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On Unions of Knots

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Introduction

If two knots κ and κ' with a common arc α, of which κ lies inside a cube Q and κ' outside of it, α lying naturally on the boundary of Q, are joined together along α, that is, if α is deleted to obtain a single knot out of κ and κ', then we have by definition the product of κ and κ'. If a knot κ cannot be the product of any two non trivial knots, then κ is said to be prime. It is H. Schubert [9] who showed that the genus of the product of two knots is equal to the sum of their genera and that every non trivial knot is decomposable in a unique way into prime knots.

Now a close inspection through the table of knots by Alexander and Briggs [1] as reproduced in the book of Reidemeister [8], where only prime knots are given, or rather a simple experiment by a thread, will show that there are a number of knots composed of prime knots in a more complicated way. Thus 8_5, 8_{10}, 8_{15}, $8_{19(n)}$, $8_{20(n)}$, $8_{21(n)}$, 9_{16}, 9_{24} and 9_{28} of the Alexander Briggs table are all composed of two trefoil knots 3_1 in the following way (Fig. 1 indicates the composition of 8_{19} out of two 3_1) : First join the trefoil knots κ and κ' together along their arcs $\widehat{AB}$

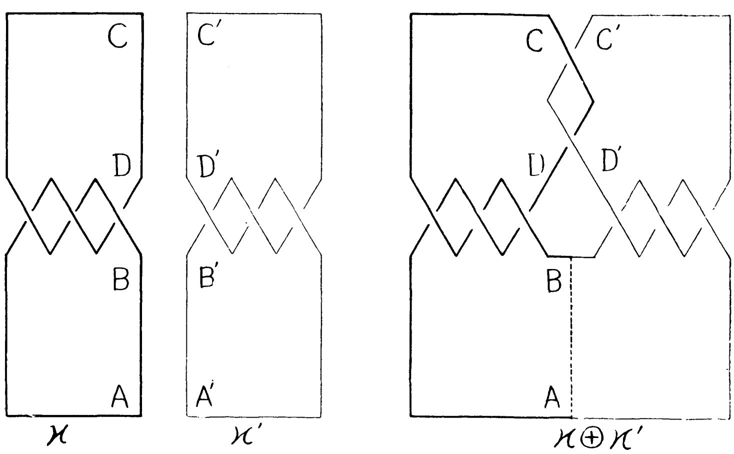

Fig. 1.

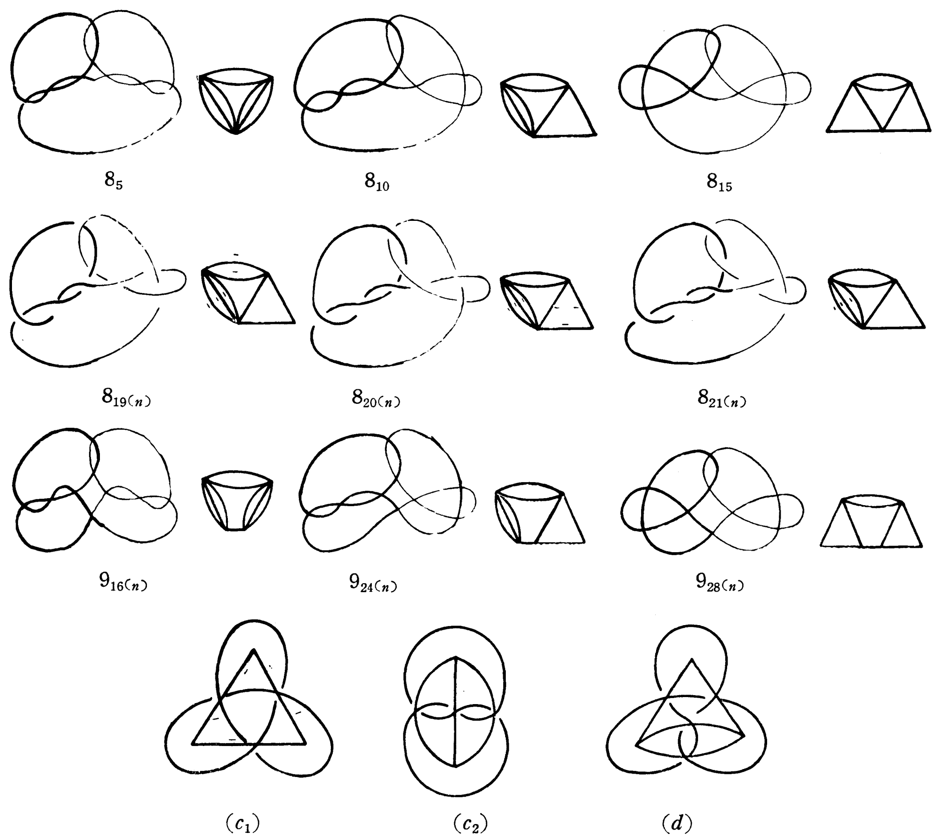

Fig. 2.

and $\widehat{A'B'}$ as the usual product making and then wind them together in the neighbourhood of the arcs $\widehat{CD}$ and $\widehat{C'D'}$. Likewise 9_{22}, 9_{25}, 9_{30}, 9_{36}, $9_{42(n)}$, $9_{43(n)}$, $9_{44(n)}$ and $9_{45(n)}$ are composites of the trefoil knot 3_1 and the knot 4_1 (Fig. 3).

Such a composition of knots will best be described if we make use of the graphs of knots [15]: The graph of a knot is a linear graph on a plane, where the vertices represent the alternating, the so-called white or black, domains ([8], p. 9), and the edges the crossing points, of the ordinary regular projection of the knot. To every edge is attached further a sign $+$, which may well be understood, or $-$, according as the knot passes the crossing point right-handedwise or left-handedwise. Every knot has a pair of graphs, dual to each other. Thus, the knots $8_5 - 9_{45}$ are represented by the annexed graphs in Fig. 2 and Fig. 3, where (c_1) and (c_2) are the dual forms of a trefoil knot, and (d) represent the knot 4_1 which is self-dual if we disregard its sign.

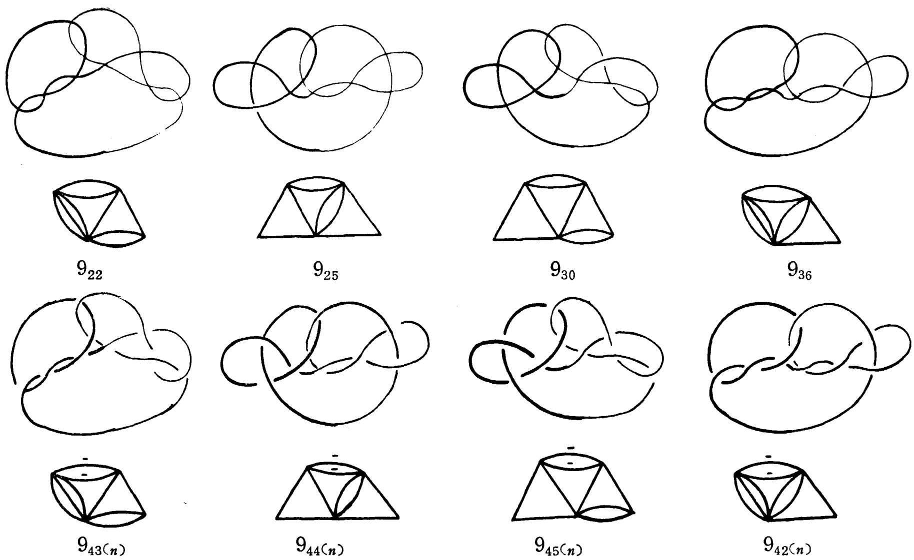

Fig. 3.

In general, if (k) and (k') are the graphs of the knots κ and κ', bring a pair of vertices A and A' together to a coincidence and connect another

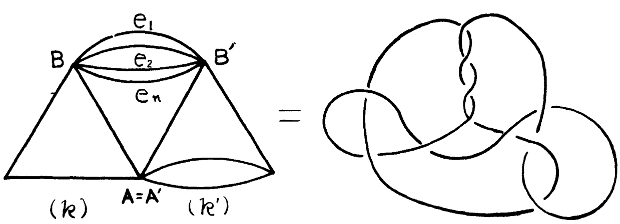

Fig. 4.

pair of vertices B and B' with n edges $e_1, e_2, \cdots, e_n$ attached with one and the same sign (Fig. 4). We shall call the knot whose graph is represented by such a composition a *union* of κ and κ' with the *winding number* n if n is even and a *skew union* with the *winding number* n if n is odd. A circumstance revealed to the authors that if two knots κ and κ' are symmetrically situated and if we perform the above composition, the Alexander polynomial [2] remains the same irrespective of the winding number and that equals the square of the Alexander polynomial of κ if it is a union: One of the authors conjectured[1] namely that every knot could be obtained by joining together a pair of (in general linking) trivial knots (=circles) along a common arc, and as the simplest cases made tentatively such joinings with non linking circles as shown in

1) Meanwhile, this conjecture turned out to be true. See Appendix.

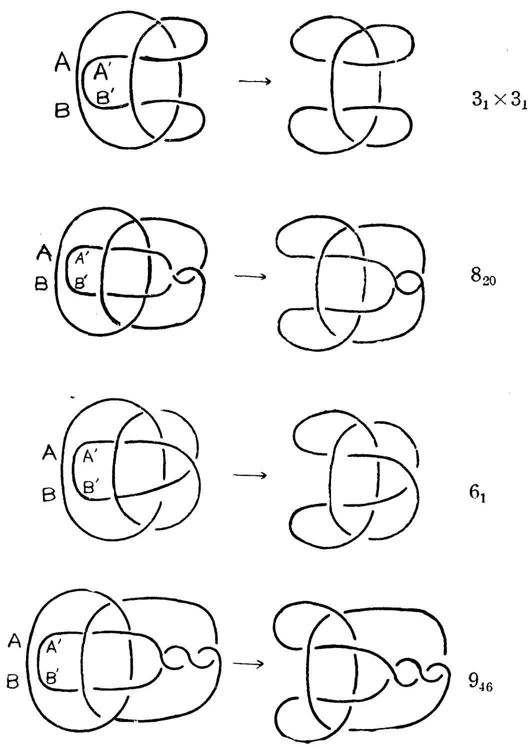

Fig. 5.

Fig. 5. Thus we obtained e. g. the product of trefoil knots, the knot 8_{20} and a knot of 10 double points, all with the Alexander polynomial $(x^2-x+1)^2$ if we gave 0, 2 or 4 windings beforehand, and the knots 6_1 and 9_{46} with the Alexander polynomial $2x^2-5x+2$, if these were 1 and 3. On writing these compositions by graphs, the other of the authors generalised the fact to symmetric unions and skew unions.

The purpose of the present note is to show among others that every union of two non trivial knots is always non trivial (Theorem 1) and that the Alexander polynomial is independent of the winding number for any symmetric union (skew union) of a knot (Theorems 2 and 3). As an application we give an example of a non trivial knot of eleven crossings with the Alexander polynomial $\Delta(x)=1$, the number of crossings being smaller than those of the examples hitherto known of H. Seifert [13] and of J. H. C. Whitehead [14]. In Appendix is further proved that every knot is a sum of two trivial knots.

§ 1. Union of knots.

In the following we use freely such expressions as arcs, circles, disks, spheres, mappings, etc. but it should be understood that all figures we shall be talking about are simplicial and mappings are all semi-linear.

Let Q be a solid cube and let κ be a knot which has two disjoint segments AD and BC in common with the surface $\dot{Q}$ of Q, the remaining arcs $\widehat{AB}=\alpha_1$ and $\widehat{CD}=\alpha_2$ of κ lying wholly within Q except for their endpoints. Similarly for another cube Q' and a knot κ', the corresponding points and arcs being primed. It will be convenient to call Q and Q' the *cubes of union.*

To define a union of κ and κ' we proceed as follows:

Place Q and Q' apart and parallel to each other, one face Σ of Q facing the face Σ' of Q' in such a way that Σ and Σ' make together a pair of opposite faces of a rectangular parallelepiped. If E is the plane containing a face of the last named parallelepiped different from Σ and Σ', then Q and Q' are seen to be lying on E, when E is thought of as a ground plane.

By suitable semi-linear transformations of Q and Q' onto themselves bring κ and κ' to such positions that the segments AD and BC and the segments $A'D'$ and $B'C'$ lie respectively on Σ and Σ' and that A, B, C, D and A', B', C', D' lie in these orders along straight lines parallel to the ground plane E, and finally that AA', BB', CC' and DD' are perpendicular to Σ and Σ'.

Now let β and β' be disjoint arcs connecting B with C and B' with C' respectively outside Q and Q', thus $BC\cup\beta$ and $B'C'\cup\beta'$ making together a link, such that their projections on E are each a polygon contained within the projection of the rectangle $BCC'B'$, cutting each other alternatively one above the other in the same sense, as indicated in Fig. 6. Speaking more exactly, the projections of β and β' on E intersect at $2n$ points $P_1, P_2, \cdots, P_{2n}$ arranged in this order on either of the projections of β and β', the original β passes above (or always under) β' at every second crossing point P_{2i+1} beginning with P_1.

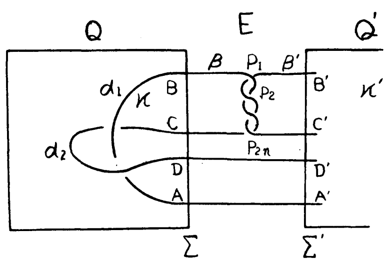

Fig. 6.

Replace now the segments AD and $A'D'$ with the segments AA' and DD', and the segments BC and $B'C'$ with β and β' respectively. Then the knot thus obtained will be called a *union* of κ and κ' *joined* along AD and $A'D'$ and *winded* along BC and $B'C'$ with the *winding number* $2n$. n should be taken >0 if β passes above P_1, otherwise <0.

Likewise for a *skew union* with the *winding number* $2n+1$.

A union (skew union) depends in general on the locality of its winding and on its winding number. It is uniquely determined if the winding number is 0, in which case the union is merely the usual product.

REMARK. More general union can be defined if we make windings at several places, not only one. Even in this generalised union Theorem 2 is seen to hold, and most probably Theorem 1 also holds.

First we prove the following

Lemma 1. *Let κ be a knot consisting of an arc γ on the surface $\dot{Q}$ of a solid cube Q and an arc α connecting the endpoints of γ within Q. If $\kappa=\alpha\cup\gamma$ can be spanned by a disk Δ, then κ can also be spanned by a disk Δ' wholly contained within Q, provided that Δ lies wholly within Q in some neighbourhood of γ.*

Proof. On account of the assumption there is a polygonal region P on $\dot{Q}$ containing γ in its interior and two disks S_1 and S_2 bounded by the boundary $\dot{P}$ of P, of which S_1 lies outside and S_2 inside $\dot{Q}$, such that the sphere $S_1\cup P$ has no point in common with the disk Δ except along γ. By a suitable semi-linear mapping which fixes all points of S_2 map the region outside $S_1\cup S_2$ into the region bounded by S_2 and by $\dot{Q}-P$. Then the image of the disk Δ is just the desired one spanning κ within Q.

Every knot has a representation by bridges (Schubert [10]), the smallest number of its bridges being an invariant. We shall call κ a knot *with n bridges* if κ has a representation by n bridges but not less. According to this definition every non trivial knot is a knot with at least two bridges. We shall say that a knot κ has an *n-bridged form with respect to a cube Q* if κ is decomposed by Q into $2n$ arcs $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$ such that α_i lie wholly inside Q and β_i lie wholly outside Q except for their endpoints and that $\alpha_i (i=1, 2, \cdots, n)$ and $\beta_i (i=1, 2, \cdots, n)$ can be spanned by disjoint disks within and without Q respectively, i.e., if κ makes inside and outside Q n simple chords in Schubert's terminology ([11], p. 135).

Then we have

Lemma 2. *Let κ'' be a union of two non trivial knots κ and κ' with the cubes of union Q and Q'. Then, if κ is not of the two-bridged form with respect to Q, κ is a non trivial knot.*

Proof. Retaining our earlier notations, suppose that the contrary were true and let σ be a disk spanning the union $\kappa''=\kappa\oplus\kappa'$, i.e., let σ be a semi-linear image $f(K)$ of a circular region K, the image $f(\dot{K})$ of the boundary $\dot{K}$ of K being κ''. By a usual consideration we can modify the mapping f, if necessary, so that $\sigma = f(K)$ "intersects" the surface $\dot{Q}$ of the cube Q along arcs γ_1 and γ_2 connecting two pairs of points of A, B, C and D and eventually along polygons, which together with γ_1 and γ_2 are all disjoint from one another.

Denote the inverse images of A, B, C, D, A', B', C', D' and γ_1, γ_2 by a, b, c, d, a', b', c', d' and w_1, w_2 respectively. We notice in the first place that neither of γ_1 and γ_2 is a path connecting A and D on $\dot{Q}$, for if γ_1, say, were such a path, then $w_1 = f^{-1}(\gamma_1)$ would be a path leading from a to d in K, and the image by f of the region bounded by w_1 and by the arc $\widehat{ad}$ of $\dot{K}$ containing a', b', c', d' would be a disk spanning the knot $\alpha'_1 \cup \beta' \cup \alpha_2' \cup D'D \cup \gamma_1 \cup AA'$, which is equivalent to the knot κ', contrary to the supposition that κ' was a non trival knot.

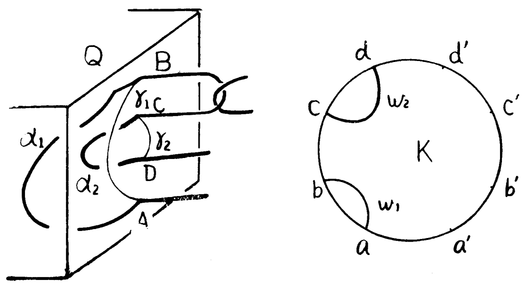

Fig. 7.

We suppose therefore that γ_1 connects A and B and γ_2 connects C and D on $\dot{Q}$, and consequently that w_1 and w_2 connects a with b and c with d respectively in K.

If σ intersects $\dot{Q}$ besides γ_1 and γ_2 along some polygons, we shall call each one of the latter an *intersection-polygon*. Now let $\dot{\Pi}$ be one of the intersection-polygons and let Π be one of the polygonal regions bounded by $\dot{\Pi}$ on $\dot{Q}$ such that Π has no point in common with σ. It does not matter whether such an intersection-polygon actually exists or not. Let $\dot{P}$ be further the inverse image of $\dot{\Pi}$ in K: $f(\dot{P}) = \dot{\Pi}$, and let P be the region bounded by $\dot{P}$ in K. Then, since $\Pi \cup f(P)$ is clearly a sphere, we can by a usual consideration (see e.g. [4], [9]) modify the mapping f in the neighbourhood of $\dot{P}$ and within $\dot{P}$, so that the new semi-linear mapping f' of K coincides with f outside a certain polygon $\dot{P}'$ containing P in its interior and that $f'(P')$, where P' denotes the region bounded by $\dot{P}'$, runs in the neighbourhood of Π without having any point in common with Q. Then $f'(K)$ intersects $\dot{Q}$ other than γ_1 and γ_2 along polygons whose number is diminished by one in comparison with $f(K)$.

Operating with this modification as long as there remains any intersection-polygon of the above type, we get finally a mapping, which we denote again by the same letter f, such that every intersection-polygon, if any, has on either side γ_1 or γ_2. Let $\dot{P}$ be the inverse image of one of these polygons such that there is no inverse image of any of the intersection-polygons within $\dot{P}$. If P denotes the region bounded by $\dot{P}$, then $f(P)$ has points in common with $\dot{Q}$ solely along $f(\dot{P})$. $f(P)$ lies moreover within Q, for otherwise the arc $AA' \cup \alpha'_1 \cup BB' (\subset \kappa'')$ would have point in common with $f(P)$, which is impossible. Thus the interior of Q is divided by $f(P)$ into two domains Q_1 and Q_2. If T_1 and T_2 denote two regions of $\dot{Q}$ divided by $f(P)$, of which T_1 contains γ_1 and T_2 contains γ_2, then Q_1 and Q_2 are the domains bounded by the sphere $f(P) \cup T_1$, which contains α_1, and the domain bounded by $f(P) \cup T_2$, which contains α_2, in their interiors. Since $\alpha_1 \cup \gamma_1$ is spanned by the disk which is the image by f of the region in K bounded by $\widehat{ab} \cup w_1$, so $\alpha_1 \cup \gamma_1$ can also be spanned by a disk within $f(P) \cup T_1$, on account of Lemma 1. Likewise $\alpha_2 \cup \gamma_2$ can be spanned by a disk within $f(P) \cup T_2$. Thus the knot turned out to be equivalent to the knot

$$\gamma_1 \cup (AA' \cup A'D' \cup D'D) \cup \gamma_2 \cup \beta,$$

which is a knot of two-bridged form with respect to Q, contradicting the original assumption of our Lemma. If there is no intersection-polygon on $\dot{Q}$, that is, if $f(K)$ does not intersect $\dot{Q}$ except along γ_1 and γ_2, then, since $\alpha_1 \cup \gamma_1$ and $\alpha_2 \cup \gamma_2$ are spanned within Q by disks disjoint from each other, namely by disks which are the images of the regions bounded by $\widehat{ab} \cup w_1$ and by $\widehat{cd} \cup w_2$ in K, κ would be equivalent to a knot of two-bridged form with respect to Q, contradicting again the assumption.

Thus the proof of Lemma 2 is complete.

Next we prove the following

Lemma 3. *Let κ'' be a union of two non trivial knots κ and κ' with the cubes of union Q and Q'. Then, even if κ and κ' are of the two-bridged form with respect to Q and Q', κ'' is a non trivial knot.*

Proof. Retaining again our earlier notations, cut Q by a plane parallel to the face Σ in a square $\bar{\Sigma}$. By a semi-linear transformation which maps Q onto itself bring the knot κ into its normal form (α, β) ([11], p. 140) represented by two disjoint arcs $\bar{\alpha}_1 = \widehat{\bar{A}\bar{B}}$ and $\bar{\alpha}_2 = \widehat{\bar{C}\bar{D}}$ lying on $\bar{\Sigma}$ with the bridges $\bar{A}A \cup AD \cup D\bar{D}$ and $\bar{B}B \cup BC \cup C\bar{C}$, where $\bar{A}A$, $\bar{B}B$, $C\bar{C}$, $D\bar{D}$ are perpendiculars between Σ and $\bar{\Sigma}$. The same for the knot κ' and its cube of union Q', and make a union $\kappa'' = \kappa \oplus \kappa'$ with κ and κ'.

We are now going to compute the fundamental group of the two-fold covering space branched along κ'' to show that it is not trivial.

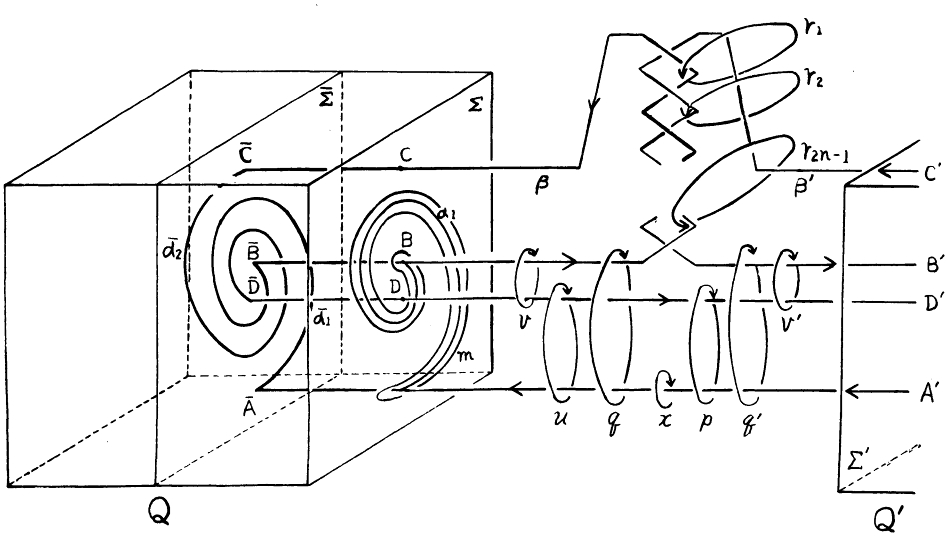

Fig. 8.

To this end consider first the two-fold covering manifold Q_2 of Q branched along κ'', which is a solid torus ([11], p. 146). Let α_i be the "projection" of $\bar{\alpha}_i$ on Σ. If m is a meridian on the boundary $\dot{Q}_2$ of Q_2 given by a closed path on Σ including the arc α_1 in its interior and excluding α_2 outside of it, and if l is a longitude of $\dot{Q}_2$, then the fundamental group $F(Q_2)$ of Q_2 will be given by the presentation

Generators: l, m.
Relation: $m=1$,

where l and m stand for the elements of the path group corresponding to the path l and m respectively.

Similarly the fundamental group $F(Q_2')$ of the two-fold covering manifold Q'_2 of Q' branched along κ'' is given by

Generators: l', m'.
Relation: $m'=1$,

where l' and m' have the similar meaning.

Finally let us compute the fundamental group of the two-fold covering manifold R_2 branched along κ'' of the exterior R of Q and Q', the boundaries of Q and Q' being included. Let x, p, q be closed path in $R-\kappa''$, encircling respectively AA', $AA' \cup DD'$, $AA' \cup DD' \cup BB'$, the same for q', and let $r_1, r_2, \cdots, r_{2n-1}$ be closed paths in $R-\kappa''$ passing respectively through where the $2n$ times windings were performed, thus linking with both $\beta\cup BC$ and $\beta'\cup B'C'$, as shown in Fig. 8. Then the fundamental group of the space $R-\kappa''$ has the following presentation:

$$\begin{aligned} &\text{Generators:} \quad x,\ p,\ q,\ q',\ r_1,\ r_2,\ \cdots,\ r_{2n-1}.\\ &\text{Relations:} \quad q^{-1}r_1q'^{-1}=1,\\ &\qquad\qquad\quad r_1q^{-1}r_2q'^{-1}=1,\\ &\qquad\qquad\quad \cdots\cdots\cdots\\ &\qquad\qquad\quad r_{2n-1}q^{-1}pq'^{-1}=1, \end{aligned}$$

if the winding number of the union $2n$ is $\geqq 0$. This presentation can be transformed into

$$\begin{aligned} &\text{Generators:} \quad x,\ p,\ q,\ q'.\\ &\text{Relation:} \quad p=(qq'^{-1})^n(q^{-1}q')^n. \end{aligned}$$

It should be remarked that this presentation holds true even if n is negative.

Now, if we introduce on Σ a circular path u encircling the segment AD and excluding B and C outside of it and a circular path v encircling BD and excluding A and C outside of it, and similarly v' on Σ', then, since

$$p=u,\quad q=vx,\quad q'=v'x,$$

the last obtained presentation can further be transformed into

$$\begin{aligned} &\text{Generators:} \quad x,\ u,\ v,\ v'\\ &\text{Relation:} \quad u=(vv'^{-1})^n(x^{-1}v^{-1}v'x)^n. \end{aligned}$$

Observing the relations

$$xux^{-1}=u^{-1},\quad xvx^{-1}=v^{-1},\quad xv'x^{-1}=v'^{-1},$$

which hold in R_2, we have as a presentation of the fundamental group $F(R_2)$ the following (cf. [3], [6]):

$$\begin{aligned} &\text{Generators:} \quad u,\ v,\ v'\\ &\text{Relations:} \quad u=(vv'^{-1})^{2n},\\ &\qquad\qquad\quad u^{-1}=(v^{-1}v')^{2n}. \end{aligned}$$

We are now in a position to calculate the fundamental group $F(M_2)$ of the two-fold covering space M_2 of S^3 branched along κ'', which is the sum of the two-fold covering manifolds Q_2, R_2 and Q'_2 considered above.

On the torus $Q_2 \frown R_2$ we have indeed the relations:

$$u = l^{\alpha} m^{\mu_1}, \quad v = l^{\beta} m^{\mu_2},$$

where α and β are the torsion and the crossing-class of the knot κ'' in its normal form (α, β) introduced in the beginning of our proof, and where μ_1 and μ_2 are some integers.

Similarly we have

$$u' = l'^{\alpha'} m'^{\mu_3}, \quad v' = l'^{-\beta'} m'^{\mu_4}$$

on the torus $Q'_2 \frown R_2$.

The fundamental group $F(M_2)$ assumes consequently (Seifert and Threlfall [12]) the following presentation:

Generators: u, v, v', l, l'.

Relations: $u = l^{\alpha}, \quad v = l^{\beta}$

$u = l'^{\alpha'}, \quad v' = l'^{-\beta'}$

$u = (vv'^{-1})^{2n}, \quad u^{-1} = (v^{-1}v')^{2n}$,

which can be transformed into:

Generators: l, l'.

Relations: $l^{\alpha} = l'^{\alpha'} = (l^{\beta} l'^{\beta'})^{2n} = (l'^{\beta'} l^{\beta})^{2n}$.

Since α and α' are both $\geqq 3$, because κ and κ' are non trivial knots, $F(M_2)$ is seen to be non trivial[2]. Thus the proof of Lemma 3 is complete.

Combining Lemma 1 and Lemma 2 we have immediately

Theorem 1. *Every union of two non trivial knots is a non trivial knot.*

§ 2. Symmetric union and symmetric skew union.

If two knots κ and κ' with their cubes of union Q and Q' are situated symmetric to each other, then a union (skew union) of κ and κ' will be called a *symmetric union* (*skew union*) of κ. It is not assumed that κ is non trivial.

Then one of the theorems to be proved is the following

Theorem 2. *If* $\Delta_{\kappa}(x)$ *denotes the Alexander polynomial of a knot* κ, *then the Alexander polynomial* $\Delta_{\kappa''}(x)$ *of every symmetric union* κ'' *of* κ *is*

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2) If we add a relation $l^{\alpha}=1$ to the above ones, then we have a presentation

$$\{y, y'; y^{\alpha}, y'^{\alpha'}, (yy')^{2n}\}$$

of a subgroup of $F(M_2)$, where $y=l^{\beta}$, $y'=l'^{\beta'}$, showing the group to be non trivial.

equal to the square of $\Delta_\kappa(x)$:

$$\Delta_{\kappa''}(x) = (\Delta_\kappa(x))^2, \tag{1}$$

and is therefore independent of the winding number and the locality of winding of the union.

Proof. Let $2n$ be the winding number of the union κ''. If $n=0$, then $\kappa''(x)$ is the product of κ with its symmetric image κ', and (1) is immediate.

Suppose now that n is positive. Then the projection κ''_E of κ'' on the ground plane E assumes the form as shown in Fig. 9. κ''_E decomposes E into domains, of which those contained within the left-hand square Q_E are denoted by $c_1, c_2, \cdots, c_m$, the symmetric image of $c_i\,(i=1, 2, \cdots, m)$ by c_i', and the remaining ones by $a_0, a_1, \cdots, a_{2n}, b, c_0$ and c'_0. Since $\Delta_{\kappa''}(x)$ is independent of the choice of orientation of κ'', we may suppose that κ'' is oriented as in Fig. 9. Then the Alexander matrix of κ'' will take the following form[3]:

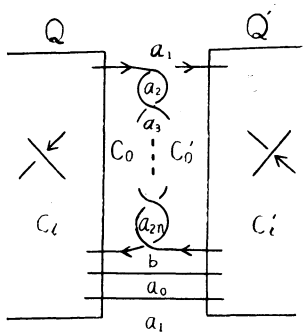


Fig. 9.

$$\begin{array}{cc|ccc|c|c|ccc|c|ccc}
a_0 & a_1 & a_2 & \cdots\cdots & a_{2n} & b & c_0 & c_1 & \cdots & c_m & c_0' & c_1' & \cdots & c_m' \\
1 & 0 & 0 & \cdots\cdots\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots\cdots\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & x & 1 & 0\ \cdots & 0 & 0 & -x & & & & -1 & & & \\
0 & 0 & 1 & x\ \cdots & 0 & 0 & -1 & & & & -x & & & \\
\vdots & & & \ddots & & \vdots & \vdots & & 0 & & \vdots & & 0 & \\
0 & 0 & 0 & \cdots\ x & 1 & 0 & -x & & & & -1 & & & \\
0 & 0 & 0 & \cdots\ 0 & 1 & x & -1 & & & & -x & & & \\
\hline
* & & & 0 & & b_1 \ \vdots\ b_{m+1} & 0 & & c_{ij} & & & & 0 & \\
\hline
* & & & 0 & & -b_1 \ \vdots\ -b_{m+1} & 0 & & 0 & & & & -c_{ij} &
\end{array} \tag{2}$$

3) The matrix defined originally by Alexander was not a square one. But some consideration of the presentation of the knot group will convince us of the naturality of using the square matrix of the following shape.

where $i=1, 2, \cdots, m+1$ and $j=0, 1, \cdots, m$.

Subtract the $(2n+1)$-th row from the $(2n+2)$-th row and then the $2n$-th row from the $(2n+1)$-th row, etc.; omitting the unnecessary rows and columns from this matrix in order to obtain the Alexander polynomial, we have

$$
(3)\qquad
\begin{array}{c}
\begin{array}{c|cccc|cccc}
b & c_0 & c_1 & \cdots & c_m & c'_0 & c'_1 & \cdots & c'_m
\end{array}\\
\left(\begin{array}{c|c|c}
-x & -nx+n\ \ 0\ \cdots\ 0 & nx-n\ \ 0\ \cdots\ 0 \\ \hline
\begin{array}{c} b_1 \\ \vdots \\ b_{m+1} \end{array} & c_{ij} & 0 \\ \hline
\begin{array}{c} -b_1 \\ \vdots \\ -b_{m+1} \end{array} & 0 & -c_{ij}
\end{array}\right)
\end{array}.
$$

Adding each $(m+3+i)$-th column $(i=0, 1, \cdots, m)$ to the $(2+i)$-th column and then each $(1+i)$-th row $(i=1, 2, \cdots, m+1)$ to the $(m+2+i)$-th row respectively, we have further

$$
(4)\qquad
\begin{array}{c}
\begin{array}{c|cccc|cccc}
b & c_0 & c_1 & \cdots & c_m & c'_0 & c'_1 & \cdots & c'_m
\end{array}\\
\left(\begin{array}{c|c|c}
-x & 0\ \ \ 0\ \cdots\ 0 & nx-n\ \ 0\ \cdots\ 0 \\ \hline
\begin{array}{c} b_1 \\ \vdots \\ b_{m+1} \end{array} & c_{ij} & 0 \\ \hline
0 & 0 & -c_{ij}
\end{array}\right)
\end{array}.
$$

We have therefore

$$\Delta_{\kappa''}(x) = \pm x^p |c_{ij}|^2 ,$$

where p is a suitably chosen integer.

On the other hand, since the Alexander matrix of κ takes the form

$$
(5)\qquad
\begin{array}{c}
\begin{array}{cc|ccc}
a_0 & a_1 & c_0 & \cdots & c_m
\end{array}\\
\left(\begin{array}{cc|ccc}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\ \hline
\multicolumn{2}{c|}{*} & \multicolumn{3}{c}{c_{ij}}
\end{array}\right)
\end{array},
$$

where $i = 1, 2, \cdots, m+1$ and $j = 0, 1, \cdots, m$, we have

$$\Delta_\kappa(x) = \pm x^{p'} |c_{ij}| \,,$$

where p' is a suitably chosen integer. We have therefore

$$\Delta_{\kappa''}(x) = (\Delta_\kappa(x))^2 \,,$$

which was to be proved.

Since our method yields the same result even if n is negative, the proof of Theorem 2 is thus complete.

Concerning a symmetric skew union we have only the following weaker result:

Theorem 3. *If κ_1 and κ_2 are symmetric skew unions of a knot κ with the same place of joinings and windings, their Alexander polynomials $\Delta_{\kappa_1}(x)$ and $\Delta_{\kappa_2}(x)$ coincide:*

$$\Delta_{\kappa_1}(x) = \Delta_{\kappa_2}(x) \,,$$

even if the winding numbers are different from each other.

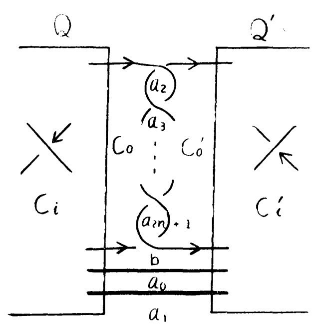


Fig. 10.

Proof. Let κ'' be a symmetric skew union of κ with the winding number $2n+1$. Suppose first that $2n+1$ is positive, and let the projection κ''_E of κ'' on the ground plane E assume the form as shown in Fig. 10. κ''_E decomposes E into domains as indicated in the figure. Then we are only to prove that the determinant of the matrix

$$\begin{array}{c|cccc|cccc}
b & c_0 & c_1 & \cdots & c_m & c'_0 & c'_1 & \cdots & c'_m \\
x & -(n+1)x+n & 0 & \cdots & 0 & nx-(n+1) & 0 & \cdots & 0 \\
\hline
\begin{matrix} b_1 \\ \vdots \\ b_{m+1} \end{matrix} & & c_{ij} & & & & 0 & & \\
\hline
\begin{matrix} -b_1 \\ \vdots \\ -b_{m+1} \end{matrix} & & 0 & & & & -c_{ij} & &
\end{array}$$

corresponding to (3) of the proof of Theorem 2 is independent of n. The matrix corresponding to (4) takes the form

$$
(6)\qquad
\begin{array}{c}
\begin{array}{c|ccc|ccc}
b & c_0 & c_1 \cdots c_m & & c_0' & c_1' \cdots c_m' & \\
\end{array}\\
\left(\begin{array}{c|ccc|ccc}
x & -x-1 & 0 \cdots 0 & & -(n+1)\,x+n & 0 \cdots 0 & \\
\hline
\begin{matrix} b_1 \\ \vdots \\ b_{m+1} \end{matrix} & & c_{ij} & & & 0 & \\
\hline
0 & & 0 & & & -c_{ij} &
\end{array}\right)
\end{array}.
$$

Adding the first column to the second and then the second column multiplied by x to the first, we have

$$
\left(\begin{array}{c|ccc|ccc}
0 & -1 & 0 \cdots 0 & & -(n+1)\,x+n & 0 \cdots 0 & \\
\hline
\begin{matrix} b_1' \\ \vdots \\ b_{m+1}' \end{matrix} & & c_{ij}' & & & 0 & \\
\hline
0 & & 0 & & & -c_{ij} &
\end{array}\right),
$$

the determinant of which is seen to be independent of n. Since our method yields the same result even if $2n+1$ is negative, the proof of Theorem 3 is thus complete.

Theorem 4. *If κ'' is a symmetric union or a symmentric skew union of a knot κ, then*

$$T_2(\kappa'') = (T_2(\kappa))^2,$$

where $T_2(\kappa)$ denotes the product of torsion numbers of the two-fold branched covering space of κ.

Proof. Since $T_2(\kappa) = |\Delta_\kappa(-1)|$ (cf. [5], [6]) for every knot κ, we are only to prove that

$$|\Delta_{\kappa''}(-1)| = |\Delta_\kappa(-1)|^2.$$

If κ'' is a symmetric union, this equality follows immediately from Theorem 2. Therefore let κ'' be a symmetric skew union. Then from (5) we have

$$|\Delta_\kappa(-1)| = \pm\, |c_{ij}(-1)|$$

and from (6), where we put c'_{ij} instead of c_{ij},

$$|\Delta_{\kappa''}(-1)| = |c'_{ij}(-1)|^2.$$

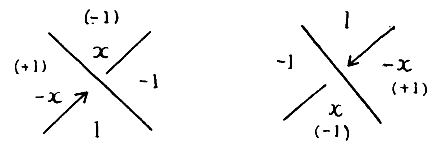


Fig. 11.

It is easy to see that for each $i\,(i=1, 2, \cdots, m-1)$ (see Fig. 11)

$$(c_{i,0}(-1)\,,\ c_{i,1}(-1)\,,\ \cdots,\ c_{i,m}(-1)) = \pm(c'_{i,0}(-1)\,,\ c'_{i,1}(-1)\,,\cdots,\ c'_{i,m}(-1))\,.$$

Therefore

$$|\Delta_{\kappa''}(-1)| = |\Delta_{\kappa}(-1)|^2$$

and the proof is complete.

§ 3. Union of knots with the same Alexander polynomial.

We have mentioned in Introduction that the knots 8_5, 8_{10}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{24} and 9_{28} of Alexander and Briggs table are unions of two trefoil knots. Now, if we decompose the Alexander polynomials of these knots into factors, we find a remarkable fact that they have a factor x^2-x+1 in common:

Knot	Alexander polynomial	Factorization[4]
8_5	$1-\ 3+\ 4-\ 5$	$=(1-1)\,(1-2+1)$
8_{10}	$1-\ 3+\ 6-\ 7$	$=(1-1)^3$
8_{15}	$3-\ 8+11$	$=(1-1)\,(3-5)$
8_{19}	$1-\ 1+\ 0+\ 1$	$=(1-1)\,(1+0-1)$
8_{20}	$1-\ 2+\ 3$	$=(1-1)^2$
8_{21}	$1-\ 4+\ 5$	$=(1-1)\,(1-3)$
9_{16}	$2-\ 5+\ 8-\ 9$	$=(1-1)\,(2-3+3)$
9_{24}	$1-\ 5+10-13$	$=(1-1)^2(1-3)$
9_{28}	$1-\ 5+12-15$	$=(1-1)\,(1-4+7)$.

But this is not a mere contingency. We have in fact

Theorem 5. *If κ and κ' are knots with the same Alexander polynomial $\Delta(x)$, then every union of κ and κ' has a factor $\Delta(x)$ in its Alexander polynomial.*

Proof. Since the theorem is clear in the case $2n=0$, first let $2n>0$.

4) $(1-1)\,(1-2+1)$ means e.g. $(x^2-x+1)(x^4-2x^3+x^2-2x+1)$.

As in the proof of Theorem 2 the Alexander matrix takes the following form:

$$
\begin{array}{cc|cccc|c|c|ccc|c|ccc}
a_0 & a_1 & a_2 & \cdots & & a_{2n} & b & c_0 & c_1 & \cdots & c_m & c'_0 & c'_1 & \cdots & c'_{m'} \\
\hline
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & x & 1 & 0 & \cdots & 0 & 0 & -x & & & & -1 & & & \\
0 & 0 & 1 & x & \cdots & 0 & 0 & -1 & & 0 & & -x & & 0 & \\
\vdots & & & & \ddots & & \vdots & \vdots & & & & \vdots & & & \\
0 & 0 & 0 & \cdots & x & 1 & 0 & -x & & & & -1 & & & \\
0 & 0 & 0 & \cdots & 0 & 1 & x & -1 & & & & -x & & & \\
\hline
* & & & 0 & & & * & & c_{ij} & & & & 0 & & \\
\hline
* & & & 0 & & & * & & 0 & & & & c'_{ij} & &
\end{array}
,$$

where

$$
\begin{array}{cc|ccc}
a_0 & a_1 & c_0 & \cdots & c_m \\
\hline
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\hline
* & & & c_{ij} &
\end{array}
\quad , \quad
\begin{array}{cc|ccc}
a_0 & a_1 & c'_0 & \cdots & c'_{m'} \\
\hline
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\hline
* & & & c'_{ij} &
\end{array}
$$

represent the Alexander matrices of κ and κ' respectively.

For the sake of the generality of speaking let $c_1(x)$ be the greatest common factor of elements in the c_1-column. Divide the c_1-column by $c_1(x)$ and multiply the c_0-column by $c_1(x)$. Then by elementary transformations operating on rows bring all elements but one, say c_{1j}, of the c_1-column to 0, letting c_{1j} be equal to 1. Again by elementary transformations operating this time on columns bring all elements of the row of c_{1j} except for this one equal to 0.

Proceeding in this way with c_2-column, etc. and interchanging row and columns suitably, we have finally

$$
\begin{array}{c|c|c|c|c}
a_0\ a_1 & a_2 \cdots\cdots a_{2n} & b & c_0 \cdots\cdots & c'_0 \cdots\cdots \\
\hline
\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & 0 & 0 & 0 \\
\hline
* & * & * & * \quad 0 & * \quad 0 \\
\hline
\begin{matrix} * \\ 0 \end{matrix} & 0 & \begin{matrix} * \\ 0 \end{matrix} & \begin{matrix} \pm x^p \Delta(x) & & 0 \\ & 1 & \\ & & \ddots \\ & & & 1 \end{matrix} & 0 \\
\hline
* & 0 & * & 0 & \begin{matrix} \pm x^q \Delta(x) & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{matrix}
\end{array}
$$

the determinant of which, by deleting the first two rows and columns and by expanding along the last $2(m+1)$ rows, shows immediately that it has indeed a factor $\Delta(x) = \pm x^s |c_{ij}| = \pm x^r |c'_{ij}|$.

Similarly for the case $2n < 0$ and the proof is complete.

§ 4. Knots with the Alexander polynomial equal to unity.

H. Seifert [13] and J. H. C. Whitehead [14] determined some types of knots whose Alexander polynomials are equal to 1. Here another type of such knots will be given.

The basic idea of the construction is this: If the Alexander polynomial of a knot κ is unity, then by Theorem 2 that of every symmetric union of κ is also unity; therefore, if we take as κ a trivial knot, we would obtain a knot of the above property. The main difficulty is, how to determine the place of joining and winding in order that the union be a non trivial knot.

Let κ be a trivial knot given in its graphical representation (k) by $2p+1$ sided polygon $A_0A_1 \cdots A_{2p}$ of p consecutive sides A_0A_1, $A_1A_2, \cdots,$ $A_{p-1}A_p$ with negative signs. Let $A'_0A'_1 \cdots A'_{2p}$ be another polygon of consecutive sides $A'_pA'_{p+1}$, $A'_{p+1}A'_{p+2}$, $\cdots$, $A'_{2p-1}A'_{2p}$ with negative signs, symmetric to $A_0A_1 \cdots A_{2p}$ with respect to a straight line. Bring A_0 and A'_0 to a coincidence and connect A_p and A'_p with $2n$ arcs $e_1, e_2, \cdots, e_{2n}$. The knot represented by this graph, which is a symmetric union of κ with the winding number $2n$, will be denoted by $\kappa(p, 2n)$. n should be taken negative if the signs of e_i are negative.

Then we have

Theorem 6. *$\kappa''=\kappa(p, 2n)$ is a non trivial knot with the Alexander polynomial $\Delta_{\kappa''}(x)=1$, whenever $p\geqq 2$ and $n\neq 0$. Especially $\kappa(2, 2)$ is a knot of eleven crossings with the Alexander polynomial equal to unity.*

Proof. As will be seen from the structure of $\kappa(p, 2n)$ it suffices to prove the theorem only for the case $n>0$. It is also immaterial whether p is even or odd, but first let p be an odd number. Then, passing over from the original graph to its dual form, we see that $\kappa(p, 2n)$ is a union of knots κ_1 and κ_2 whose graphs (k_1) and (k_2) consist respectively of:

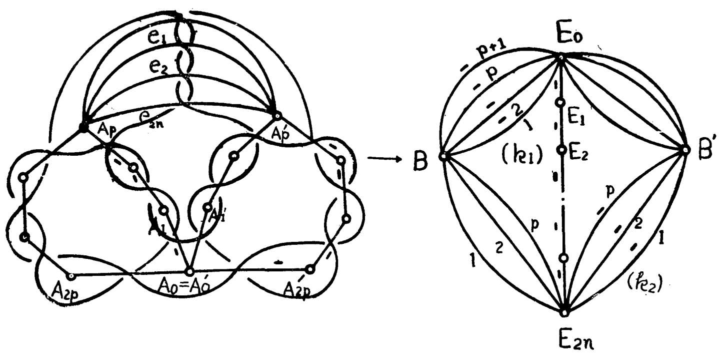


Fig. 12.

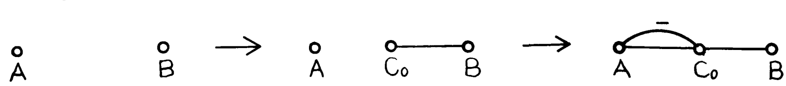
(k_1): vertices E_0, E_1, $\cdots$, E_{2n} and B and $2n$ consecutive edges with negative signs E_0E_1, E_1E_2, $\cdots$, $E_{2n-1}E_{2n}$, a bundle of $p+1$ edges with negative signs connecting B and E_0 and a bundle of p edges connecting B and E_{2n}.

(k_2): vertices E_{2n} and B' and a bundle of p edges with negative signs connecting these vertices.

Now the knot κ_2 represented by (k_2) is alternating and so by Bankwitz's theorem ([3], see [8], p. 34) a non trivial knot.

In order to show that κ_1 is also non trivial, we proceed as follows[5]:

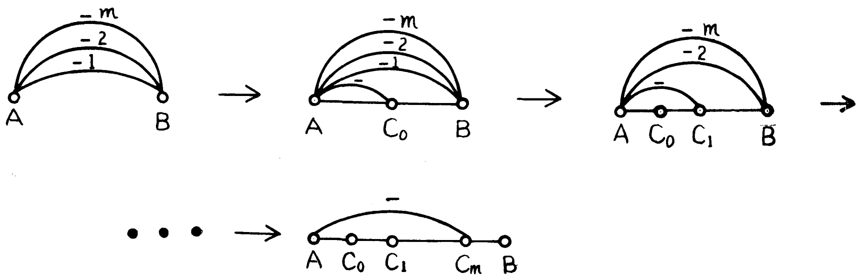
(1) A graph can be carried over to an equivalent one, if we insert between two vertices A and B of the graph two edges AC_0 and C_0B with positive signs (which are understood) and an edge AC_0 with a negative sign:



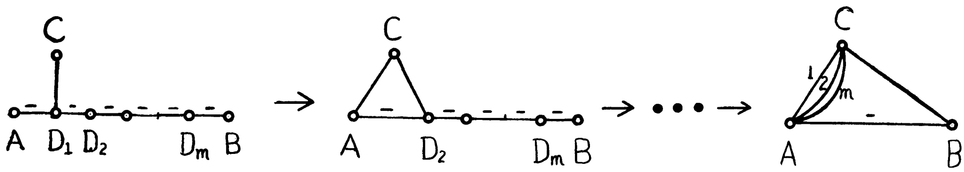
(2) By inserting between the vertices A and B of a graph three

5) For the reduction of graphs see [15].

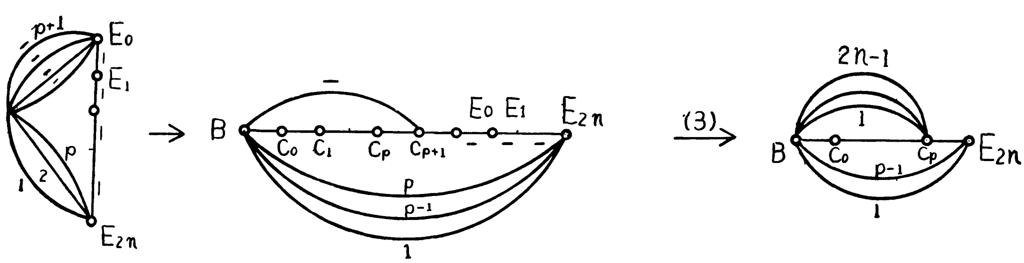
edges of the type of (1), a bundle of m edges with negative signs connecting A and B can be substituted by $m+2$ edges AC_0, $C_0C_1, \cdots, C_mB$ and by an edge AC_m with a negative sign:



(3) If two vertices A and B of a graph is connected by $m+1$ consecutive edges AD_1, $D_1D_2, \cdots, D_mB$ with negative signs and another vertex C is further connected with D_1 by an edge CD_1, then it will be shown by repetition of a fundamental operation on graphs [15] that these edges can be substituted by a bundle of m edges connecting A and C, by the edge BC and by the edge AB with a negative sign.



(4) Applications of (2) and (3) shows finally the following reduction of (k_1) to an alternating knot:



Again by Bankwitz's theorem the knot κ_1 is seen to be a non trivial one.

Hitherto we have assumed that p was an odd number, but if p is even, a similar decomposition of (k'') into knots will be obtained if we exchange $E_0, E_1, \cdots, E_{2n}, B, B', p$ and $p+1$ seriatim by $E_{2n}, E_{2n-1}, \cdots, E_0, B', B, p+1$ and p in the definitions of (k_1) and (k_2).

The knot κ'' is thus seen to be a union of two non trivial knots κ_1 and κ_2, and consequently it is by Theorem 1 non trivial.

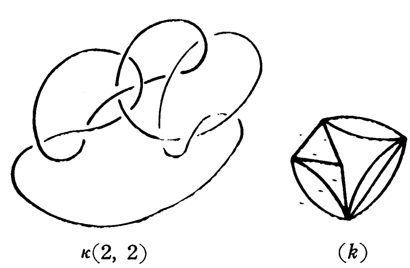


Fig. 13.

By a similar reduction of graph it can be proved without difficulty that if $p=2$ and $n=1$, $\kappa(2, 2)$ is a knot of 11 crossings with the graph (k) as shown in Fig. 13.

REMARK. The non triviality of knots κ_1 and κ_2 can also be proved directly and without difficulty by means of the calculation of the determinant of knot.

APPENDIX

The sum of knots modulo 2.

Let κ_1 and κ_2 be two knots having a single arc γ as their meet: $\kappa_1 \frown \kappa_2 = \gamma$; then the sum modulo 2 of κ_1 and κ_2, that is the point set $\kappa_1 \cup \kappa_2 - (\gamma)$, where (γ) denotes the open arc of γ, will be called the *sum* of κ_1 and κ_2, and denoted by $\kappa_1(+)\kappa_2$. We are going to prove that any knot can always be represented as a sum of two trivial knots.

A knot κ will be said to be *decomposable into trivial knots* by a pair of different points A and B on κ, if there is an arc γ, a *decomposition-arc*, connecting A and B outside κ such that κ is represented as the sum of knots $\alpha \cup \gamma$ and $\beta \cup \gamma$, where α and β are two arcs of κ of endpoints A and B.

According to this definition, if a knot κ is the sum of two trivial knots κ_1 and κ_2, then there must be a pair of points A and B on κ such that κ is decomposable into trivial knots by A and B; for the endpoints of the common arc $\kappa_1 \frown \kappa_2$ have this property.

Then we have

Lemma 1. *If a knot κ is decomposable into trivial knots by two points A and B on κ, then κ can also be decomposable into trivial knots by any two points A' and B' of κ.*

Proof. Let $\widehat{AA'}$ be the arc of κ which does not contain B. Then, if γ is a decomposition-arc of endpoints A and B, an arc γ_1 connecting

A' and B sufficiently near $\widehat{AA'}\cup\gamma_1$ will be found as a decomposition-arc of endpoints A' and B. By the same reason there can be found further a decompotition-arc of endpoints A' and B' sufficiently near γ_2, q.e.d.

Lemma 2. *Let Q be a solid cube that has two segments in common with a knot κ and let γ be an arc that connects two points X and Y of κ lying outside Q and that has no points in common with κ except for X and Y. Then X and Y can be connected wholly outside Q by an arc that is isotopic to γ in the complementary $E^3-\kappa$ of κ.*

Proof. $Q\cap\kappa$ can be thought of as consisting of two perpendiculars AB and CD between a pair of opposite faces Σ and Σ' of Q. Suppose γ intersect Σ in a point P. Then we can substitute a sufficiently small arc β of γ containing P in its interior by an arc β' that is isotopic to β in $E^3-\kappa$ and that has no point in common with Σ, so that the newly obtained arc γ' intersects Σ in points less by one in number as compared with γ. Proceeding in the same way with all intersection points of γ and Σ, which we may suppose finite in number, we can obtain finally an arc γ'' that is isotopic to γ in $E^3-\kappa$ and that has no points in common with Σ. Then by a homeomorphism of E^3 leaving all points of κ and Σ fixed "drive away" γ'' out of Q in the direction from Σ to Σ' so that the image of γ'' becomes the desired arc lying wholly outside Q.

We are now in a position to prove

Theorem. *Every knot can be represented as the sum of two trivial knots.*

Proof. We prove the theorem by induction on the unknotting number of a knot [7], that is the minimum number of cuts in order to change the given knot to a trivial one.

First suppose that the theorem holds true if the unknotting number of a knot is $n-1$, and let κ be a knot of unknotting number $n(\geqq 1)$. Then there are by the definition of the unknotting number an equivalent of κ, which we denote again by the same letter κ, and a solid cube Q having two segments AB and CD in common with κ such that if AEB is a triangle within Q having a point in common with CD and if we substitute the arc AB of κ by $AE\cup EB$, the newly obtained knot κ' becomes a knot of the unknotting number $n-1$. By our assumption κ' is the sum of two trivial knots, and so κ' is decomposable into trivial knots by two points X and Y on κ'. Then by Lemma 1 X and Y can be taken outside Q. Let γ be a decomposition-arc of endpoints X and Y: by Lemma 2 γ can be so chosen that it lies wholly outside Q. If α and β are arcs of κ' of endpoints X and Y, of which β contains the segment CD, then $\alpha \cup \gamma$ and $\beta \cup \gamma$ are trivial knots. But since γ has no points in common with Q, $(\alpha(+)\Delta AEB) \cup \gamma$ is also a trivial knot. Thus κ turns out to be the sum of two trivial knots $(\alpha(+)\Delta AEB) \cup \gamma$ and $\beta \cup \gamma$.

Since the theorem is true if $n = 0$, that is, if the knot is itself trivial, the theorem is thus proved to be true for any knot.

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