (E_{t/n} *)^{n-1}(z, y) dz \\ &\leq \left(1 - \frac{1}{2} \varepsilon\right)^{n-m-1} \int_{\mathcal{E}} E_{(n-1)t/n}(x, z) E_{t/n}(z, y) dz \\ &\quad + \int_{|\zeta| \geq c} E_1(0, \zeta) d\zeta A \\ &= \left(1 - \frac{1}{2} \varepsilon\right)^{n-m-1} E_t(x, y) + \int_{|\zeta| \geq c} E_1(0, \zeta) d\zeta A \end{aligned}$$

where

$$A = \sup_{z \in D, \frac{1}{2} \leq h \leq t} E_h(z, y).$$

Now let  $c$  vary and go to infinity slow enough as  $n$  tends to infinity, that  $\varepsilon$  depending on  $c$  by (29) satisfies

$$(30) \quad \lim_n \left(1 - \frac{1}{2} \varepsilon\right)^{n-m} = 0.$$

For instance if we take  $c$ 's in such a way that  $\varepsilon = \varepsilon(n) = n^{-1/2}$ , (30) holds.

Under these modifications we can commute

$$(31) \quad \begin{aligned} \lim_n (E_{t/n} *)^n(x, y) &\leq \lim_n \left(1 - \frac{1}{2} \varepsilon\right)^{n-m} E_t(x, y) \\ &\quad + \lim_n \int_{|\zeta| \geq c} E_1(0, \zeta) d\zeta A = 0 \end{aligned}$$

which prove the lemma.

REMARK. Lemma 7 holds for a regularly open  $D$  with a smooth boundary and the same kind of proof works for it. Moreover we have

$$(32) \quad \lim_n (E_{r/n} *)^n(x, y) = 0$$

when either  $x$  or  $y$  is outside of  $D$ .

**Lemma 8.** *Let  $\phi(y)$  be a continuous function over the closure of  $D$ , and  $x$  be a point on  $D$ . Then*

$$(33) \quad \lim_{h \rightarrow 0} \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy = \phi(x).$$

Proof. Let  $S(h)$  be the solid sphere of radius  $\sqrt[3]{4h}$  around  $x$ . Set

$$(34) \quad A(h) = \sup_{y \in S(h)} \phi(y)$$

and

$$B(h) = \inf_{y \in S(h)} \phi(y).$$

Then from (5), considering  $S(h)$  as  $Q$ ,

$$(35) \quad \begin{aligned} & \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy \\ & \leq A(h) \int_{S(h)} E_h(x, y) dy + \int_{D-S(h)} E_h(x, y) \phi(y) dy \\ & \leq A(h) + 0 \left( 2^d \exp \left( -\frac{1}{4} (h)^{-2/3} \right) \right). \end{aligned}$$

However from (15),

$$(36) \quad \begin{aligned} & \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy \\ & \geq B(h) (1 - 0 (\exp(-h^{-1/2}) \int_D E_h(x, y) dy)) \end{aligned}$$

Since the right hand sides of both (35) and (36) approaches  $\phi(x)$  when  $h$  goes to zero, we have

$$\lim_{h \rightarrow 0} \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy = \phi(x).$$

**Corollary.**

$$(37) \quad \lim_{h \rightarrow 0} \int_D \underline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy = \phi(x).$$

The proof is obtained by changing  $\overline{\lim}$  to  $\underline{\lim}$  in Lemma 8.

**Lemma 9.** *Suppose that  $\phi(y)$  be a  $C^2$  function over the closure of  $D$ . Then,*



$$(38) \quad \lim_{h \rightarrow 0} (1/h) \left( \int_D \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right) = \Delta \phi$$

Proof.

$$(39) \quad \int_{S(h)} \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) = \int_{S(h)} E_h(x, y) \phi(y) dy - \phi(x) + 0 (\exp(-h^{-1/2})) \int_{S(h)} \phi(y) dy.$$

On the other hand by (5)

$$(40) \quad \int_{D-S(h)} \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy \leq \sup_{y \in D} |\phi(y)| 2^d \exp\left(-\frac{1}{4} (4h)^{-2/3}\right).$$

Hence

$$(41) \quad (1/h) \left[ \int_D \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right] = (1/h) \left( \int_{S(h)} E_h(x, y) \phi(y) dy - \phi(x) \right) + 0(h^{-1} \exp(-h^{-1/2})).$$

It is well known that

$$(42) \quad \lim_{h \rightarrow 0} (1/h) \left( \int_{S(h)} E_h(x, y) \phi(y) dy - \phi(x) \right) = (\Delta \phi)_x.$$

Hence we have

$$\lim_{h \rightarrow 0} (1/h) \left[ \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right] = (\Delta \phi)_x$$

**Corollary.**

$$(43) \quad \lim_{h \rightarrow 0} (1/h) \left[ \int_D \underline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right] = (\Delta \phi)_x.$$

**Lemma 10.**

$$(44) \quad \overline{\lim}_n (E_{t/n} *)^n(x, y) = \underline{\lim}_n (E_{t/n} *)^n(x, y) \text{ exists.}$$

We denote this by  $K(x, y; t)$ .

Proof. From (16) it follows that

$$(E_{t/mn} *)^m(x, y) \leq E_{t/n}(x, y)$$

and hence

$$(E_{t/mn} *)^{mn}(x, y) \leq (E_{t/n} *)^n(x, y).$$

Therefore the limit in the sense of Moore-Smith exists for integers  $2^n$ 's when we introduce an partial order in such a way that  $m < p$  when  $m$  is a divisor of  $p$ .

In general, first we approximate  $1/m$  with  $p/n$ . Then  $E_{t/m}(x, y)$  is uniformly approximated in such a way that for a preassigned  $\delta$

$$E_{t/m}(x, y) \geq E_{t_{pI(n)}}(x, y)e^{-\delta/m}.$$

Then

$$(E_{t/m} *)^m(x, y) \geq (E_{t_{pI(n)}} *)^m(x, y)e^{-\delta} \geq \lim_j (E_{t_{pI(n+j)}} *)^{m_2^j}(x, y)e^{-\delta}.$$

Since this holds for any  $\delta$ ,

$$(E_{t/m} *)^m(x, y) \geq \lim_j (E_{t_{I(j)/m}} *)^{2_m^j}(x, y) = \lim_j (E_{t_{I(j)}} *)^{2^j}(x, y).$$

Conversely, when  $m$  is large, we have

$$E_{t_{p/m}}(x, y) \leq E_{t_{I(n)}}(x, y)e^{\delta I(n)}$$

with a suitable integer  $p$ .

Hence it follows that

$$\overline{\lim} (E_{t/m} *)^m(x, y) \leq \overline{\lim} (E_{p t/m} *)^{2^n}(x, y) \leq \lim_n (E_{t_{I(n)}} *)^{2^n}(x, y)e^{\delta}$$

which hold for any  $\delta$ . Hence

$$\lim (E_{t/m} *)^m(x, y) = \lim_n (E_{t_{I(n)}} *)^{2^n}(x, y)$$

has been proved. This proves the lemma.

Introduce a one parameter family of operators  $K_t$  by

$$(45) \quad (K_t \phi)(x) = \int_D K(x, y; t) \phi(y) dy.$$

**Lemma 11.**  $\{K_t\}$  forms a one parameter semi-group of operators.

Proof. For two positive reals  $t$  and  $s$

$$\begin{aligned} (46) \quad (K_t(K_s \phi))(x) &= \int_D K(x, y; t) \int_D K(y, z; s) \phi(z) dz dy \\ &= \int_D \left( \int_D K(x, y; t) K(y, z; s) dy \right) \phi(z) dz \\ &= \int_D \lim_n (E_{(t+s)/n} *)^n(x, z) \phi(z) dz \\ &= (K_{t+s} \phi)(x) \end{aligned}$$

which proves the lemma.

**Lemma 12.** Suppose that for a  $C^2$  continuous  $\phi(x)$ ,

$$\lim_{h \rightarrow 0} (1/h) ((K_h \phi)(x) - \phi(x))$$

exists everywhere. Then the limit is equal to  $(\Delta \phi)_x$ .

Proof. Take a point  $x$  and denote by  $S(h)$  the solid sphere of radius  $\sqrt{4h}$  around  $x$ . Then from (19)

$$\begin{aligned}
 \int_D K(x, y; h)\phi(y)dy &= \int_{S(h)} K(x, y; h)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
 (46) \quad &= \int_{S(h)} E_h(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
 &= \int_{S(h)} \int_{\varepsilon}^h \frac{\partial}{\partial t} E_t(x, y)dt\phi(y)dy + \int_{S(h)} E_{\varepsilon}(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
 &= \Delta_x \int_{S(h)} \int_{\varepsilon}^h E_t(x, y)\phi(y)dy dt + \int_{S(h)} E_{\varepsilon}(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})),
 \end{aligned}$$

for some small  $\varepsilon$ .

Hence

$$\begin{aligned}
 (47) \quad 1/h((E_h\phi)(x) - \phi(x)) &= \Delta_x \int_{S(h)} (1/h) \int_{\varepsilon}^h E_t(x, y)\phi(y)dydt \\
 &\quad + (1/h) \left( \int_{S(h)} E_{\varepsilon}(x, y)\phi(y)dy - \phi(x) + 0(h^{-1} \exp(-h^{-1/2})) \right).
 \end{aligned}$$

When  $\varepsilon$  goes to zero,

$$\begin{aligned}
 (48) \quad (1/h)(K_h\phi)(x) - \phi(x) &= \Delta_x \int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dt dy \\
 &\quad + 0(h^{-1} \exp(-h^{-1/2})).
 \end{aligned}$$

and therefore

$$(49) \quad \lim_{h \rightarrow 0} (1/h)((K_h\phi)(x) - \phi(x)) = \Delta_x \lim_h \int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dt dy$$

which proves that  $(\Delta\phi)_x$  exists and is equal to  $\frac{\partial}{\partial h}(K_h\phi)(x)$ . Here we use the fact that  $\int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dt dy$  approaches  $\phi(x)$  uniformly together with its second derivatives with the order of  $h^{-1} \exp(-h^{-1/2})$ .

**Lemma 13.** *Let  $\phi(x)$  be a  $C^2$  function over the closure of  $D$ . Then for any  $x$  in  $D$  and for any  $t$*

$$(50) \quad \left[ \frac{\partial}{\partial h} (K_{t+h}\phi) \right]_{h=0} (x) = \Delta_x (K_t\phi)$$

exists.

Proof. By Lemma 9,

$$(51) \quad \lim_h (1/h)(K_h\phi - \phi) = \Delta_x \phi.$$

However

$$(52) \quad K_{t+h}\phi - K_t\phi = K_t(K_h\phi - \phi).$$

Hence

$$\left( \frac{\partial}{\partial h} K_{t+h}\phi \right) (x) = [K_t \lim \frac{1}{h} (K_h\phi - \phi)](x) = (K_t \Delta\phi)(x).$$

By the previous Lemma

$$(K_t \Delta \phi)(x) = (\Delta K_t \phi)(x) = \left( \frac{\partial}{\partial h} K_{t+h} \phi \right)(x)$$

because  $\frac{\partial}{\partial h} K_{t+h} \phi = \lim_{h \rightarrow 0} (1/h)(K_{t+h} \phi - K_t \phi)$  exists. This completes the proof.

**Corollary.**  $\int_D K(x, y; t) \phi(y) dy$  is a solution of a differential equation  $\frac{\partial}{\partial t} U = \Delta U$ , for any continuous function  $\phi(x)$ .

*Proof.* When  $\phi(x)$  is  $C^2$ , then  $U$  is a solution of  $\partial U / \partial t = \Delta U$ . Now let  $\phi_n(x)$ 's converge to  $\phi(x)$  where all  $\phi_n$ 's are  $C^2$ . Then the corresponding  $U_n$ 's converge to a weak solution which, by a theorem by Nirenberg (1), is a genuine solution. This completes the proof.

**Lemma 14.**  $K(x, y; t)$  is  $C^2$  both in  $x$  and in  $y$ .

*Proof.* From the previous corollary it is evident that  $\int K(x, y; t) \phi(y) dy$  is  $C^2$  in  $x$  for any  $t > 0$ . Hence for an  $h < t$

$$(53) \quad K(x, y; t) = \int_D K(x, z; h) \int_D K(z, y; t-h) dz$$

is  $C^2$  in  $x$ . By the construction of  $K(x, y; t)$

$$(54) \quad K(x, y; t) = K(y, x; t).$$

Therefore  $K(x, y; t)$  is  $C^2$  in  $y$ .

**Theorem 1.** Suppose that  $D$  is a regularly open set with either smooth or rectilinear boundary. Then  $K(x, y; t)$  defined in Lemma 10 is the kernel function of the differential equation

$$(55) \quad \frac{\partial}{\partial t} U = \Delta U$$

over  $D$ .

*Proof.* By Lemma 7

$$(56) \quad K(x, y; t) = 0$$

if either  $x$  or  $y$  be on  $\partial D$ . From its construction it follows directly that  $K(x, y; t) > 0$  for  $x \neq y$ . Lemma 14 says  $K(x, y; t)$  is  $C^2$  both in  $x$  and in  $y$ .

Therefore the result in Lemma 12, i.e.

$$(57) \quad \frac{\partial}{\partial t} \int_D K(x, y; t) \phi(y) dy = \Delta_x \int K(x, y; t) \phi(y) dy$$

implies  $\frac{\partial}{\partial t} K(x, y; t-h) = \Delta_x K(x, y; t-h)$ , which proves the theorem.

NOTICE. Theorem 1 holds when any point on  $\partial D$  is on a boundary of an open convex set  $D'$  disjoint with  $D$ .

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