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On the Estimation of the Quality of a Group of Lots by the Single Sampling Inspection in Destructive Case

By Sadao IKEDA

Summary. A. Kolmogorov [1] considered the estimation of the quality of a group of lots by the single sampling inspection in destructive case. The sampling inspection must be used to improve the quality of the group of lots through the inspection, and the detailed plan must be devised in each practical case, but it may be useful in many cases to estimate the qualities of the group of lots before and after the inspection. In this paper we will try to construct the estimates to evaluate the improvement in the quality through the sampling inspection adopted.

For this purpose, in Section 1 we extend the results of M. A. Girshick, F. Mosteller and L. J. Savage [2] from binomial case to hypergeometric case only for the finite regions. Some properties of the operating characteristic curve are stated in Section 2, and we investigate in Section 3 the estimates of three important qualities of the group of lots inspected.

1. Unbiased estimates in hypergeometric sampling

In this section we show that the unbiased estimates obtained in hypergeometric case analogously to those of M. A. Girshick, F. Mosteller, and L. J. Savage [2] are unique for the simple regions. The contents of this section are the direct extension of the results in the papers, [2], L. J. Savage [3] and J. Wolfowitz [4] for the finite regions. The estimates obtained here are useful for the unbiased estimation in the sampling where the binomial approximation is impossible.

We start with the following definitions.

DEFINITION 1. A *region* R is a subset of all two dimensional non-negative integer points which contains the origin, i. e.,

$$R=\{\alpha=(x, y)\,|\,x, y: \text{non-negative integers}, (0, 0)\ \varepsilon\ R\}.$$

DEFINITION 2. The *path* $\phi(\alpha, \beta)$ is a set of points $\{\alpha_i^{\xi_i}\,|\,i=0, 1,$

$2, \dots, n\}$ satisfying the following conditions:

$$\begin{aligned}
&\alpha = \alpha_0^{\varepsilon_0},\ \beta = \alpha_n^{\varepsilon_n},\ \{\alpha_0^{\varepsilon_0}, \alpha_1^{\varepsilon_1}, \dots, \alpha_{n-1}^{\varepsilon_{n-1}}\} \subset R,\\
&\alpha_i^{\varepsilon_i} = (x_i, y_i),\ \varepsilon_i = 0 \text{ or } \varepsilon_i = 1,\\
&\varepsilon_i = 0 \text{ implies } x_{i+1} = x_i + 1,\ y_{i+1} = y_i,\\
&\varepsilon_i = 1 \text{ implies } x_{i+1} = x_i,\ y_{i+1} = y_i + 1\ (i = 0, 1, \dots, n-1).
\end{aligned}$$

The set of all $\phi(\alpha, \beta)$ is denoted by $K(\alpha, \beta)$.

DEFINITION 3. The probability of path $\phi(\alpha, \beta)$ is defined as follows:

$$P\{\phi\} = \prod_{i=0}^{n-1} p(\alpha_i^{\varepsilon_i}) q(\alpha_i^{\varepsilon_i}),$$

where for $i = 0, 1, \dots, n-1$,

$$\begin{aligned}
0 \leqq p(\alpha_i^1) \leqq 1,\ p(\alpha_i^0) = 1,\\
0 \leqq q(\alpha_i^0) \leqq 1,\ q(\alpha_i^1) = 1,\\
p(\alpha_i^1) + q(\alpha_i^0) = 1.
\end{aligned}$$

Briefly we denote it by $\prod_{\phi} p(\alpha^\varepsilon) q(\alpha^\varepsilon)$.

DEFINITION 4. If there exists a path $\phi(0, \alpha)$ for a point α in R, then α is called an *accessible point*, and $\bar{R}$ is the totality of the accessible points in R.

If there exists a path $\phi(0, \alpha)$ for a point α not in R, then α is called a *boundary point* of R, and the totality of boundary points of R is denoted by B.

DEFINITION 5. For any point $\alpha = (x, y)$, $I(\alpha) = x + y$ is called the *index of* α, and

$$\operatorname{Sup}_{\alpha \in \bar{R}} \{I(\alpha)\}$$

is called the *index of* R.

We say a region R is *finite* if the index of R is finite.

DEFINITION 6. If the equality

$$\sum_{\alpha \in B} \sum_{\phi \in K(0,\alpha)} P\{\phi\} = 1$$

holds, then R is called a *closed* region.

DEFINITION 7. For any two paths $\phi(\alpha, \beta)$ and $\phi'(\beta, \gamma)$, the path $\phi''(\alpha, \gamma)$ which coincides with $\phi(\alpha, \beta)$ between α and β and with $\phi'(\beta, \gamma)$

between β and γ is denoted by

$$\phi''(\alpha, \gamma) = \phi(\alpha, \beta)\cdot\phi'(\beta, \gamma),$$

or briefly by $\phi\cdot\phi'$.

Under these definitions we prove the following theorems.

Theorem 1. *If R is a finite region, then R is closed.*

PROOF. For any non-negative integer n, we define:

$$\begin{aligned} E(n) &= \{\alpha \mid I(\alpha) = n\}, \\ \bar{R}_n &= E(n)\cdot\bar{R}, \\ R^*_{n+1} &= \{\alpha^* \mid \alpha^* = (x^*, y^*) : x^* = x+1,\ y^* = y \text{ or } x^* = x, \\ &\qquad y^* = y+1 \text{ for } \alpha = (x, y)\varepsilon\bar{R}_n\}, \end{aligned}$$

then the boundary of index $n+1$ is clearly

$$B_{n+1} = R^*_{n+1} - \bar{R}_{n+1}\,.$$

In general, the boundary of R is as follows:

$$B = \sum_{n=0}^{\infty} B_{n+1}\,,$$

and in particular if the region is finite, there exists a positive integer N such that

$$B = \sum_{n=0}^{N} B_{n+1}\,. \tag{1}$$

The probability (with regard to R) of the point α is denoted by

$$P\{\alpha\} = \sum_{\phi \in K(0,\alpha)} P\{\phi\}\,,$$

and for any set A we define

$$P\{A\} = \sum_{\alpha \in A} P\{\alpha\}\,,$$

then the following equalities hold:

$$P\{R_1^*\} = 1, \tag{2}$$

$$P\{\bar{R}_n\} = P\{R^*_{n+1}\},\ (n = 0, 1, 2, \ldots), \tag{3}$$

$$P\{B_{n+1}\} = P\{R^*_{n+1}\} - P\{\bar{R}_{n+1}\},\ (n = 0, 1, 2, \ldots)\,. \tag{4}$$

From (1), (2), (3) and (4), we have

$$\begin{aligned}
P\{B\} &= \sum_{n=0}^{N} P\{B_{n+1}\} \\
&= \sum_{n=0}^{N} (P\{R_{n+1}^{*}\} - P\{\bar{R}_{n+1}\}) \\
&= \sum_{n=0}^{N} (P\{\bar{R}_n\} - P\{\bar{R}_{n+1}\}) \\
&= P\{\bar{R}_0\} = P\{R_1^*\} = 1 .
\end{aligned}$$

Thus R is closed and the proof of our theorem is complete.

Lemma 1. *If R is a closed region and a region R' is included in R, then R' is also closed.*

PROOF.

$$\begin{aligned}
P\{B\} &= \sum_{n=0}^{\infty} P\{B_{n+1}\} \\
&= \sum_{n=0}^{\infty} (P\{\bar{R}_n\} - P\{\bar{R}_{n+1}\}) \\
&= \lim_{n\to\infty} (P\{\bar{R}_0\} - P\{\bar{R}_n\}) \\
&= 1 - \lim_{n\to\infty} P\{\bar{R}_n\} ,
\end{aligned}$$

and thus the closedness of R is equivalent to the equality

$$\lim_{n\to\infty} P\{\bar{R}_n\} = 0 .$$

Let $\bar{R}_n'$ be the totality of accessible points of index n in R', then clearly we have:

$$\bar{R}_n' \subset \bar{R}_n \qquad (n = 0, 1, 2, \dots) ,$$

and if we denote the probability with regard to R' by $P'\{\cdot\}$,

$$0 \leqq P'\{\bar{R}_n'\} \leqq P\{\bar{R}_n\} \qquad (n = 0, 1, 2, \dots) .$$

Thus, we can conclude that

$$\lim_{n\to\infty} P\{\bar{R}_n'\} = 0 ,$$

and this implies the closedness of R'.

Theorem 2. *If R is a closed region, and $\tau \in \bar{R}$, then the equality*

$$\sum_{\alpha\in B} \sum_{\phi'\in K(0,\tau)} \sum_{\phi\in K(\tau,\alpha)} P\{\phi'\cdot\phi\} = \sum_{\phi'\in K(0,\tau)} P\{\phi'\}$$

holds.

PROOF. If we define

$$Q(\tau)=\sum_{\alpha\in B}\sum_{\phi\in K(\tau,\alpha)}P\{\phi\}\ ,$$

then we have

$$\sum_{\alpha\in B}\sum_{\phi'\in K(0,\tau)}\sum_{\phi\in K(\tau,\alpha)}P\{\phi'\cdot\phi\}=Q(\tau)\sum_{\phi'\in K(0,\tau)}P\{\phi'\}\ .$$

Let $R'=R-\{\tau\}$, then $\tau\in B'$, where B' is the boundary of R'. Then

$$(5)\quad \begin{aligned}&\sum_{\alpha\in B}\sum_{\phi\in K(0,\alpha)}P\{\phi\}\\ &=\sum_{\alpha\in B}\sum_{\phi\in K'(0,\alpha)}P\{\phi\}+\sum_{\alpha\in B}\sum_{\varphi'\in K(0,\tau)}\sum_{\phi\in K(\tau,\alpha)}P\{\phi'\cdot\phi\}\\ &=\sum_{\alpha\in B}\sum_{\phi\in K'(0,\alpha)}P\{\phi\}+Q(\tau)\sum_{\phi'\in K(0,\tau)}P\{\phi'\}\ ,\end{aligned}$$

where $K'(0,\alpha)$ is the set of all paths $\phi(0,\alpha)$ in R'. On the other hand, we have

$$(6)\quad \sum_{\alpha\in B'}\sum_{\phi\in K'(0,\alpha)}P\{\phi\}=\sum_{\alpha\in B}\sum_{\phi\in K'(0,\alpha)}P\{\phi\}+\sum_{\phi'\in K(0,\tau)}P\{\phi'\}\ .$$

R' is closed by Lemma 1, and the left hand members of the equalities (5) and (6) are both unity, i. e.,

$$(7)\quad \sum_{\alpha\in B}\sum_{\phi\in K(0,\alpha)}P\{\phi\}=\sum_{\alpha\in B'}\sum_{\phi\in K'(0,\alpha)}P\{\phi\}=1\ .$$

(5), (6) and (7) implies the equality

$$Q(\tau)=1\ ,$$

and this completes the proof of our theorem.

The proof of above theorem shows only that

$$Q(\tau)=1\ ,$$

so that we can replace the equality in the theorem by the equality:

$$\sum_{\alpha\in B}\sum_{\phi\in K(\tau,\alpha)}P\{\phi'\cdot\phi\}=P\{\phi'\}$$

for any ϕ' in $K(0,\tau)$.

Theorem 3. *Let R be closed, and $\tau\in\bar{R}$. If, for any $\alpha\in B$, the function of α:*

$$\varphi(\alpha)=\sum_{\phi'\in K(0,\tau)}\sum_{\phi\in K(\tau,\alpha)}P\{\phi'\cdot\phi\}\Big/\sum_{\phi\in K(0,\alpha)}P\{\phi\}$$

is independent of any $p(\alpha^{\varepsilon})$, then $\varphi(\alpha)$ is an unbiased estimate of $f(p(\alpha^{\varepsilon}),\ q(\alpha^{\varepsilon}))$:

$$f(p(\alpha^{\varepsilon}),\ q(\alpha^{\varepsilon}))=\sum_{\phi'\in K(0,\tau)}P\{\phi'\}\ .$$

PROOF. By the result of Theorem 2, it is easy to show that

$$\sum_{\alpha \in B} \varphi(\alpha) \sum_{\phi \in K(0,\alpha)} P\{\phi\} = \sum_{\phi' \in K(0,\tau)} P\{\phi'\} .$$

This means that $\varphi(\alpha)$ is an unbiased estimate of $f(p(\alpha^{\varepsilon}),\ q(\alpha^{\varepsilon}))$.

Theorem 4. *If the probability of any path $\phi \in K(0, \alpha)$ depends only on* α, i. e.,

$$P\{\phi\} = \prod_{\phi} p(\alpha^{\varepsilon}) q(\alpha^{\varepsilon})$$

is constant for any path in $K(0, \alpha)$, then $\varphi(\alpha)$ does not depend on any $p(\alpha^{\varepsilon})$.

PROOF. We can easily conclude the result of the theorem from the fact that

$$\phi' \cdot \phi \in K(0,\ \alpha)$$

and the condition of our theorem.

If the condition of Theorem 4 is fulfilled, then

$$\varphi(\alpha) = k(0,\ \tau) k(\tau,\ \alpha) / k(0,\ \alpha),$$

and

$$f(p(\alpha^{\varepsilon}),\ q(\alpha^{\varepsilon})) = k(0,\ \tau) \bar{P}(\tau) ,$$

where $k(\alpha,\ \beta)$ is the number of paths contained in $K(\alpha,\ \beta)$, and for ϕ' in $K(0,\ \tau)$

$$\bar{P}(\tau) = \prod_{\phi'} p(\alpha^{\varepsilon}) q(\alpha^{\varepsilon}) .$$

This occurs for example in the binomial case.

Now we show that the simplicity of R is the necessary and sufficient condition for the uniqueness of $\varphi(\alpha)$ under suitable conditions. The definition of simplicity is as follows:

DEFINITION 8. A region R is *simple* if all the points of index n between any two accessible points of index n are also accessible for all n, i. e., if $x+y=n$, and for some positive integer k,

$$\alpha_0 = (x,\ y) \in \bar{R} ,\ \alpha_k = (x-k,\ y+k) \in \bar{R} ,$$

then for all $j : (1 \leqq j < k)$,

$$\alpha_j = (x-j,\ y+j) \in \bar{R} .$$

Theorem 5. *Suppose that the conditions* (i) *to* (iv) *are satisfied*:

(i) *R is finite.*

(ii) *The condition of Theorem* 4 *is fulfilled.*

(iii) *The class of probabilities is such that we can choose arbitrarily small values of* $p(\alpha^1)$ *and* $q(\alpha^0)$ *uniformly in* α.

(iv) *If there exists at least one boundary point whose index is greater than* n, *then, for any sequence of boundary points of index* n, $\{\alpha_0, \alpha_1, \dots, \alpha_t\}$, *such that*

$$\alpha_j = (x_j, y_j),\ x_{j+1} = x_j - 1,\ y_{j+1} = y_j + 1, \quad (j = 0, 1, \dots, t),$$

there exist two sets of constants which are not all zero $\{c_0, c_1, \dots c_t\}$, $\{b_0, b_1, \dots, b_u\}$ *and a sequence of points* $\{\beta_0, \beta_1, \dots, \beta_u\}$ *in* $(\bar{R}+B)$ *of which* $I(\beta_i) \geqq n$, *satisfying the equality*

$$\sum_{j=0}^{t} c_j \bar{P}(\alpha_j) = \sum_{i=0}^{u} b_i \bar{P}(\beta_i) \tag{8}$$

identically in $p(\alpha^\varepsilon)$, *where* $\phi \in K(0, \alpha)$ *and*

$$\bar{P}(\alpha) = \prod_{\phi} p(\alpha^\varepsilon) q(\alpha^\varepsilon).$$

Then, in order that the estimate $\varphi(\alpha)$ *is the unique proper unbiased estimate of* $f(p(\alpha^\varepsilon), q(\alpha^\varepsilon))$, *it is necessary and sufficient that the region* R *be simple.*

PROOF. (1°) Necessity.

Suppose that $\varphi(\alpha)$ is the unique proper unbiased estimate of $f(p(\alpha^\varepsilon), q(\alpha^\varepsilon))$ and R is not simple. (The notion *proper* is the same as *bounded.*) Then there exists a sequence of boundary points $\{\alpha_0, \alpha_1, \dots, \alpha_t\}$ such as

$$\alpha_j = (x_j, y_j),\ x_{j+1} = x_j - 1,\ y_{j+1} = y_j + 1,\ I(\alpha_j) = n, \quad (j = 0, 1, \dots, t),$$
$$\alpha_* = (x_0 + 1,\ y_0 - 1) \in \bar{R},\ \alpha^* = (x_t - 1,\ y_t + 1) \in \bar{R}.$$

We can choose the minimum value of n which satisfies the above conditions. Then, clearly the boundary B of R contains at least one boundary point of index $> n$.

By the condition (iv) of our theorem there exist two sets of constants $\{c_0, c_1, \dots, c_t\}$, $\{b_0, b_1, \dots, b_u\}$ and a sequence $\{\beta_0, \beta_1, \dots, \beta_u\}$ in $(\bar{R}+B)$ of which $I(\beta_i) \geqq n$, satisfying the equality (8). Now we define a new function on B such that

$$m(\alpha_j) = c_j / k(0, \alpha_j), \quad (j = 0, 1, \dots, t),$$
$$m(\alpha) = -\sum_{i=0}^{u} b_i k(\beta_i, \alpha) / k(0, \alpha), \quad (\alpha \in B - \{\alpha_0, \alpha_1, \dots, \alpha_t\}).$$

Then, $m(\alpha)$ is not identically zero, and

$$\begin{aligned}
&\sum_{\alpha\in B} m(\alpha)k(0,\alpha)\bar{P}(\alpha)\\
&\quad=\sum_{j=0}^{t} m(\alpha_j)k(0,\alpha_j)\bar{P}(\alpha_j)+\sum_{\alpha\in B-\{\alpha_0,\alpha_1,\dots,\alpha_t\}} m(\alpha)k(0,\alpha)\bar{P}(\alpha)\\
&\quad=\sum_{j=0}^{t} c_j\bar{P}(\alpha_j)-\sum_{i=0}^{u} b_i \sum_{\alpha\in B-\{\alpha_0,\alpha_1,\dots,\alpha_t\}} k(\beta_i,\alpha)\bar{P}(\alpha)\\
&\quad=\sum_{j=0}^{t} c_j\bar{P}(\alpha_j)-\sum_{i=0}^{u} b_i\bar{P}(\beta_i)=0\,.
\end{aligned}$$

This means that $m(\alpha)+\varphi(\alpha)$ is another proper unbiased estimate of $f(p(\alpha^\varepsilon),\ q(\alpha^\varepsilon))$, which contradicts the uniqueness of $\varphi(\alpha)$.

(2°) Sufficiency.

Suppose that R is simple, and $\varphi(\alpha)$, $\varphi'(\alpha)$ are two not identically equal proper unbiased estimates of $f(p(\alpha^\varepsilon),\ q(\alpha^\varepsilon))$. Then

$$m(\alpha)=\varphi(\alpha)-\varphi'(\alpha)$$

is not identically zero, and

$$\sum_{\alpha\in B} m(\alpha)k(0,\alpha)\bar{P}(\alpha)=0\,. \tag{9}$$

Since R is simple, we can choose the boundary point α_0 without loss of generality such that

(a) $m(\alpha_0)=0$, $\alpha_0=(x_0,y_0)$, and $I(\alpha_0)=n_0$,

(b) $m(\alpha)=0$ for any α in B that is $I(\alpha)<n_0$,

(c) if $I(\alpha)=n_0$, $m(\alpha)\neq 0$, and $\alpha=(x,y)$ is in B, then $y>y_0$,

(d) if $I(\alpha)=n_0$, $\alpha\in\bar{R}$, $y>y_0$.

That is, α_0 is the boundary point of lowest index and lowest y-coordinate such as $m(\alpha_0)=0$.

Now, from (9), we have

$$\begin{aligned}
&\sum_{\alpha\in B} m(\alpha)k(0,\alpha)\bar{P}(\alpha)\\
&\quad=\sum_{\alpha\in B-\{\alpha_0\}} m(\alpha)k(0,\alpha)\bar{P}(\alpha)+m(\alpha_0)k(0,\alpha_0)\bar{P}(\alpha_0)=0\,,
\end{aligned}$$

which leads us to

$$|m(\alpha_0)k(0,\alpha_0)|=\Big|\sum_{\alpha\in B-\{\alpha_0\}} m(\alpha)k(0,\alpha)\bar{P}(\alpha)/\bar{P}(\alpha_0)\Big| \tag{10}$$

for any value of $p(\alpha^\varepsilon)$.

Here we consider the value of $\bar{P}(\alpha)/\bar{P}(\alpha_0)$. If there exists a point α_1 in B such as

$$I(\alpha_1) = n_0, \ m(\alpha_1) \neq 0, \ \alpha_1 = (x_1, y_1),$$

then, by the simplicity of R, we have

$$(x_1, y_0) \in \bar{R},$$

and all the points $\alpha = (x, y)$ such as $x_1 < x < x_0$, $y_0 = y$ or $x = x_0$, $y_0 < y < y_1$ are accessible points. Thus, if we define the paths

$$\phi = \phi(0, (x_1, y_0)), \ \phi_1 = \phi_1((x_1, y_0), \alpha_1),$$
$$\phi_0 = \phi_0((x_1, y_0), \alpha_0),$$

then the equality

$$\bar{P}(\alpha_1)/\bar{P}(\alpha_0) = \prod_{\phi\,\phi_1} p(\alpha^\varepsilon)q(\alpha^\varepsilon) / \prod_{\phi\cdot\phi_0} p(\alpha^\varepsilon)q(\alpha^\varepsilon) = \prod_{\phi_1} p(\alpha^1) / \prod_{\phi_0} q(\alpha^0) \tag{11}$$

holds.

For any point α in B such as $m(\alpha) \neq 0$ and $I(\alpha) \neq n_0$, the inequality $I(\alpha) > n_0$ holds. Then there exists a point α_1 in $\bar{R}$ such as $I(\alpha_1) = n_0$, $\alpha_1 = (x_1, y_1)$ and

$$\phi(0, \alpha) = \phi'(0, \alpha_1) \cdot \phi''(\alpha_1, \alpha).$$

From the simplicity of R, (x_1, y_0) is in $\bar{R}$, and if we define the paths

$$\phi_1 = \phi_1((x_1, y_0), \alpha), \ \phi_0 = \phi_0((x_1, y_0), \alpha_0),$$

then the equality

$$\bar{P}(\alpha)/\bar{P}(\alpha_0) = \prod_{\phi_1} p(\alpha^\varepsilon)q(\alpha^\varepsilon) / \prod_{\phi_0} q(\alpha^0) \tag{12}$$

holds. By the condition (d) stated above, $\prod_{\phi_1} p(\alpha^\varepsilon)q(\alpha^\varepsilon)$ contains at least one $p(\alpha^1)$.

From the condition (i), (iii) and the equalities (11), (12), it is clear that the equality (10) can not hold. Thus, $\varphi(\alpha)$ must be unique.

This completes the proof of our theorem.

In the case of binomial sampling the conditions of Theorem 5 are satisfied, and also in the case of hypergeometric sampling they are all fulfilled. In the latter case the probability at the point $\alpha = (x, y)$ of index n is as follows:

$$p(\alpha^1) = \frac{N}{N-n}\left(p - \frac{y}{N}\right), \ p(\alpha^0) = 1, \quad 0 \leqq p \leqq 1,$$
$$q(\alpha^0) = \frac{N}{N-n}\left(q - \frac{x}{N}\right), \ q(\alpha^1) = 1, \quad p + q = 1,$$
$$p(\alpha^1) + q(\alpha^0) = 1.$$

Since

$$\bar{P}(\alpha) = \prod_{\phi} p(\alpha^{\varepsilon})q(\alpha^{\varepsilon})$$
$$= \frac{N^n}{N(N-1)\dots(N-n+1)} \prod_{i=0}^{y-1}\left(p-\frac{i}{N}\right)\prod_{j=0}^{x-1}\left(q-\frac{j}{N}\right),$$

the condition of Theorem 4 is satisfied.

The condition (iv) of Theorem 5 is satisfied as follows:

$$\sum_{m=y_0}^{y_t} \frac{N^n(-1)^m}{N(N-1)\dots(N-n+1)} \prod_{i=0}^{m-1}\left(p-\frac{i}{N}\right)\prod_{j=0}^{n-m-1}\left(q-\frac{j}{N}\right)$$
$$= \sum_{m=y_0}^{y_t}\left[\frac{N^n(-1)^m}{N(N-1)\dots(N-n+1)} \prod_{i=0}^{m-1}\left(p-\frac{i}{N}\right)\prod_{j=0}^{n-m-1}\left(q-\frac{j}{N}\right)\right]$$
$$\left[\frac{N}{N-n}\left(p-\frac{m}{N}\right)+\frac{N}{N-n}\left(q-\frac{n-m}{N}\right)\right]$$
$$= \sum_{m=y_0+1}^{y_t+1} \frac{N^{n+1}(-1)^{m-1}}{N(N-1)\dots(N-n)} \prod_{i=0}^{m-1}\left(p-\frac{i}{N}\right)\prod_{j=0}^{n-m}\left(q-\frac{j}{N}\right)$$
$$+ \sum_{m=y_0}^{y_t} \frac{N^{n+1}(-1)^{m}}{N(N-1)\dots(N-n)} \prod_{i=0}^{m-1}\left(p-\frac{i}{N}\right)\prod_{j=0}^{n-m}\left(q-\frac{j}{N}\right)$$
$$= \frac{N^{n+1}(-1)^{y_t}}{N(N-1)\dots(N-n)} \prod_{i=0}^{y_t}\left(p-\frac{i}{N}\right)\prod_{j=0}^{n-y_t-1}\left(q-\frac{j}{N}\right)$$
$$+ \frac{N^{n+1}(-1)^{y_0}}{N(N-1)\dots(N-n)} \prod_{i=0}^{y_0-1}\left(p-\frac{i}{N}\right)\prod_{j=0}^{n-y_0}\left(q-\frac{j}{N}\right),$$

thus, if we put

$$\alpha_j = (x_j,\ y_j),\ I(\alpha_j) = n,\ c_j = (-1)^{y_j},\quad (j = 0, 1, 2, \dots, t),$$
$$u = 1,\ b_0 = (-1)^{y_t},\ b_1 = (-1)^{y_0},$$
$$\beta_0 = (x_t,\ y_t+1),\ \beta_1 = (x_0+1,\ y_0),$$

then the equality (8) is satisfied.

Now we construct the unbiased estimates of p and pq in the following example.

Unbiased estimates in multiple sampling inspection are obtained by the same principle.

EXAMPLE. *Unbiased estimates in the single hypergeometric sampling inspection*

In this case we have

$$R = \{\alpha = (x,\ y)\,|\,x+y < n\}$$
$$B = \{\alpha = (x,\ y)\,|\,x+y = n\}\ .$$

Clearly R is simple and $R=\bar{R}$.

Unbiased estimate of p is $\varphi(\alpha)=\dfrac{y}{n}$, since $\tau=(0,1)$, $k(0,\tau)=1$ and

$$\sum_{\phi'\in K(0,\tau)}\prod_{\phi'} p(\alpha^{\varepsilon})q(\alpha^{\varepsilon})=p\,.$$

Unbiased estimate of pq is $\varphi(\alpha)=\dfrac{N-1}{N}\dfrac{y(n-y)}{n(n-1)}$, since $\tau=(1,1)$, $k(0,\tau)=2$ and

$$\sum_{\phi'\in K(0,\tau)}\prod_{\phi'} p(\alpha^{\varepsilon})q(\alpha^{\varepsilon})=2\frac{N}{N-1}pq\,.$$

For large N, these estimates coincide with those in binomial sampling.

2. Some properties of operating characteristic curve

In this section we remark two properties of the operating characteristic curve. Property 1 is mentioned by A. Kolmogorov [1], which is useful together with Property 2 when we determine the detailed sampling inspection plan specifying the lot tolerance fraction-defective.

First we define our inspection plan and operating characteristic curve.

Definition 1. We use the single sampling inspection plan (N, n, c, d), where N, n, c, and d are respectively size of lot, size of sample, acceptance number in inspection, and rejection number in inspection. Here $d=c+1$, and $N>n>c\geqq 0$.

Definition 2. Conditional probability that the number of defectives inspected is m when the fraction-defective in lot is q, is denoted by $p_m(q)$,

$$p_m(q)=\binom{N-qN}{n-m}\binom{qN}{m}\Big/\binom{N}{n}$$
$$=\begin{cases}\dfrac{n!}{m!(n-m)!}\dfrac{(N-n)!N^n}{N!}\displaystyle\prod_{i=0}^{m-1}\left(q-\frac{i}{N}\right)\prod_{j=0}^{n-m-1}\left(1-q-\frac{j}{N}\right), \\ \qquad\qquad \left(\dfrac{m}{N}\leqq q\leqq 1-\dfrac{n}{N}+\dfrac{m}{N}\right), \\ 0, \qquad\qquad (\text{otherwise}).\end{cases}$$

For the inspection plan (N, n, c, d), the function of q:

$$L(q)=\sum_{m\leqq c}p_m(q)$$

is called the operating characteristic function.

PROPERTY 1. *$L(q)$ is monotone decreasing, and if we put* $q' = q + \frac{1}{N}$, *then the function*

$$F(q) = L(q) - L(q')$$

is uni-modal, and takes its maximum value at the points

$$q = \frac{c}{n-1} - \frac{1}{N} \quad \text{and} \quad q = \frac{c}{n-1}.$$

This means that $L(q)$ decreases most rapidly in the neighborhood of the point $q_0 = \frac{c}{n-1}$.

PROPERTY 2. *If,* $q_0 = \frac{c_1}{n_1 - 1} = \frac{c_2}{n_2 - 1}$ *and* $c_1 < c_2$, *then*

$$F_1(q_0) < F_2(q_0)$$

holds, where $F_1(q)$ and $F_2(q)$ correspond to (N, n_1, c_1, d_1) and (N, n_2, c_2, d_2) respectively. That is, when the sample size n increases under the condition that $\frac{c}{n-1}$ is constant, the operating characteristic curve decreases more rapidly in the neighborhood of $q_0 = \frac{c}{n-1}$.

3. Estimation of the quality of a group of lots by the single sampling inspection

Suppose there are s lots of size N each, whose fraction-defective are

$$q_1, q_2, \cdots, q_s,$$

and we adopt the single sampling inspection (N, n, c, d) for each lot, where the unit is destroyed by the inspection.

Let the number of defectives before inspection be

$$y_1, y_2, \cdots, y_s,$$

at every lot, and the number of defectives in inspection be

$$x_1, x_2, \ldots, x_s,$$

respectively.

Total number of units contained in the group of s lots is

$$R = sN,$$

and the number of defectives in the group is

$$Y=\sum_{r=1}^{s} y_r .$$

Then the quality of the group before inspection is of fraction-defective

$$q_{cp}=\frac{Y}{R}=\frac{1}{s}\sum_{r=1}^{s} q_r . \tag{1}$$

If s' lots are accepted through the sampling inspection adopted, then the total number of defectives accepted is

$$Y'=\sum_{r=1}^{s}(y_r-x_r)z_r ,$$

where z_r are the chance variables defined by

$$z_r=\begin{cases}1 & \text{if } x_r \leqq c,\\ 0 & \text{if } x_r \geqq d, \qquad (r=1,2,\dots,s).\end{cases}$$

The ratio of Y' to R, i. e.,

$$q^*_{cp}=\frac{Y'}{R}=\frac{1}{s}\sum_{r=1}^{s}\left(q_r-\frac{x_r}{N}\right)z_r \tag{2}$$

must be considered when we discuss the decrease of the defectives in the group by the inspection.

Now, if we put

$$R'=s'(N-n) ,$$

then the fraction-defective after inspection in the group is

$$q'_{cp}=\frac{Y'}{R'}=\frac{R}{R'}q^*_{cp} . \tag{3}$$

In the following, we construct the estimates of q_{cp}, q^*_{cp} and q'_{cp}. In general, we cannot evaluate the value of q'_{cp} by the value of q^*_{cp}, since $\frac{R'}{R}$ is a chance variable depending on the state of control in the production and on the sampling inspection plan adopted.

Estimates of q_{cp} in the binomial approximation case and of q^*_{cp} in Poisson approximation case were already obtained by A. Kolmogorov [1], but they are stated here again together with the estimates in other cases.

I. ESTIMATION OF q_{cp}

Ia. *Binomial approximation case*

When $N \gg n$, the approximation

$$p_m(q) \sim \binom{n}{m} q^n (1-q)^{n-m}$$

is applicable, and we call this case the *binomial approximation case.*

Unbiased estimate of q is $\varphi(x) = \frac{x}{n}$, and from the independence of $x_1, x_2, \dots, x_s$, we obtain the estimation

$$q_{cp} = \frac{1}{s} \sum_{r=1}^{s} q_r \approx \varphi_{cp} = \frac{1}{s} \sum_{r=1}^{s} \varphi(x_r) , \tag{4}$$

where the R. H. S. of $\approx$ is an estimate of the L. H. S..

The variance of $\varphi(x)$ is

$$\begin{aligned} D^2\varphi(x) &= \sum_{m=0}^{n} \left(\frac{m}{n} - q\right)^2 p_m(q) \\ &= \frac{q(1-q)}{n} , \end{aligned}$$

and from the result obtained by M. A. Girshick, F. Mosteller and L. J. Savage [2], the unbiased estimate of this variance is

$$\psi^2(x) = \frac{x(n-x)}{n^2(n-1)} ,$$

hence we have

$$D^2\varphi_{cp} = \frac{1}{s^2} \sum_{r=1}^{s} D^2\varphi(x_r) \approx \Delta^2 = \frac{1}{s^2} \sum_{r=1}^{s} \psi^2(x_r) . \tag{5}$$

If we can expect a positive a-priori probability that q_r falls in the interval $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$, then the Liapounov's condition is fulfilled, and for sufficiently large s,

$$P\left\{ \left| \frac{q_{cp} - \varphi_{cp}}{\Delta} \right| \leq t \right\} \sim \frac{2}{\sqrt{2\pi}} \int_0^t e^{-\frac{t^2}{2}} dt \tag{6}$$

holds. This shows us the precision of the estimate φ_{cp}.

Ib. *Hypergeometric sampling case*

When binomial approximation is not applicable, the unbiased estimate of q_{cp} is as follows. From the result obtained in Section 1, unbiased estimate of q is $\varphi(x) = \frac{x}{n}$, and

$$q_{cp} = \frac{1}{s} \sum_{r=1}^{s} q_r \approx \varphi_{cp} = \frac{1}{s} \sum_{r=1}^{s} \varphi(x_r) . \tag{7}$$

The variance of the estimate $\varphi(x)$ is

$$D^2\varphi(x) = \sum_{m=0}^{n}\left(\frac{m}{n} - q\right)^2 p_m(q)$$
$$= \frac{N-n}{N-1}\frac{q(1-q)}{n},$$

and by the result obtained in Section 1, the unbiased estimate of this variance is

$$\psi^2(x) = \frac{N-n}{N}\frac{x(n-x)}{n^2(n-1)}.$$

Thus, the unbiased estimate of $D^2\varphi_{cp}$ is

$$(8) \qquad D^2\varphi_{cp} = \frac{1}{s^2}\sum_{r=1}^{s} D^2\varphi(x_r) \approx \Delta^2 = \frac{1}{s^2}\sum_{r=1}^{s}\psi^2(x_r).$$

Since Liapounov's theorem holds in this case too, we have

$$(9) \qquad P\left\{\left|\frac{q_{cp}-\varphi_{cp}}{\Delta}\right| \leq t\right\} \sim \frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t2}{2}}\,dt$$

for sufficiently large s.

II. Estimation of q^*_{cp}

If we put

$$y^* = (y-x)z,\ z = 1 \text{ if } x \leqq c, = 0 \text{ if } x \geqq d,$$
$$q^* = \frac{y^*}{N},$$

then

$$q^*_{cp} = \frac{1}{s}\sum_{r=1}^{s} q^*_r,$$

and the expectation of q^*,

$$Q(q) = \sum_{m=0}^{n}\left(q - \frac{m}{N}\right)z p_m(q)$$
$$= \sum_{m=0}^{c}\left(q - \frac{m}{N}\right)p_m(q)$$

is a polynomial of degree $n+1$.

Thus, in general we can not get the unbiased estimate of q^* from the sample of size n, but when Poisson approximation is possible, we can obtain the unbiased estimate of the approximated value for q^*.

IIa. *Poisson approximation case*

When $N \gg n \gg c$, and q is sufficiently small, the approximation

$$p_m(q) \sim \frac{n^m}{m!} q^m e^{-nq}$$

is applicable, and we call this case *Poisson approximation case.*

In this case the approximation

$$q^* \sim \begin{cases} q & (x \leqq c), \\ 0 & (x \geqq d) \end{cases}$$

is possible, so that we have

$$\begin{aligned} Q(q) &= \sum_{m=0}^{c} q p_m(q) \\ &= \sum_{m=0}^{d} \frac{m}{n} p_m(q). \end{aligned}$$

Thus the unbiased estimate of q^* is

$$\varphi^*(x) = \begin{cases} \dfrac{x}{n} & (x \leqq d), \\ 0 & (x \geqq d+1), \end{cases}$$

and hence we obtain the unbiased estimate of q^*_{cp} as follows:

$$q^*_{cp} = \frac{1}{s}\sum_{r=1}^{s} q^*_r \approx \varphi^*_{cp} = \frac{1}{s}\sum_{r=1}^{s} \varphi^*(x_r). \tag{10}$$

By the equality

$$q p_m(q) = \frac{m+1}{n} p_{m+1}(q),$$

the variance of $\varphi^*(x)$ is calculated as follows: since

$$q^* - \varphi^*(x) = \begin{cases} q - \dfrac{x}{n} & (x \leqq c), \\ -\dfrac{d}{n} & (x = d), \\ 0 & (x \geqq d+1), \end{cases}$$

we have

$$\begin{aligned} D^2\varphi^*(x) &= E(q^* - \varphi^*(x))^2 \\ &= \sum_{m=0}^{c}\left(q - \frac{m}{n}\right)^2 p_m(q) + \left(\frac{d}{n}\right)^2 p_d(q) \\ &= \frac{1}{n^2}\left\{\sum_{m=1}^{d} m p_m(q) + d(d+1) p_{d+1}(q)\right\}. \end{aligned}$$

The unbiased estimate of this variance is given by

$$\psi_*^2(x) = \begin{cases} \dfrac{x}{n^2} & (x \leqq d) \\ \dfrac{d(d+1)}{n^2} & (x = d+1) \\ 0 & (x \geqq d+1), \end{cases}$$

and, for the variance of φ_{cp}^* it is given by

$$D^2\varphi_{cp}^* = \frac{1}{s^2}\sum_{r=1}^{s} D^2\varphi^*(x_r) \approx \Delta_*^2 = \frac{1}{s^2}\sum_{r=1}^{s} \psi_*^2(x_r)\,. \tag{11}$$

Liapounov's theorem leads us to

$$P\left\{\left|\frac{q_{cp}^* - \varphi_{cp}^*}{\Delta_*}\right| \leqq t\right\} \sim \frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t^2}{2}}\,dt \tag{12}$$

for sufficiently large s.

IIb. *The case when Poisson approximation is not applicable*

In this case the expectation of q^*,

$$Q(q) = \sum_{m=0}^{c}\left(q - \frac{m}{N}\right)p_m(q)\,,$$

is a polynomial of degree $n+1$. Now we consider the following sampling inspection plan $(N, n, c, d, n+1, n+2)$, i. e., we inspect a sample of size $n+2$, and suppose the first n units in the sample contains x defectives, the first $n+1$ units x' defectives and all the $n+2$ units x'' defectives, where if $x \leqq c$ then the lot is accepted and if $x \geqq d$ then it is rejected. We will call this inspection plan an *over-sampling inspection plan.*

Put

$$\begin{aligned} p_m(q) &= P\{x = m \mid q\}\,, \\ p'_m(q) &= P\{x' = m \mid q\}\,, \\ p''_m(q) &= P\{x'' = m \mid q\}\,, \end{aligned}$$

then we have the following lemma, whose proof is easy and omitted.

Lemma 1. *For any function of the form,*

$$f(q) = \sum_{m=0}^{n} \varphi_0(m)p_m(q) + \sum_{m=0}^{n+1} \varphi_1(m)p'_m(q) + \sum_{m=0}^{n+2} \varphi_2(m)p''_m(q)\,, \tag{13}$$

the function of x, x' and x'',

$$\varphi(x, x', x'') = \varphi_0(x) + \varphi_1(x') + \varphi_2(x'') \tag{14}$$

is the unique unbiased estimate.

In the following, we construct the estimates exactly in binomial aproximation case and in the general hypergeometric sampling case.

IIb. 1. *Binomial approximation case*

In this case we have

$$p_m(q) \sim \binom{n}{m} q^m (1-q)^{n-m}\,,$$

$$p'_m(q) \sim \binom{n+1}{m} q^m (1-q)^{n+1-m}\,,$$

$$p''_m(q) \sim \binom{n+2}{m} q^m (1-q)^{n+2-m}\,,$$

and, since

$$\begin{aligned} Q(q) &= \sum_{m=0}^{c} \left(q - \frac{m}{N}\right) p_m(q) \\ &= \sum_{m=0}^{d} \frac{m}{n+1} p'_m(q) - \sum_{m=0}^{c} \frac{m}{N} p_m(q)\,, \end{aligned}$$

the unbiased estimate of q^* is given by

$$\varphi^*(x, x') = \begin{cases} \dfrac{x'}{n+1} - \dfrac{x}{N} & (x \leqq c,\ x' \leqq d), \\ \dfrac{d}{n+1} & (x = x' = d), \\ 0 & (\text{otherwise}). \end{cases} \tag{15}$$

Thus the unbiased estimate of q^*_{cp} is

$$q^*_{cp} = \frac{1}{s} \sum_{r=1}^{s} q^*_r \approx \varphi^*_{cp} = \frac{1}{s} \sum_{r=1}^{s} \varphi^*(x_r, x_r')\,. \tag{16}$$

From the equality

$$q^* - \varphi^*(x, x') = \begin{cases} q - \dfrac{x'}{n+1} & (x \leqq c,\ x' \leqq d)\,, \\ -\dfrac{d}{n+1} & (x = x' = d)\,, \\ 0 & (\text{otherwise})\,, \end{cases}$$

we have the variance of $\varphi^*(x, x')$

$$\begin{aligned} D^2\varphi^*(x, x') &= \sum_{x,x'} (q^* - \varphi^*(x, x'))^2 P(x, x') \\ &= \sum_{m=1}^{c} \frac{m^2}{(n+1)^2} p_m(q) - \sum_{m=1}^{d} \frac{m(2nm-2n-1)}{(n+1)^3} p'_m(q) \\ &\quad + \frac{d^2(n-d+1)}{(n+1)^3} p_d'(q) + \sum_{m=2}^{d+1} \frac{(n-1)\, m(m-1)}{(n+1)^2(n+2)} p''_m(q)\,, \end{aligned}$$

so that the unbiased estimate of this variance is given by

$$\text{(17)}\quad \psi_*^2(x, x', x'') = \frac{x^2}{(n+1)^2} Z[1, c] - \frac{x'(2nx'-2n-1)}{(n+1)^3} Z'[1, d] - \frac{d^2(n-d-1)}{(n+1)^3} Z'[d] + \frac{(n-1)\,x''(x''-1)}{(n+1)^2(n+2)} Z''[2, d+1],$$

where

$$Z[I] = \begin{cases} 1 & (x \in I), \\ 0 & (x \bar{\in} I), \end{cases} \quad Z'[I] = \begin{cases} 1 & (x' \in I), \\ 0 & (x' \bar{\in} I), \end{cases} \quad Z''[I] = \begin{cases} 1 & (x'' \in I), \\ 0 & (x'' \bar{\in} I), \end{cases}$$

for any interval I. The unbiased estimate of the variance of φ_{cp}^* is

$$\text{(18)}\quad D^2\varphi_{cp}^* = \frac{1}{s^2}\sum_{r=1}^{s} D^2\varphi^*(x_r, x_r') \approx \Delta_*^2 = \frac{1}{s^2}\sum_{r=1}^{s}\psi_*^2(x_r, x_r', x_r''),$$

and Liapounov's theorem shows that, for sufficiently large s,

$$\text{(19)}\quad P\left\{\left|\frac{q_{cp}^* - \varphi_{cp}^*}{\Delta_*}\right| \leq t\right\} \sim \frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t^2}{2}}dt$$

IIb. 2. *Hypergeometric sampling case*

In this case

$$p_m(q) = \frac{(N-n)!N^n}{N!}\frac{n!}{m!(n-m)!}\prod_{i=0}^{m-1}\left(q-\frac{i}{N}\right)\prod_{j=0}^{n-m-1}\left(1-q-\frac{j}{N}\right),$$

$$p_m'(q) = \frac{(N-n-1)!N^{n+1}}{N!}\frac{(n+1)!}{m!(n+1-m)!}\prod_{i=0}^{m-1}\left(q-\frac{i}{N}\right)\prod_{j=0}^{n+1-m-1}\left(1-q-\frac{j}{N}\right),$$

$$p_m''(q) = \frac{(N-n-2)!N^{n+2}}{N!}\frac{(n+2)!}{m!(n+2-m)!}\prod_{i=0}^{m-1}\left(q-\frac{i}{N}\right)\prod_{j=0}^{n+2-m-1}\left(1-q-\frac{j}{N}\right),$$

and, since

$$Q(q) = \sum_{m=0}^{c}\left(q-\frac{m}{N}\right)p_m(q) = \frac{N-n}{N}\frac{1}{n+1}\sum_{m=1}^{d} m p_m'(q),$$

we can get the unbiased estimate of q^*

$$\text{(20)}\quad \varphi^*(x') = \begin{cases} \dfrac{N-n}{N}\dfrac{x'}{n+1} & (x' \leq d), \\ 0 & (\text{otherwise}). \end{cases}$$

Thus the unbiased estimate of q_{cp}^* is given by

$$\text{(21)}\quad q_{cp}^* = \frac{1}{s}\sum_{r=1}^{s} q_r^* \approx \varphi_{cp}^* = \frac{1}{s}\sum_{r=1}^{s}\varphi^*(x_r').$$

From the equality

$$q^* - \varphi^*(x') = \begin{cases} q - \dfrac{x}{N} - \dfrac{N-n}{N}\dfrac{x'}{n+1} & (x \leqq c,\ x' \leqq d)\,, \\ \qquad\quad - \dfrac{N-n}{N}\dfrac{d}{n+1} & (x = x' = d)\,, \\ 0 & (\text{otherwise})\,, \end{cases}$$

we have

$$D^2\varphi^*(x') = E(q^* - \varphi^*(x'))^2$$
$$= \sum_{m=1}^{c} \frac{(N-n)^2 m + (n+1)^2(m^2-m-1)}{N^2(n+1)} \frac{m}{n+1} p_m(q)$$
$$- \sum_{m=1}^{d} \frac{2n(N+1)(N-n)(m-1) - (N-n)^2 - (N+n)(n+1)m(nm+m-n-3)}{N^2(n+1)^2}$$
$$\frac{m}{n+1} p'_m(q)$$
$$+ \left(\frac{N-n}{N}\frac{d}{n+1}\right)^2 \frac{n-d+1}{n+1} p'_d(q)$$
$$+ \sum_{m=2}^{d+1} \frac{(N-n)(N-n-1)(n-1)}{N^2(n+1)} \frac{m(m-1)}{(n+1)(n+2)} p''_m(q)$$

and the unbiased estimate of this variance is given by

$$\psi_*^2(x, x', x'') = \frac{(N-n)^2 x + (n+1)^2(x^2-x-1)}{N^2(N+1)} \frac{x}{n+1} Z[1, c] \tag{22}$$
$$- \frac{(x'-1)2n(N+1)(N-n) - (N-n)^2 - (N+n)(n+1)x'(nx'+x'-n-3)}{N^2(n-1)^2}$$
$$\frac{x'}{n+1} Z'[1, d]$$
$$+ \left(\frac{N-n}{N}\frac{d}{n+1}\right)^2 \frac{n-d+1}{n+1} Z'[d]$$
$$+ \frac{(N-n)(N-n-1)(n-1)}{N^2(n+1)} \frac{x''(x''-1)}{(n+1)(n+2)} Z''[2, d+1]\,.$$

Thus the unbiased estimate of $D^2\varphi^*_{cp}$ is

$$D^2\varphi^*_{cp} = \frac{1}{s^2}\sum_{r=1}^{s} D^2\varphi^*(x'_r) \approx \Delta_*^2 = \frac{1}{s^2}\sum_{r=1}^{s} \psi_*^2(x_r, x'_r, x''_r)\,. \tag{23}$$

Liapounov's theorem leads us, for sufficiently large s, to

$$P\left\{\left|\frac{q^*_{cp} - \varphi^*_{cp}}{\Delta_*}\right| \leqq t\right\} \sim \frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t^2}{2}} dt\,. \tag{24}$$

IIb. 3. *Relation between* q^*_{cp} *and* $q^{*\prime\prime}_{cp}$ *and the estimation of* $q^{*\prime\prime}_{cp}$

In the case of over-sampling inspection we must consider $q^{*\prime\prime}_{cp}$ instead of q^*_{cp} such as

$$q^{*\prime\prime}_{cp} = \frac{1}{s}\sum_{r=1}^{s} q^{*\prime\prime}_r ,$$

where

$$q^{*\prime\prime}_r = \begin{cases} \dfrac{y_r - x''_r}{N} & (x_r \leqq c), \\ 0 & (x_r \geqq d), \qquad (r = 1, 2, \dots, s) . \end{cases}$$

Clearly

$$q^{*\prime\prime}_{cp} = q^*_{cp} - \frac{1}{s}\sum_{r=1}^{s} \frac{x''_r - x_r}{N} z_r ,$$

and thus

$$\varphi^{*\prime\prime}_{cp} = \varphi^*_{cp} - \frac{1}{s}\sum_{r=1}^{s} \frac{x''_r - x_r}{N} z_r \tag{25}$$

is the unbiased estimate of $q^{*\prime\prime}_{cp}$, where φ^*_{cp} is the unbiased estimate of q^*_{cp} given in (16) or (21). Since

$$0 \leqq q^*_{cp} - q^{*\prime\prime}_{cp} \leqq \frac{2}{N} ,$$

it is possible to regard φ^*_{cp} as an estimate of $q^{*\prime\prime}_{cp}$ when N is large.

From (19) or (24), we have for sufficiently large s,

$$P\left\{ \left| \frac{q^{*\prime\prime}_{cp} - \varphi^{*\prime\prime}_{cp}}{\Delta_*} \right| \leqq t \right\} \sim \frac{2}{\sqrt{2\pi}} \int_0^t e^{-\frac{t^2}{2}} dt , \tag{26}$$

where Δ^2_* is given in (18) or (23).

III. ESTIMATION OF q'_{cp}

When the over-sampling inspection is adopted we must consider the following quantities:

$$q''_{cp} = \frac{R}{R''} q^{*\prime\prime}_{cp} , \quad R'' = (N - n - 2)s' , \tag{27}$$

instead of the quantities:

$$q'_{cp} = \frac{R}{R'} q^*_{cp} , \quad R' = (N - n)s' .$$

It is difficult to construct the estimate of q'_{cp} directly, and so we shall obtain it from the unbiased estimate of q^*_{cp}.

First we investigate the behavior of $\dfrac{R'}{R}$.

IIIa. *Distribution of* $\frac{R'}{R}$

Since, for $r=1, 2, \dots, s$,

$$z_r = \begin{cases} 1 & (x_r \leqq c)\,, \\ 0 & (x_r \geqq d)\,, \end{cases}$$

we have

$$P\{z_r = 1 | q\} = L(q_r)\,,$$
$$P\{z_r = 0 | q\} = 1 - L(q_r)\,.$$

Then the number of lots accepted:

$$s' = \sum_{r=1}^{s} z_r$$

is a chance variable following the generalized binomial distribution of Poisson (cf., H. Cramér, Mathematical Methods of Statistics, p. 206 and p. 217), and its mean and variance are given by

$$E(s') = \sum_{r=1}^{s} L(q_r)\,,$$
$$D^2(s') = \sum_{r=1}^{s} L(q_r)(1-L(q_r))\,.$$

Now, we suppose that the a-priori distribution of q is normal $N(\tilde{q}, \tilde{\sigma}^2)$ with density $\phi(q)$. Since

$$P\left\{\frac{R'}{R} \leqq u | q_1, q_2, \dots, q_s\right\}$$
$$= \sum_{k=0}^{\left[\frac{Ns}{N-n}u\right]} \sum_{\{r_1,\dots,r_k\}} \prod_{i=1}^{k} L(q_{r_i}) \prod_{j \neq r_i} (1-L(q_j))\,,$$

we have

$$\int_0^1 \cdots \int_0^1 P\left\{\frac{R'}{R} \leqq u | q_1, q_2, \dots, q_s\right\} \phi(q_1)\phi(q_2)\dots\phi(q_s) dq_1 dq_2 \dots dq_s \tag{28}$$
$$= \sum_{k=0}^{\left[\frac{Ns}{N-n}u\right]} \binom{s}{k} \left(\int_0^1 L(q)\phi(q)dq\right)^k \left(1-\int_0^1 L(q)\phi(q)dq\right)^{s-k}.$$

That is, the distribution of $\frac{R'}{R}$ when a-priori distribution of q exists is a binomial distribution, of which the mean and variance are

$$E\left(\frac{R'}{R}\right) = \frac{N-n}{N}\int_0^1 L(q)\phi(q)dq\,, \tag{29}$$
$$D^2\left(\frac{R'}{R}\right) = \left(\frac{N-n}{N}\right)^2 \frac{1}{s}\int_0^1 L(q)\phi(q)dq\left(1-\int_0^1 L(q)\phi(q)dq\right).$$

In the binomial approximation case we get, for sufficiently small $\tilde{\sigma}$,

$$(30)\quad E\left(\frac{R'}{R}\right)\sim\frac{N-n}{N}\int_{-\infty}^{\infty}\sum_{m=0}^{c}\binom{n}{m}q^m(1-q)^{n-m}\frac{1}{\sqrt{2\pi}\tilde{\sigma}}e^{-\frac{(q-\tilde{q})^2}{2\tilde{\sigma}^2}}dq$$

$$=\frac{N-n}{N}\sum_{m=0}^{c}\binom{n}{m}\sum_{k=0}^{n-m}(-1)^{n-m-k}\binom{n-m}{k}\sum_{\substack{i=0\\ i:\text{even}}}^{n-k}\binom{n-k}{i}\left(\frac{\tilde{\sigma}}{\sqrt{2}}\right)^i\tilde{q}^{n-k-i}\frac{i!}{\left(\frac{i}{2}\right)!},$$

and in Poisson approximation case we have

$$(31)\quad E\left(\frac{R'}{R}\right)\sim\frac{N-n}{N}\int_{-\infty}^{\infty}\sum_{m=0}^{c}\frac{n^m}{m!}q^m e^{-nq}\frac{1}{\sqrt{2\pi}\tilde{\sigma}}e^{-\frac{(q-\tilde{q})^2}{2\tilde{\sigma}^2}}dq$$

$$=\frac{N-n}{N}e^{-\tilde{q}n+\frac{n^2\tilde{\sigma}^2}{2}}\sum_{m=0}^{c}\frac{n^m}{m!}\sum_{\substack{k=0\\ k:\text{even}}}^{m}\left(\frac{\tilde{\sigma}}{\sqrt{2}}\right)^k(\tilde{q}-n\tilde{\sigma}^2)^{m-k}\frac{k!}{\left(\frac{k}{2}\right)!}.$$

The values of $E\left(\frac{R'}{R}\right)$ in Poisson approximation case are given below for $\tilde{q}=0.01$, $q_0=\frac{c}{n}=0.02$.

($\tilde{\sigma}=0.001$)	$N=1000$	$N=2000$	$N=5000$	$N=8000$	$N=10000$
$n=50$	0.863959	0.886695	0.900336	0.903747	0.904883
$n=100$	0.826917	0.872857	0.900421	0.907312	0.909609
$n=150$		0.862988	0.904971	0.915467	0.918965
$n=200$		0.851002	0.907735	0.921919	0.926646
$n=250$			0.908070	0.925993	0.931967
$n=300$			0.906400	0.928096	0.935328

($\tilde{\sigma}=0.002$)	$N=1000$	$N=2000$	$N=5000$	$N=8000$	$N=10000$
$n=50$	0.862860	0.885567	0.899191	0.902597	0.903732
$n=100$	0.824450	0.870253	0.897734	0.904605	0.906895
$n=150$		0.859208	0.901007	0.911457	0.914908
$n=200$		0.846365	0.902789	0.916895	0.921597
$n=250$			0.902501	0.920314	0.926251
$n=300$			0.900363	0.921915	0.929098

($\tilde{\sigma}=0.003$)	$N=1000$	$N=2000$	$N=5000$	$N=8000$	$N=10000$
$n=50$	0.860983	0.883640	0.897235	0.900633	0.901766
$n=100$	0.820470	0.866052	0.893401	0.900238	0.902517
$n=150$		0.853244	0.894754	0.905131	0.908590
$n=200$		0.839188	0.895134	0.909121	0.913783
$n=250$			0.893878	0.911521	0.917401
$n=300$			0.899414	0.920943	0.928119

($\tilde{\sigma}=0.004$)	$N=1000$	$N=2000$	$N=5000$	$N=8000$	$N=10000$
$n=50$	0.858247	0.880832	0.894383	0.897771	0.898900
$n=100$	0.815180	0.860468	0.887640	0.894434	0.896698
$n=150$		0.845556	0.886691	0.896975	0.900403
$n=200$		0.830115	0.885456	0.899291	0.903903
$n=250$			0.883002	0.900429	0.906239
$n=300$			0.879455	0.900506	0.907523

$(\tilde{\sigma}=0.005)$	$N=1000$	$N=2000$	$N=5000$	$N=8000$	$N=10000$
$n=50$	0.828250	0.850046	0.863124	0.866393	0.867483
$n=100$	0.808968	0.853911	0.880876	0.887618	0.889865
$n=150$		0.837609	0.878358	0.888545	0.891941
$n=200$		0.820001	0.874668	0.888335	0.892890
$n=250$			0.873479	0.890719	0.896465
$n=300$			0.869709	0.890527	0.897466

If the number of lots in the group is large, it will be seen from (29) that the value of $D^2\left(\frac{R'}{R}\right)$ is considerably small

IIIb. *Estimation of* q'_{cp}

First we state two lemmas useful to construct the estimates of q'_{cp}. The proofs are easy and omitted.

Lemma 2. *If there are three events A, B, and C, such that*

$$P(A)=\alpha,\ P(B)=\beta,\ \textit{and}\ A\cdot B\subset C,$$

then it holds that

$$P(C)\geqq \max(\alpha,\beta)+\alpha+\beta-2\,. \tag{32}$$

Lemma 3. *Suppose that two sequences of chance variables* $\{\xi_s\}$, $\{\eta_s\}$ $(s=1,2,\dots)$ *satisfy the following conditions* (i) *to* (iv),

(i) $\eta_s(x_1, x_2, \dots, x_s)>0$ *for all* $(x_1, x_2, \dots, x_s)$, $(s=1,2,\dots)$,

(ii) $P\{\eta_s(x_1, x_2, \dots, x_s)\leqq\varepsilon\}\sim 1$ *for any* $\varepsilon>0$ *and sufficiently large s,*

(iii) $|\xi_s(x_1, x_2, \dots, x_s)|\leqq K$ *with probability* 1, $(s=1,2,\dots)$,

(iv) $P\left\{\left|\frac{\xi_s}{\eta_s}\right|\leqq t\right\}\sim\frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t^2}{2}}dt$ *for sufficiently large s.*

Then we have

$$\xi_s\sim 0\quad \text{(in probability)} \tag{33}$$

for sufficiently large s.

Now we consider the estimation of q'_{cp}.

IIIb. 1. *In Poisson approximation case, if a-priori distribution of q is completely specified*, then for given α we can take two positive numbers λ_1, λ_2 such that

$$P\left\{\lambda_1\leqq\frac{R'}{R}\leqq\lambda_2\right\}=\alpha\,, \tag{34}$$

$$\lambda_2-\lambda_1:\ \text{minimum}\,.$$

From (12), (34) and Lemma 2 we have, for sufficiently large s,

$$P\left\{\frac{\varphi^*_{cp}-3\Delta_*}{\lambda_2} \leqq q'_{cp} \leqq \frac{\varphi^*_{cp}+3\Delta_*}{\lambda_1}\right\} \geqq \max(\alpha, \beta)+\alpha+\beta-2,$$

where φ^*_{cp} and Δ^2_* are the unbiased estimates of q^*_{cp} and the variance of φ^*_{cp} respectively given in (10) and (11), while

$$\beta = \frac{2}{\sqrt{2\pi}}\int_0^3 e^{-\frac{t^2}{2}}\,dt \sim 0.9973\,.$$

Thus we can obtain the following confidence interval for sufficiently large s:

$$\frac{\varphi^*_{cp}-3\Delta_*}{\lambda_2} \leqq q'_{cp} \leqq \frac{\varphi^*_{cp}+3\Delta_*}{\lambda_1}\,. \tag{35}$$

IIIb. 2. *In Poisson approximation case, if a-priori distribution of* q *is unknown* and s is sufficiently large, we can get the estimate of q'_{cp} as follows. From (12), we have

$$P\left\{\left|\frac{\frac{R}{R'}q^*_{cp}-\frac{R}{R'}\varphi^*_{cp}}{\frac{R}{R'}\Delta_*}\right| \leqq t\right\} \sim \frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t^2}{2}}\,dt\,, \tag{36}$$

and Lemma 3 leads us to

$$q'_{cp} = \frac{R}{R'}\,q^*_{cp} \approx \varphi'_{cp} = \frac{R}{R'}\,\varphi^*_{cp}\,. \tag{37}$$

For sufficiently large s, we have the following confidence interval by 3σ-method,

$$\varphi'_{cp}-3\Delta' \leqq q'_{cp} \leqq \varphi'_{cp}+3\Delta', \tag{38}$$

where $\Delta' = \frac{R}{R'}\Delta_*$ for Δ^2_* given in (11).

IIIb. 3. *In the case of over-sampling inspection*, we must estimate the value of q''_{cp} given in (27). The distribution function of $\frac{R''}{R}$ is obtained from that of $\frac{R'}{R}$ replacing n by $n+2$, and we can take two positive numbers λ_1, λ_2 such that

$$P\left\{\lambda_1 \leqq \frac{R''}{R} \leqq \lambda_2\right\} = \alpha\,,$$

$$\lambda_2-\lambda_1: \text{ minimum},$$

and we obtain the confidence interval analogous to (35) : for sufficiently large s,

$$\frac{\varphi_{cp}^{*\prime\prime}-3\Delta_*}{\lambda_2} \leqq q_{cp}^{*\prime\prime} \leqq \frac{\varphi_{cp}^{*\prime\prime}+3\Delta_*}{\lambda_1}, \tag{39}$$

whose coefficient of confidence is not less than $\max(\alpha,\ 0.9973)+\alpha+0.9973-2$.

From (26) and Lemma 3, we obtain the estimate of $q_{cp}^{\prime\prime}$ analogous to (37) :

$$q_{cp}^{\prime\prime} = \frac{R}{R^{\prime\prime}} q_{cp}^{*\prime\prime} \approx \varphi_{cp}^{\prime\prime} = \frac{R}{R^{\prime\prime}} \varphi_{cp}^{*\prime\prime}, \tag{40}$$

where $\varphi_{cp}^{*\prime\prime}$ is given in (25). From (26), we have the following confidence interval by 3σ–method,

$$\varphi_{cp}^{\prime\prime}-3\Delta^{\prime\prime} \leqq q_{cp}^{\prime\prime} \leqq \varphi_{cp}^{\prime\prime}+3\Delta^{\prime\prime},$$

where $\Delta^{\prime\prime} = \frac{R}{R^{\prime\prime}}\Delta_*$ for Δ_*^2 given in (18) or (23) .

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