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On the Pseudo-Harmonic Functions

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Introduction. Let F be an orientable surface. Let $u(p)$ be a real-valued function in a neighborhood N_{p_0} of p_0 on F where N_{p_0} corresponds to the unit circular disc in the complex plane by the topological mapping $z=T_{p_0}(p)$, $z=x+iy$.

Set
$$u(p)=u(T_{p_0}(p))=U(z)\,.$$

Then $u(p)$ is termed *pseudo-harmonic* at p_0, if $U(z)$ is harmonic and not identically constant in $|z|<1$. A real-valued function on F is termed *pseudo-harmonic* if it is pseudo-harmonic on each point of F. In this paper we will prove that there exist the local parameters such that F is a Riemann surface with respect to them and $u(p)$ is harmonic on F.

1. Terminologies and notations.

Let $u(p)$ be a pseudo-harmonic function on F. By the *level-curve* of $u(p)$ with the *height* c, we mean the locus of the equation $u(p)=c$. It is well known that with each point $p_0 \in F$, there exists a suitably chosen neighborhood N_{p_0} of p_0 and a topological mapping $z=T_{p_0}(p)$ of N_{p_0} onto $|z|<|1$ under which p_0 goes into $z=0$ and the level-curves of $u(p)$ in N_{p_0} go into the level-curves of $Re\, z^n$ in $|z|<1$[1]. we shall term this N_{p_0} a *canonical neighborhood* of p_0. When $n=1$, we shall call p_0 a *regular point* and N_{p_0} a *simple canonical neighborhood*. When $n \geqq 2$, we shall call p_0 a *saddle-point* of *order* n. A real-valued function $v(p)$ on F is called "*pseudo-conjugate to a pseudo-harmonic function* $u(p)$", if it satisfies the following condition.

There exists a topological mapping $z=T_{p_0}(p)$ by which N_{p_0} corresponds to $|z|<1$, and $U(z)=u(T_{p_0}(p))$ is conjugate-harmonic to $V(z)=v(T_{p_0}(p))$ in $|z|<1$.

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1) Y. Tôki, A topological characterization of pseudo-harmonic functions, Osaka Mathematical J. **3** (1951), 101–122. See also J. Jenkins and M. Morse, Topological methods on Riemann surface, pseudoharmonic function. Contributions to the theory of Riemann surfaces 1953 p. 114.

2. The triangulation of a surface.

Let F be an orientable surface and $u(p)$ be a pseudo-harmonic function on it. In the first place, we can easily triangulate the surface F such that each saddle-point of $u(p)$ is a vertex of a triangle and each triangle of F is contained in a canonical neighborhood, especially any triangle without the saddle-points is contained in a simple canonical neighborhood. We shall prove the following lemmas on this triangulation.

Lemma 1. *We can triangulate the surface F such that each side of any triangle of F intersects every one of the level-curves of $u(p)$ at most at the finite number of points.*

Proof. Let Δ be any triangle on F and a, b, c, be the three vertices of it. Let L_i $(i=1, 2 \cdots n)$ and M_j $(j=1, 2 \cdots m)$ be the sides of the triangles with the common vertex a and b respectively: especially L_1 denotes the arc $\widehat{ab}$, M_1 denotes the arc $\widehat{ba}$. There exists a canonical neighborhood N_Δ $(N_\Delta \supset \Delta)$ and a topological mapping $z=T_\Delta(p)$ under which Δ is mapped onto a curvilinear triangle Δ' in $|z|<1$. Let the points a', b', c', be the three vertices of Δ' and L_i' $(i=1, 2 \cdots n)$ and M_j' $(j=1, 2 \cdots m)$ the mapped images of the arc L_i $(i=1, 2 \cdots n)$ and M_j $(j=1, \cdots m)$ in N_Δ. Let $C_{a'}$ and $C_{b'}$ be the sufficiently small circles with the center a', b' and contained in $|z|<1$ respectively. Let a_i' $(i=1, 2, \cdots n)$ be the points at which the arc L_i' cut the circle $C_{a'}$ for the last time. We can choose the points b_j' $(j=1, 2 \cdots m)$ on $C_{b'}$ similarly. We can connect a_1' and b_1' by a polygon without intersecting L_i' $(i=1, 2 \cdots n)$ and M_j' $(j=1, 2 \cdots m)$ out side of the circles

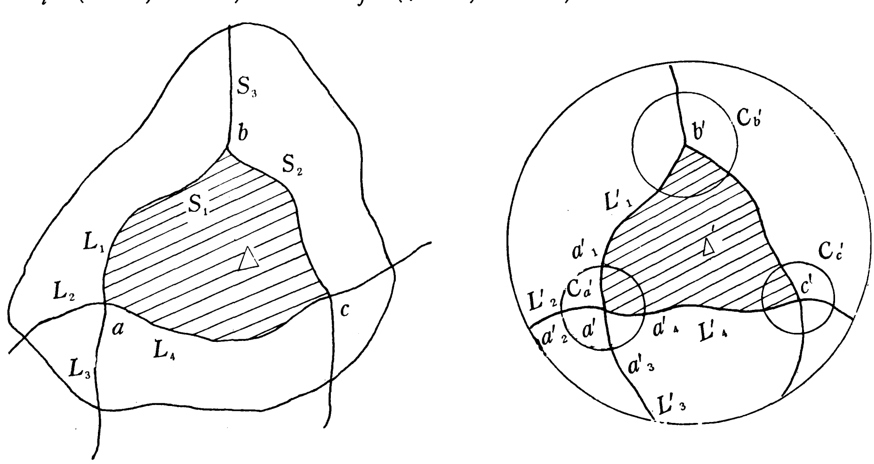


Fig. 2.

$C_{a'}$ and $C_{b'}$. We also connect a_j' and a' by the radius in the circle $C_{a'}$. We connect b_j' and b' similarly. We repeat this deformation with respect to every side of the traingles on F. In this repetition, each side of the triangles are varied in finite times: for instance, side $\widehat{ab}$ varies in $(m+n-1)$–times. When some part of a side of a triangle lies on a level-curve, then we can deform slightly it such that each one of sides of the deformed triangle cut the level-curves at most once.

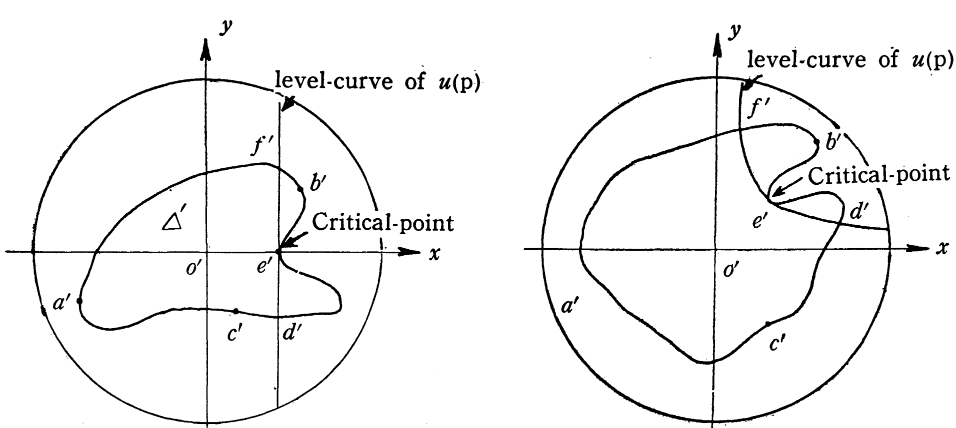
Therefore we have after a finite number of time the desired triangulation. A point p on the sides of a triangles is termed a critical point when the side through the point p is on one side of the level-curve $u(p)$ except to the point p in the neighborhood of p from now.

Lemma 2. *We can triangulate the surface F such that each side of any triangle of F intersects every one of the level-curves of $u(p)$ at most at one point.*

Proof. Let Δ be any triangle of F such that each side of it intersects every one of the level-curves of $u(p)$ at most at a finite number of points. When Δ have ciritical points or saddle-points on its boundary. Let us subdivide Δ into triangles and polygonal domains by the level-curves through the critical points and the saddle-point.

Let one of these polygonal domains be $\sum$. The polygonal domain $\sum$ can be mapped onto a rectangle $\sum^*$ by the topological mapping $z=S_{\Sigma}(p)$ under which the level-lines in $\sum$ go into the lines parallel to the y–axies and the vertices of $\sum$ go into points on the boundary of $\sum^*$.

The polygonal domain $\sum^*$ can be subdivided into triangles by lines connecting the center of $\sum^*$ to the vertices. Let us subdivide $\sum$ into triangles which are the inverse images of the triangles of $\sum^*$.



simple-canonical neighborhood. Fig. 1. Saddle-point of order 2.

Subdivide each polygonal domain of F into triangles similarly. We can easily deform the above triangulation slightly such that each side of the triangles intersect the level-lines at most once.

Theorem. *Let $u(p)$ be pseudo-harmonic on F. We can associate the local parameters of F such that F is a Riemann surface with respect to them and $u(p)$ is harmonic on it.*

Proof. By the lemma 2, we can subdivide the surface F such that each side of any triangle of F intersects every one of the level-curves of $u(p)$ at most at one point. Therefore each triangle of $\{\Delta\}$ can be mapped onto the rectilinear one in the z-plane and at the same time the level-curves of $u(p)$ can be mapped onto the lines parallel to the y-axis.

Let these transformations be $z=\tau_{\Delta}(p)$. It is clear that the function $u(\tau_{\Delta}^{-1}(z))$ is harmonic. Let p_0 be any point on F and Δ_{p_0} be a triangle such that $\Delta_{p_0}\ni p_0$. The following three cases will arise:

(i) p_0 is contained in Δ_{p_0}.

(ii) p_0 lies on one of the sides of Δ_{p_0}.

(iii) p_0 is a vertex of Δ_{p_0}.

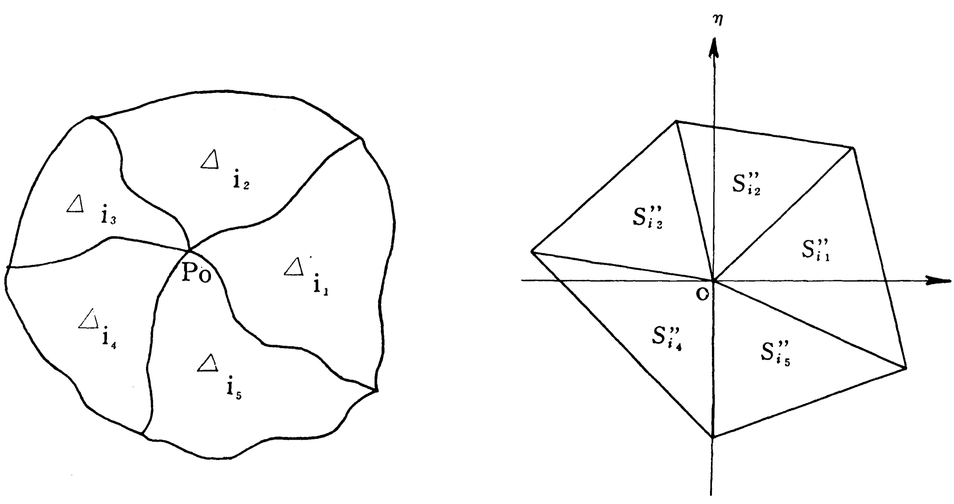
We can associate the local parameters as follows, corresponding to the above three cases.

(i) We associate the function $z=\tau_{\Delta_{P_0}}(p)$ as a local parameter to p_0.

(ii) There exists the two neighboring triangles Δ_j and Δ_k such that the point p_0 is contained in the common side of Δ_j and Δ_k. We can transform Δ_j and Δ_k onto the rectilinear ones S_j and S_k by the transformation $z=\tau_{\Delta_j}(p)$ and $z=\tau_{\Delta_k}(p)$ respectively. We can also map S_j and S_k onto the triangles R_j and R_k lying on the upper and the lower half-plane with common side of the interval $0\leqq x\leqq 1$ by two linear transformations resepctively. Any point on the common side of Δ_j and Δ_k is mapped on the different points on the side of S_j and S_k respectively. Since these two points lie on the same level-curve parallel to the x-axis, it is clear that these are mapped on the same point on the interval $0\leqq x\leqq 1$ by the two linear transformations. Thus we can map the curvilinear quadrilateral $\Delta_j\cup\Delta_k$ onto the rectilinear quadrilateral $R_j\cup R_k$ topologically and the common side of Δ_j and Δ_k can be mapped onto the interval $0\leqq x\leqq 1$. Let this transformation be $z=\tau_{\Delta_j,\Delta_k}(p)$. We associate this function to p_0 as a local parameter of p_0.

(iii) Let Δ_{i_1}, Δ_{i_2}, $\cdots$ Δ_{i_n} be the triangles with the common vertex p_0. Each Δ_{i_k} $(k=1, 2\cdots n)$ is mapped onto a rectilinear one S_{i_k} $(k=1, 2, \cdots, n)$ and p_0 goes into z_{i_k} by the transformation $z=T_{\Delta_{i_k}}(p)$. Let the vertical angle of z_{i_k} of S_{i_k} be α_{i_k}. The triangle S_{i_k} is mapped onto S'_{i_k}

and z_{i_k} goes into w_{i_k} by the transformation $w = z^{2\pi/(\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_n})}$. Let the vertical angle of w_{i_k} of S'_{i_k} be β_{i_k}. Then $\sum_{k=1}^{n}\beta_{i_k} = 2\pi$. Accordingly, we can map S'_{i_k} and $S'_{i_{k+1}}$ onto S''_{i_k} and $S''_{i_{k+1}}$ by linear transformations respectively such that w_{i_k} and $w_{i_{k+1}}$ go into $\zeta=0$ and the common side of the two neighboring triangles Δ_{i_k} and $\Delta_{i_{k+1}}$ goes into the common side of S''_{i_k} and $S''_{i_{k+1}}$. Thus the polygonal domain composed of Δ_{i_k} $(k=1, 2 \cdots n)$ is mapped onto the polygonal domain consisting of S''_{i_k} $(k=1, 2 \cdots n)$ in the ζ-plane. Let this mapping be $\zeta = \tau_{\Delta_{i_1}, \Delta_{i_2} \cdots \Delta_{i_n}}(p)$.



the surface F. ζ-plane $(\zeta=\xi+i\eta)$

Fig. 3.

We associatet he function $\zeta = \tau_{\Delta_{i_1}, \ldots \Delta_{i_n}}(p)$ to p_0 as a local parameter. These local parameters $\tau_{\Delta}(p)$, $\tau_{\Delta_i, \Delta_i}(p)$ and $\tau_{\Delta_{i_1}, \ldots \Delta_{i_n}}(p)$ satisfy the conformal neighboring relation and $u(p)$ is harmonic on F with respect to them.

Corollary. *Let $u(p)$ be a pseudo-harmonic function on F. Then there exists always a conjugate pseudo-harmonic function to $u(p)$ on F.*

Proof. We can assume that the function $u(p)$ is harmonic on F with respect to the suitably chosen local parameters by the theorem. Then there exists always a conjugate harmonic function to $u(p)$ on F. The corollary follows at once. This conjugate pseudo-harmonic function $v(p)$ is multiple-valued on F in general.

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