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On the Homogeneous Linear Partial Differential Equation of the First Order

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§ 1. Introduction

In this paper, we shall treat the following partial differential equation

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_{\mu}(x, y_1, \cdots, y_n)\frac{\partial z}{\partial y_{\mu}}=0 \qquad (n \geqq 1)$$

without the usual condition of the total differentiability on the solution $z(x, y_1, \cdots, y_n)$.

For early contributions by R. Baire and P. Montel to this problem in the special case $n=1$, cf. Baire [1], Montel [6]. Our method is entirely different from theirs and gives more general results even for the case $n=1$, cf. Kasuga [4]. Also notwithstanding Baire's statement[1] in his paper, it seems to us that their methods cannot be generalized to the case $n>1$ immediately.

We have not yet succeeded in treating the more general non-homogeneous partial differential equation

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_{\mu}(x, y_1, \cdots, y_n, z)\frac{\partial z}{\partial y_{\mu}}=g(x, y_1, \cdots, y_n, z)$$

in a similar way, except for the case $n=1$. For this case, cf. Kasuga [5].

1. In this paper, we shall use for points in R^n, R^{n+1} or R^{n+2} and for their functions, abbreviations such as:

$$y=(y_1, \cdots, y_n), \qquad (x; y)=(x, y_1, \cdots, y_n),$$
$$\eta=(\eta_1, \cdots, \eta_n), \qquad (\xi; \eta)=(\xi, \eta_1, \cdots, \eta_n),$$
$$(x, \xi; \eta)=(x, \xi, \eta_1, \cdots, \eta_n), \qquad z(x; y)=z(x, y_1, \cdots, y_n),$$

and if $\varphi_{\lambda}(x, \xi; \eta)=\varphi_{\lambda}(x, \xi, \eta_1, \cdots, \eta_n)$ $\lambda=1, \cdots, n$ are n functions of $(x, \xi; \eta)$,

1) Cf. Baire [1], p. 120.

$$\varphi(x, \xi; \eta) = (\varphi_1(x, \xi; \eta), \cdots, \varphi_n(x, \xi; \eta)),$$
$$z(x; \varphi(x, \xi; \eta)) = z(x, \varphi_1(x, \xi; \eta), \cdots, \varphi_n(x, \xi; \eta)).$$

Also we use the following notations:
For sets of points A, B in R^m $(m=1, 2, \cdots, n+1)$,

$\bar{A}$ = closure in R^m of A, A^0 = interior in R^m of A,
A^b = boundary in R^m of A, $A \cdot B$ = intersection of A and B,
$A[x]$ = the set of the points $(y_1, \cdots, y_n)$ in R^n such that for a fixed x $(x, y_1, \cdots, y_n) \in A$, if $A \subset R^{n+1}$.

For two points $y' = (y_1', \cdots, y_n')$, $y'' = (y_1'', \cdots, y_n'')$ in R^n,

$$\| y' - y'' \| = \sum_{\mu=1}^{n} |y_\mu' - y_\mu''|, \quad y' + y'' = (y_1' + y_1'', \cdots, y_n' + y_n'').$$

In this paper, the so-called degenerated intervals are also included, when we use the word "interval" (open, closed, or half-open). Thus the interval $a < x < a$ or the interval $a \leqq x \leqq a$ will mean degenerated interval which is empty or is composed of only one point respectively. Similarly for the interval $a \leqq x < a$ or $a < x \leqq a$.

2. In the following, we shall denote by G a fixed open set in R^{n+1}, by $f_\lambda(x; y)$ $\lambda = 1, \cdots, n$ n fixed continuous functions defined on G which have continuous $\partial f_\lambda / \partial y_\mu$ $\lambda, \mu = 1, \cdots, n$.

Under the above conditions, we shall consider the partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = 0. \tag{2. 1}$$

With (2. 1), we shall associate the simultaneous ordinary differential equations

$$\frac{dy_\lambda}{dx} = f_\lambda(x; y) \qquad \lambda = 1, \cdots, n. \tag{2. 2}$$

The continuous curves representing the solutions of (2. 2) which are prolonged as far as possible on both sides in an open subset D of G, will be called *characteristic curves* of (2. 1) in D. Through any point $(\xi; \eta)$ in D, there passes one and only one characteristic curve in D[2]. We represent it by

$$y_\lambda = \varphi_\lambda(x, \xi, \eta_1, \cdots, \eta_n | D) = \varphi_\lambda(x, \xi; \eta | D) \qquad \lambda = 1, \cdots, n.$$
$$\alpha(\xi; \eta | D) < x < \beta(\xi; \eta | D).$$

2) Cf. Kamke [3], §16, Nr. 79, Satz 4,

$\alpha(\xi;\eta|D)$, $\beta(\xi;\eta|D)$ may be $-\infty$, $+\infty$ respectively. Sometimes we abbreviate it as $C(\xi;\eta|D)$. If an interval (open, closed or half-open) is contained in the interval $\alpha(\xi;\eta|D)<x<\beta(\xi;\eta|D)$, then we say that $C(\xi;\eta|D)$ is *defined for that interval.* Also when a property holds for the portion of $C(\xi;\eta|D)$ which corresponds to the values of x belonging to an interval, we say that $C(\xi;\eta|D)$ *has the property for this interval.*

We shall use the following properties of $C(\xi;\eta|D)$ and $\varphi_\lambda(x,\xi;\eta|D)$ often without special reference.

As we can easily see from the definition of $C(\xi;\eta|D)$, $(\xi;\eta)\in C(\xi;\eta|D)$ and if $(x';y')\in C(\xi;\eta|D)$, then $C(\xi;\eta|D)=C(x';y'|D)$ and so $(\xi;\eta)\in C(x';y'|D)$.

In terms of the functions φ_λ, this means:

$$\eta=\varphi(\xi,\xi;\eta|D),$$

and if $y'=\varphi(x',\xi;\eta|D)$, then

$$\alpha(\xi;\eta|D)=\alpha(x';y'|D)=\alpha,\quad \beta(\xi;\eta|D)=\beta(x';y'|D)=\beta$$

and

$$\varphi(x,\xi;\eta|D)=\varphi(x,x';y'|D)\quad\text{for}\quad \alpha<x<\beta,$$

especially

$$\eta=\varphi(\xi,x';y'|D).$$

Also $C(\xi;\eta|D_1)\supset C(\xi;\eta/D_2)$, if $D_1\supset D_2$ and $(\xi;\eta)\in D_2$.

We denote by D^* the set of the points $(x,\xi;\eta)$ in R^{n+2} such that $(\xi;\eta)\in D$ and $\alpha(\xi;\eta|D)<x<\beta(\xi;\eta|D)$. D^* is the domain of definition of the functions $\varphi_\lambda(x,\xi;\eta|D)$. D^* is open in R^{n+2} [3]. The functions $\varphi_\lambda(x,\xi;\eta|D)$ are continuous and have continuous partial derivatives with respect to all their arguments on D^*.[4]

A continuous function $z(x;y)$ defined on G will be called a *quasi-solution* of (2.1) on G, if it has $\partial z/\partial x$, $\partial z/\partial y_\lambda$ $\lambda=1,\cdots,n$, except at most at the points of an enumerable set in G and satisfies (2.1) almost everywhere in G. Here $\partial z/\partial x$, $\partial z/\partial y$ need not necessarily be continuous.

On the other hand, a continuous function $z(x;y)$ defined on G will be called a *solution* of (2.1) on G *in the ordinary sense,* if it is totally differentiable and satisfies (2.1) everywhere in G.

3. We shall prove the following three theorems in § 3, § 4.

3) Cf. Kamke [3], §17, Nr. 84, Satz 4.

4) Cf. Kamke [3], §17, Nr. 84, Statz 4 and §18, Nr. 87, Satz 1.

Theorem 1. *A quasi-solution* $z(x;y)$ *of* (2.1) *on* G *is constant on any characteristic curve of* (2.1) *in* G.

Theorem 2. *If for a fixed number* $\xi^{(0)}$, *the family of all the characteristic curves* $C(\xi^{(0)};\eta|G)$ *such that* $\eta\in G[\xi^{(0)}]$, *covers* G *and* $\psi(\eta)=\psi(\eta_1,\cdots,\eta_n)$ *is a totally differentiable function defined on* $G[\xi^{(0)}]$, *then there is one and only one quasi-solution* $z(x;y)$ *of* (2.1) *on* G *such that* $z(\xi^{(0)};\eta)=\psi(\eta)$ *on* $G[\xi^{(0)}]$ *and this quasi-solution* $z(x;y)$ *is also a solution of* (2.1) *on* G *in the ordinary sense.*

Theorem 3. *If* $n=1$, *any quasi-solution of* (2.1) *on* G *is also a solution of* (2.1) *on* G *in the ordinary sense.*

REMARK 1. For the case $n=1$, the proof of Theorem 1 can be partly simplified, cf. Kasuga [4].

REMARK 2. In Theorem 1, the condition on $z(z;y)$ that it has $\partial z/\partial x$, $\partial z/\partial y_\lambda$ $\lambda=1,\cdots,n$ except at most at the points of an enumerable set in G, cannot be replaced by the condition that it has $\partial z/\partial x$, $\partial z/\partial y_\lambda$ $\lambda=1,\cdots,n$ almost everywhere in G, as the following example shows it.

EXAMPLE. G: $0<x<1$ $0<y<1$,
the differential equation is

$$\frac{\partial z}{\partial x}=0$$

and a function $z(x,y)$ is defined by

$$z(x,y)=\psi(x) \qquad \text{on } G$$

where $\psi(x)$ is a continuous singular function not constant on the interval $0\leqq x\leqq 1$ as given in Saks [8] p. 101.

Then $z(x,y)$ is continuous on G, has $\partial z/\partial x$, $\partial z/\partial y$ almost everywhere in G and satisfies the differential equation almost everywhere in G. But $z(x,y)$ is not constant on any characteristic curve $y=$constant.

§ 2. Some Lemmas

In this §, the notations are the same as in § 1 and we assume that $z(x;y)$ is a quasi-solution of (2.1) on G.

4. Set K and Some Lemmas. We denote by K the set of the points $(\xi;\eta)$ of G such that $z(x;y)$ is constant on the portion of the characteristic curve $C(\xi;\eta|G)$ contained in a neighbourhood of $(\xi;\eta)$.

Lemma 1. *If a characteristic curve $C(\xi;\eta|G)$ is defined for an interval I (open, closed, or half-open) and is contained in K for the open interval I^0, the interior of I, then $z(x;y)$ is constant on the portion of $C(\xi;\eta|G)$ for the interval I.*

Proof. By the definition of K, we easily see that $z(x;y)$ is constant on the portion of $C(\xi;\eta|G)$ for the open interval I^0. Then Lemma 1 follows from the continuity of $z(x;y)$ and $\varphi_\lambda(x,\xi;\eta|G)$ $\lambda=1,\cdots,n$.

Lemma 2. *Denote by D an open subset of G, and denote by D_0 the set of the points $(\xi;\eta)$ of D such that $z(x;y)$ is constant on the characteristic curve $C(\xi;\eta|D)$. Then D_0 is closed in D.*

Proof. If $C(\xi^{(0)};\eta^{(0)}|D)$ where $(\xi^{(0)};\eta^{(0)})\in D$, is defined for a closed interval $\alpha_0\leqq x\leqq\beta_0$, then $C(\xi;\eta|D)$ where $(\xi;\eta)$ is sufficiently close to $(\xi^{(0)};\eta^{(0)})$, is also defined for the interval $\alpha_0\leqq x\leqq\beta_0$ and

$$\varphi_\lambda(x,\xi;\eta|D)\rightarrow\varphi_\lambda(x,\xi^{(0)};\eta^{(0)}|D) \qquad \lambda=1,\cdots,n$$

uniformly in the interval $\alpha_0\leqq x\leqq\beta_0$ as $(\xi;\eta)\rightarrow(\xi^{(0)};\eta^{(0)})$[5]. From this and by the continuity of $z(x;y)$, we easily see that D_0 is closed in D, q.e.d.

Lemma 3. *Let D be an open subset of G. If*

$$|f_\lambda(x;\bar{y})-f_\lambda(x;y)|\leqq M\|\bar{y}-y\| \qquad \lambda=1,\cdots,n \tag{4.1}$$

for every pair of points $(x;\bar{y})$, $(x;y)\in D$ with the same x coordinate and if $C(\xi;\bar{\eta}|D)$ and $C(\xi;\eta|D)$ where $(\xi;\bar{\eta})$, $(\xi;\eta)\in D$, are both defined for an interval $\alpha_0\leqq x\leqq\beta_0$ containing ξ $(\alpha_0\leqq\xi\leqq\beta_0)$, then

$$\begin{aligned}\|\varphi(x,\xi;\bar{\eta}|D)-\varphi(x,\xi;\eta|D)\| &= \sum_{\mu=1}^{n}|\varphi_\mu(x,\xi;\bar{\eta}|D)-\varphi_\mu(x,\xi;\eta|D)|\\ &\leqq\|\bar{\eta}-\eta\|\exp(nM|x-\xi|)\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}|\varphi_\lambda(x,\xi;\bar{\eta}|D)-\varphi_\lambda(x,\xi;\eta|D)| &\leqq|\bar{\eta}_\lambda-\eta_\lambda|+\frac{1}{n}\|\bar{\eta}-\eta\|\\ &\times\{\exp(nM|x-\xi|)-1\} \qquad \lambda=1,\cdots,n\end{aligned} \tag{4.3}$$

for $\alpha_0\leqq x\leqq\beta_0$.

Proof. We abbreviate $\varphi_\lambda(x,\xi;\bar{\eta}|D)$ and $\varphi_\lambda(x,\xi;\eta|D)$ as $\bar{\varphi}_\lambda(x)$ and $\varphi_\lambda(x)$ respectively.

By (4.1) and (2.2), we have

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5) Cf. Kamke [3], §17, Nr. 84, Satz 4,

$$|\bar{\varphi}_\lambda'(x)-\varphi_\lambda'(x)| \leqq M\|\bar{\varphi}(x)-\varphi(x)\| \qquad \lambda=1, \cdots, n \tag{4.4}$$

for $\alpha_0 \leqq x \leqq \beta_0$, so that

$$\sum_{\mu=1}^{n} |\bar{\varphi}_\mu'(x)-\varphi_\mu'(x)| \leqq nM \sum_{\mu=1}^{n} |\bar{\varphi}_\mu(x)-\varphi_\mu(x)|$$

for $\alpha_0 \leqq x \leqq \beta_0$. Hence by a theorem on differential inequalities[6], taking account of $\alpha_0 \leqq \xi \leqq \beta_0$ and $\bar{\eta}_\lambda=\bar{\varphi}_\lambda(\xi)$, $\eta_\lambda=\varphi_\lambda(\xi)$ $(\lambda=1, \cdots, n)$, we obtain

$$\|\bar{\varphi}(x)-\varphi(x)\| \leqq \|\bar{\eta}-\eta\| \exp(nM|x-\xi|) \tag{4.5}$$

for $\alpha_0 \leqq x \leqq \beta_0$. Thus (4.2) is proved.

By (4.4), (4.5), we get

$$|\bar{\varphi}_\lambda'(x)-\varphi_\lambda'(x)| \leqq M\|\bar{\eta}-\eta\| \exp(nM|x-\xi|) \qquad \lambda=1, \cdots, n$$

for $\alpha_0 \leqq x \leqq \beta_0$. Hence, again taking account of $\alpha_0 \leqq \xi \leqq \beta_0$ and $\bar{\eta}_\lambda=\bar{\varphi}_\lambda(\xi)$, $\eta_\lambda=\varphi_\lambda(\xi)$ $(\lambda=1, \cdots, n)$, we have

$$|\bar{\varphi}_\lambda(x)-\varphi_\lambda(x)| \leqq |\bar{\eta}_\lambda-\eta_\lambda| + M\|\bar{\eta}-\eta\| \int_\xi^x \exp(nM|x-\xi|)dx$$

$$=|\bar{\eta}_\lambda-\eta_\lambda| + \frac{1}{n}\|\bar{\eta}-\eta\| \{\exp(nM|x-\xi|)-1\} \qquad \lambda=1, \cdots, n$$

for $\alpha_0 \leqq x \leqq \beta_0$. Thus (4.3) is also proved.

§ 3. Proof of Theorem 1.

In this §, the notations are the same as in §1 and §2 and we assume that $z(x;y)$ is a quasi-solution of (2.1) on G.

5. Set F and Domain Q. We denote by F the set $\overline{G-K}\cdot G$. Evidently F is closed in G and $K \supset G-F$.

If F is empty, that is $G=K$, we can conclude by Lemma 1 that $z(x;y)$ is constant on any characteristic curve in G and Theorem 1 is established.

Therefore we suppose in the following that $F \neq 0$ and we want to show that such supposition leads to a contradiction.

Proposition 1. *There is a positive number N and a $(n+1)$-dimensional open cube Q: $|x-a|<L$, $|y_\lambda-b_\lambda|<L$ $\lambda=1, \cdots, n$ $(L>0)$ such that*

$$\bar{Q} \subset G$$
$$(a;b) \in F$$

6) Cf. Kamke [3], §17, Nr. 85, Hilfssatz 3 and Satz 5.

and such that

$$\begin{cases} |z(x+h\,;y)-z(x\,;y)| \leqq |h|N \\ |z(x, y_1, \cdots, y_{\lambda-1}, y_\lambda+k_\lambda, y_{\lambda+1}, \cdots, y_n)-z(x\,;y)| \leqq |k_\lambda|N \\ \qquad\qquad \lambda=1, \cdots, n \end{cases} \tag{5.1}$$

whenever $(x\,;y) \in F \cdot Q$ *and* $(x+h\,;y+k) \in Q$, *where* $k=(k_1, \cdots, k_n)$.

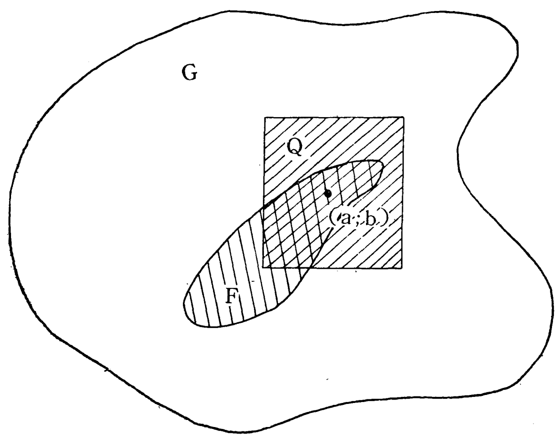

Fig. 1

Proof. We denote by H the at most enumerable set consisting of the points of G at which $z(x\,;y)$ is not derivable with respect to x and with respect to y_λ $\lambda=1, \cdots, n$ simultaneously.

If a point $(\xi^{(0)}\,;\eta^{(0)})$ of G has an open neighbourhood V such that every point of V belongs to K except at most $(\xi^{(0)}\,;\eta^{(0)})$ itself, then by Lemma 1 $z(x\,;y)$ is constant on $C(\xi^{(0)}\,;\eta\,|\,V)$ where η is any point of $V[\xi^{(0)}]$ except $\eta^{(0)}$ and so by Lemma 2, $z(x\,;y)$ is also constant on $C(\xi^{(0)}\,;\eta^{(0)}\,|\,V)$, that is, $(\xi^{(0)}\,;\eta^{(0)}) \in K$. Hence the set F which is closed in the open set G, has no isolated point.

Therefore F is a G_δ set in R^{n+1} without isolated point and so every point of F is a condensation point of F[7]. Thus since F is not empty by the supposition and H is at most enumerable, $F-H$ is not empty and

$$\overline{F-H} \supset F \tag{5.2}$$

Also the non-empty $F-H$ is a G_δ set in R^{n+1} since F is a G_δ set in R^{n+1} and H is at most enumerable. Hence $F-H$ is of the second category in itself by Baire's theorem[8].

On the other hand, if we denote by F_m for each positive integer m, the set of the points $(x\,;y)$ of G such that

$$\begin{cases} |z(x+h\,;y)-z(x\,;y)| \leqq |h|m \\ |z(x, y_1, \cdots, y_{\lambda-1}, y_\lambda+k_\lambda, y_{\lambda+1}, \cdots, y_n)-z(x\,;y)| \leqq |k_\lambda|m \\ \qquad\qquad \lambda=1, \cdots, n \end{cases} \tag{5.3}$$

7) Cf. Hausdorff [2], p. 138.
8) Cf. Hausdorff [2], p. 142.

whenever $|h|$, $|k_\lambda| \leqq 1/m$ and $(x+h;y) \in G$, $(x, y_1, \cdots, y_{\lambda-1}, y_\lambda+k_\lambda, y_{\lambda+1}, \cdots, y_n) \in G$ $\lambda=1, \cdots, n$, then the union of the sets F_m covers $F-H$ by the definition of H and each of the set F_m is closed in G by the continuity of $z(x;y)$.

Therefore there must exist a positive integer N and a $(n+1)$-dimensional open cube Q: $|x-a|<L$, $|y_\lambda-b_\lambda|<L$ $\lambda=1, \cdots, n$ $(L>0)$ such that $(a;b) \in F-H \subset F$ and

$$(F-H) \cdot Q \subset F_N . \tag{5.4}$$

Also we can take L sufficiently small so that

$$0<L<1/(2N) \tag{5.5}$$

$$\bar{Q} \subset G \tag{5.6}$$

since G is open in R^{n+1}.

By (5.4), (5.6) and by observing that Q is open in R^{n+1} and F_N is closed in G, we have

$$\overline{F-H} \cdot Q = \overline{(F-H) \cdot Q} \cdot Q \subset \bar{F}_N \cdot Q \subset \bar{F}_N \cdot G = F_N$$

so that by (5.2).

$$F_N \supset F \cdot Q .$$

Hence by (5.5), (5.6) and by the definition of F_N, the inequalities (5.3) for $m=N$ hold whenever $(x;y) \in F \cdot Q$ and $(x+h;y) \in Q$, $(x, y_1, \cdots, y_{\lambda-1}, y_\lambda+k_\lambda, y_{\lambda+1}, \cdots, y_n) \in Q$ $\lambda=1, \cdots, n$. This completes the proof of Proposition 1.

In the following, Q, L, $(a;b)$ and N have the same meanings as in Proposition 1.

6. Domains Q_1, Ω_1, G_1 and Set $\widetilde{F}$. f_λ and $\partial f_\lambda/\partial y_\mu$ $\lambda, \mu=1, \cdots, n$ are defined and continuous on $\bar{Q} \subset G$. Hence there is a positive number M_0 such that

$$|f_\lambda|,\ |\partial f_\lambda/\partial y_\mu| < M_0 \quad \lambda, \mu=1, \cdots, n \text{ on } Q. \tag{6.1}$$

Then we can easily prove

$$|f_\lambda(x;\bar{y})-f_\lambda(x;y)| \leqq M_0 \|\bar{y}-y\| \tag{6.2}$$

for any pair of points $(x;\bar{y})$, $(x;y) \in Q$ with the same x coordinate. We take a positive number L_1 such that

$$\exp(2n^2 M_0 L_1) < 2 \tag{6.3}$$

$$L_1(M_0+1) \leqq L . \tag{6.4}$$

We denote by Ω_1 the n-dimensional open cube: $|\eta_\lambda-b_\lambda|<L_1$ $\lambda=1,$

$\cdots, n$ and by Q_1 the $(n+1)$-dimensional open parallelepiped: $|x-a|<L_1$, $|y_\lambda-b_\lambda|<L$ $\lambda=1, \cdots, n$. By (6.4) $L_1<L$ and so

$$Q_1 \subset Q.$$

By (6.4), $\eta_\lambda+L_1M_0 \leqq b_\lambda+L$, $\eta_\lambda-L_1M_0 \geqq b_\lambda-L$ $\lambda=1, \cdots, n$ whenever $\eta \in \Omega_1$. Hence the characteristic curves $C(a; \eta | Q_1)$ where $\eta \in \Omega_1$, are defined just for the interval $|x-a|<L_1$ since $|f_\lambda|<M_0$ $\lambda=1, \cdots, n$ on Q_1 $(\subset Q)$ by (6.1).

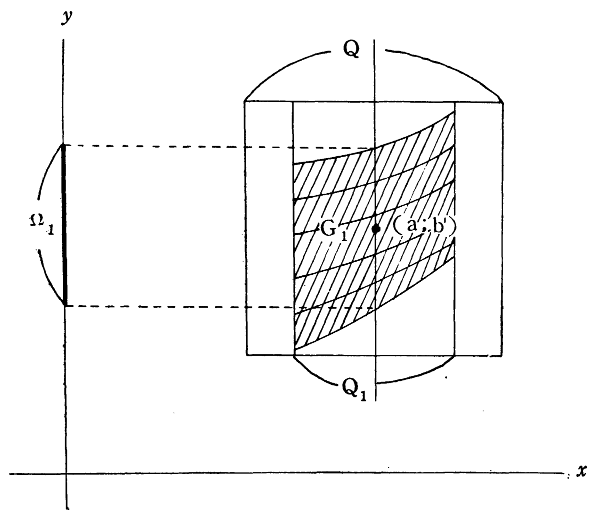

Fig. 2

We denote by G_1 the portion of Q_1 covered by the family of all the characteristic curves $C(a; \eta | Q_1)$ where $\eta \in \Omega_1$. Then by the properties of $C(\xi; \eta | Q_1)$ and $\varphi_\lambda(x, \xi; \eta | Q_1)$ as stated in §1.2, observing that Ω_1 is open in R^n, we easily prove that G_1 is open in R^{n+1} and the characteristic curves $C(\xi; \eta | G_1)$ where $(\xi; \eta) \in G_1$, are defined just for the interval $|x-a|<L_1$.

We denote by $\widetilde{F}$ the set of the points η of Ω_1 such that $C(a; \eta | G_1)$ has at least one point in common with F. $\widetilde{F}$ is not empty since $(a; b) \in F$ and $b=(b_1, \cdots, b_n) \in \Omega_1$. Now we prove

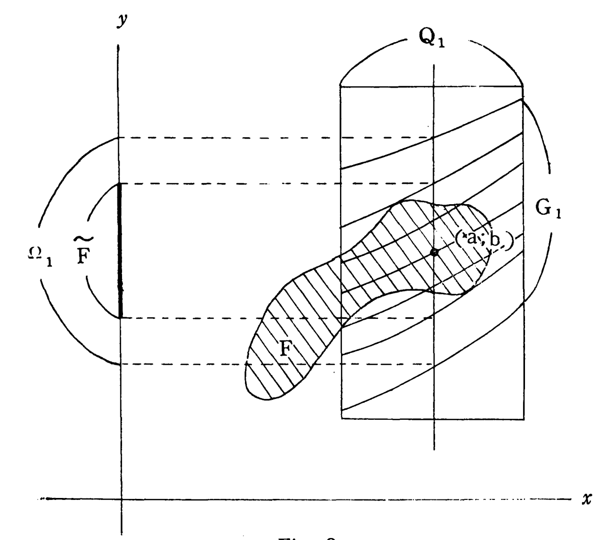

Fig. 3

Proposition 2. $\widetilde{F}^0$, *the interior of* $\widetilde{F}$ *in* R^n, *is not empty.*

Proof. Suppose, if possible, that $\widetilde{F}^0=0$.

If $\eta \in \Omega_1 - \widetilde{F}$, then $C(a; \eta | G_1)$ is contained in K by the definition of $\widetilde{F}$ and so by Lemma 1 $z(x; y)$ is constant on $C(a; \eta | G_1)$. Hence by

Lemma 2, $z(x\,;y)$ is constant on any $C(a\,;\eta|G_1)$ such that $\eta\in\overline{\Omega_1-\tilde{F}\cdot\Omega_1}$. But $\overline{\Omega_1-\tilde{F}\cdot\Omega_1}=\Omega_1$, since $\tilde{F}^0=0$ by supposition and Ω_1 is open in R^n.

Therefore $z(x\,;y)$ is constant on any $C(a\,;\eta|G_1)$ such that $\eta\in\Omega_1$ and so by the definition of K, $G_1\subset K$ since G_1 is covered by the family of all the characteristic curves $C(a\,;\eta|G_1)$ where $\eta\in\Omega_1$. Hence, observing that G_1 is open in R^{n+1}, we have

$$F=G\cdot\overline{G-K}\subset G\cdot\overline{G-G_1}\subset G-G_1\,.$$

But this is a contradiction since $(a\,;b)\in F\cdot G_1$. Thus Proposition 2 is proved.

7. Domains Ω_2, G_2. By Proposition 2, $\tilde{F}^0$ is not empty. Hence we take a point $b^{(1)}=(b_1^{(1)},\cdots,b_n^{(1)})\in\tilde{F}^0$.

Then we can construct a domain Ω_2 in R^n defined by

$$\eta:\ \|\eta-b^{(1)}\|<L_2\qquad(L_2>0)\tag{7.1}$$

such that

$$\bar{\Omega}_2\subset\tilde{F}\,.\tag{7.2}$$

Evidently

$$\bar{\Omega}_2\subset\Omega_1\tag{7.3}$$

since $\tilde{F}\subset\Omega_1$ by the definition of $\tilde{F}$.

We denote by G_2 the portion of Q_1 covered by the family of all the characteristic curves $C(a\,;\eta|Q_1)$ where $\eta\in\Omega_2$. In the same way as in the case of G_1, we easily prove that G_2 is open in R^{n+1} and so $G_2[x]$

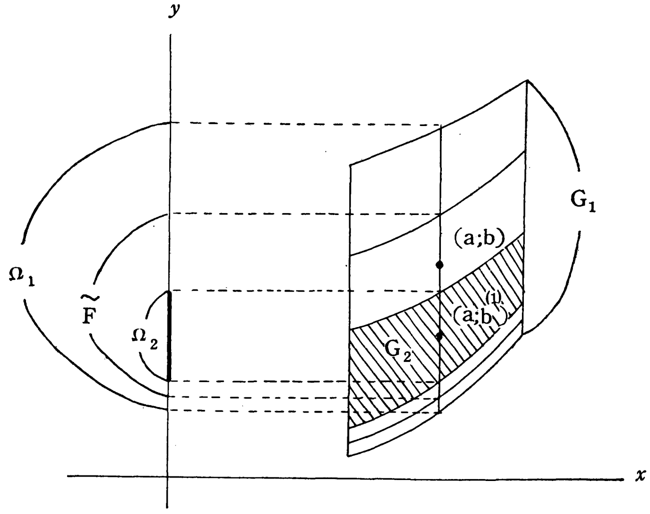

Fig. 4

is open in R^n for any x in the interval $|x-a|<L_1$. Also $C(\xi;\eta|G_2)$ where $(\xi;\eta)\in G_2$, is defined just for the interval $|x-a|<L_1$. Evidently by (7.3) and the definition of G_1, G_2,

$$G_2\subset G_1\subset Q_1\subset Q.$$

Proposition 3. *$C(\xi;\eta|G_2)$ where $(\xi;\eta)\in G_2$, has at least one point in common with $F\cdot G_2$.*

Proof. If $(\xi;\eta)\in G_2$ and $\eta^{(0)}=\varphi(a,\xi;\eta|G_2)$, then $\eta^{(0)}\in\Omega_2$ and $C(\xi;\eta|G_2)=C(a;\eta^{(0)}|G_1)$ by the definition of G_2. Then by (7.2) and the definition of $\tilde{F}$, Proposition 3 follows.

Proposition 4. *If $|\xi-a|<L_1$, then*

$$G_1[\xi]\supset(G_2[\xi])^b,\ \text{the boundary in } R^n \text{ of } G_2[\xi]. \tag{7.4}$$

Further if $|\xi-a|<L_1$ and $\eta\in(G_2[\xi])^b$, then

$\varphi(a,\xi;\eta|G_1)\in\Omega_2^b$, the boundary in R^n of Ω_2.

Proof. We consider the continuous mapping $\mathfrak{A}_\xi$ of Ω_1 onto $G_1[\xi]$ defined by

$$\eta^{(0)}\to\varphi(\xi,a;\eta^{(0)}|G_1).$$

That $\mathfrak{A}_\xi$ maps Ω_1 onto $G_1[\xi]$ follows from the definition of G_1.

By the properties of $C(\xi;\eta|G_1)$ and $\varphi_\lambda(x,\xi;\eta|G_1)$ as stated in § 1.2, we easily see that $\mathfrak{A}_\xi$ is one to one and bicontinuous and $\mathfrak{A}_\xi^{-1}$ is represented by

$$\eta\to\varphi(a,\xi;\eta|G_1) \tag{7.5}$$

We have

$$\mathfrak{A}_\xi(\Omega_2)=G_2[\xi] \tag{7.6}$$

by (7.3) and the definition of G_2. Hence again taking acconnt of (7.3), by the continuity of $\mathfrak{A}_\xi$ we have $\mathfrak{A}_\xi(\bar{\Omega}_2)\subset\overline{G_2[\xi]}$.

On the other hand, since $\bar{\Omega}_2$ is closed and bounded in R^n, its continuous image $\mathfrak{A}_\xi(\bar{\Omega}_2)$ is closed in R^n and so, taking account of (7.6), we have $\mathfrak{A}_\xi(\bar{\Omega}_2)\supset\overline{G_2[\xi]}$.

Therefore $\mathfrak{A}_\xi(\bar{\Omega}_2)=\overline{G_2[\xi]}$. Hence by (7.6), (7.3), and $\mathfrak{A}_\xi(\Omega_1)=G_1[\xi]$, observing that $\Omega_2, G_2[\xi]$ are both open in R^n, we get $\mathfrak{A}_\xi(\Omega_2^b)=(G_2[\xi])^b\subset G_1[\xi]$.

From this, taking account of the representation (7.5) of $\mathfrak{A}_\xi^{-1}$, Proposition 4 follows.

8. Classes S_λ, $S^{(l)}$ and Operations T_λ, T, $T^{(l)}$. We take two points $(x';y')$, $(x';\bar{y}')$ of R^{n+1} with the same x coordinate such that $(x';y')\in F\cdot G_2$, $(x';\bar{y}')\in G_2$ and further $(x', y_1', \cdots, y'_{\lambda-1}, \bar{y}_\lambda', y'_{\lambda+1}, \cdots, y_n')\in G_2$. In the following, we denote the class of all such ordered pairs $\{(x';y'), (x';\bar{y}')\}$ of points of R^{n+1} by S_λ $(\lambda=1, \cdots, n)$. If we put $\tilde{y}'=(y_1', \cdots, y'_{\lambda-1}, \bar{y}_\lambda', y'_{\lambda+1}, \cdots, y_n')$, then $(x';\tilde{y}')\in G_2$.

Now there is the nearest x to x' in the interval $|x-a|<L_1$ such that either $(x, \varphi(x, x';\bar{y}'|G_2))\in F\cdot G_2$ or $(x, \varphi(x, x';\tilde{y}'|G_2))\in F\cdot G_2$, since by Proposition 3 each of the continuous curves $C(x';\bar{y}'|G_2)$ and $C(x';\tilde{y}'|G_2)$ which are just defined for the interval $|x-a|<L_1$ and are contained in G_2, has at least one point in common with $F\cdot G_2$ which is closed in G_2. We denote such x by x''. If incidently two such x exist, then we take as x'' the one on the right side of x'.

Now we distinguish two cases;

i) If $(x'', \varphi(x'', x';\bar{y}'|G_2))\in F\cdot G_2$, then we put

$$y''=\varphi(x'', x';\bar{y}'|G_2) \quad\text{and}\quad \bar{y}''=\varphi(x'', x';\tilde{y}'|G_2)\,.$$

ii) If $(x'', \varphi(x'', x';\bar{y}'|G_2))\notin F\cdot G_2$ and so by the definition of x'', $(x'', \varphi(x'', x';\tilde{y}'|G_2))\in F\cdot G_2$, then we put

$$y''=\varphi(x'', x';\tilde{y}'|G_2) \quad\text{and}\quad \bar{y}''=\varphi(x'', x';\bar{y}'|G_2)\,.$$

In any case, $(x'';y'')\in F\cdot G_2$ and $(x'';\bar{y}'')\in G_2$.

We denote by T_λ $(\lambda=1, \cdots, n)$ the above operation which assigns to every (ordered) pair $\{(x';y'), (x';\bar{y}')\}$ of points of R^{n+1} belonging to the class S_λ, an (ordered) pair $\{(x'';y''), (x'';\bar{y}'')\}$ of points of R^{n+1} with the same x coordinate such that $(x'';y'')\in F\cdot G_2$ and $(x'';\bar{y}'')\in G_2$. Also we write $T_\lambda\{(x';y'), (x';\bar{y}')\}=\{(x'';y''), (x'';\bar{y}'')\}$.
If $\{(x'';y''), (x'';\bar{y}')\}\in S_\mu$, we can apply T_μ again on $\{(x'';y''), (x'';\bar{y}'')\}$.

Proposition 5. *If* $\{(x';y'), (x';\bar{y}')\}\in S_\lambda$ *and if we put* $\{(x'';y''), (x'';\bar{y}'')\}=T_\lambda\{(x';y'), (x';\bar{y}')\}$, $z'=z(x';y')$, $\bar{z}'=z(x';\bar{y}')$, $z''=z''(x'';y'')$, *and* $\bar{z}''=z(x'';\bar{y}'')$, *then*

$$|\bar{z}'-z'|\leqq|\bar{z}''-z''|+N\,\|\bar{y}'-y'\| \tag{8.1}$$

Proof. We put $\tilde{y}'=(y_1', \cdots, y'_{\lambda-1}, \bar{y}_\lambda', y'_{\lambda+1}, \cdots, y_n')$. Then $(x';y')\in F\cdot G_2\subset F\cdot Q$ and $(x';\tilde{y}')\in G_2\subset Q$ since $\{(x';y'), (x';\bar{y}')\}\in S_\lambda$. Hence by Proposition 1 (5.1), if we put $\tilde{z}'=z(x';\tilde{y}')$

$$|\tilde{z}'-z'|\leqq N|\bar{y}_\lambda'-y_\lambda'|\leqq N\,\|\bar{y}'-y'\|\,. \tag{8.2}$$

By the definition of T_λ,

$$\begin{cases} y''=\varphi(x'',x';\bar{y}'\,|\,G_2) \\ \bar{y}''=\varphi(x'',x';\tilde{y}'\,|\,G_2) \end{cases} \quad \text{or} \quad \begin{cases} y''=\varphi(x'',x';\tilde{y}'\,|\,G_2) \\ \bar{y}''=\varphi(x'',x';\bar{y}'\,|\,G_2) \end{cases} \tag{8.3}$$

On the other hand, each of $C(x';\bar{y}'\,|\,G_2)$ and $C(x';\tilde{y}'\,|\,G_2)$ has no point in common with F for the interval $x'<x<x''$ or $x''<x<x'$ by the difinition of T_λ so that they are contained in K for the interval $x''<x<x'$ or $x'<x<x''$. Therefore by Lemma 1 and (8.3)

$$\begin{cases} z''=z(x'';y'')=z(x';\bar{y}')=\bar{z}' \\ \bar{z}''=z(x'';\bar{y}'')=z(x';\tilde{y}')=\tilde{z}' \end{cases}$$

or

$$\begin{cases} z''=z(x'';y'')=z(x';\tilde{y}')=\tilde{z}' \\ \bar{z}''=z(x'';\bar{y}'')=z(x';\bar{y}')=\bar{z}' \end{cases}$$

so that

$$|\bar{z}''-z''|=|\bar{z}'-\tilde{z}'|. \tag{8.4}$$

By (8.4), (8.2) we have

$$|\bar{z}'-z'|\leqq|\bar{z}'-\tilde{z}'|+|\tilde{z}'-z'|\leqq|\bar{z}''-z''|+N\,\|\bar{y}'-y'\|,$$

q.e.d.

We denote by $T_\mu\cdot T_\lambda$ the operation which assigns to a pair $\{(x';y'),(x';\bar{y}')\}$ of points of R^{n+1}, the pair $T_\mu\{T_\lambda\{(x';y'),(x';\bar{y}')\}\}$ of points of R^{n+1} if

$$\{(x';y'),(x';\bar{y}')\}\in S_\lambda \quad \text{and} \quad T_\lambda\{(x';y'),(x';\bar{y}')\}\in S_\mu;$$

and similarly for products of any number of operations $T_\lambda(\lambda=1,\cdots,n)$.

We put $T=\overbrace{T_n\cdot T_{n-1}\cdots\cdots T_2\cdot T_1}^{n}$ and $T^m=\overbrace{T\cdot T\cdots\cdots T}^{m}$ ($T^0=$identity operator) for any non-negative integer m and $T^{[l]}=\overbrace{T_\nu\cdot T_{\nu-1}\cdots\cdots T_2\cdot T_1}^{\nu}\cdot T^m$ for any non-negative integer l if $l=mn+\nu$, $0\leqq\nu\leqq n-1$ and m, $\nu=$non-negative integer (if $\nu=0$, $T^{[l]}=T^m$).

We denote by $S^{[l]}$ $(l>0)$ the class of all the pairs $\{(x';y'),(x';\bar{y}')\}$ of points of R^{n+1} on which we can apply the operation $T^{[l]}$ $(l>0)$ and by $S^{[0]}$ the class of all the pairs $\{(x';y'),(x';\bar{y}')\}$ such that $(x';y')\in F\cdot G_2$ and $(x';\bar{y}')\in G_2$. We regard $S^{[0]}$ as the domain of definition of the identity operator $T^{[0]}=T^0$.

In the following, we put

$$M_1=\exp(2n^2M_0L_1)-1 \tag{8.5}$$

$$M_2=\exp(2nM_0L_1), \tag{8.6}$$

then by (6.3)

$$1>M_1>0. \tag{8.7}$$

Also

$$M_2>1. \tag{8.8}$$

Proposition 6. *If* $\{(x^{(0)};y^{(0)}),\ (x^{(0)};\bar{y}^{(0)})\}\in S^{[l]}$ *where* $l=mn+\nu$, $n-1\geqq\nu\geqq 0$ *and* m, $\nu=$*non-negative integer, and if we put* $T^{[l]}\{(x^{(0)};y^{(0)}),\ (x^{(0)};\bar{y}^{(0)})\}=\{(x^{(l)};y^{(l)}),\ (x^{(l)};\bar{y}^{(l)})\}$, *then*

$$\|\bar{y}^{(l)}-y^{(l)}\|\leqq(M_1+1)M_1^m\|\bar{y}^{(0)}-y^{(0)}\| \tag{8.9}$$

Proof. In the following lines, we shall prove by induction on l more precise results,

$$\|\bar{y}^{(l)}-y^{(l)}\|\leqq M_2^\nu\ M_1^m\|\bar{y}^{(0)}-y^{(0)}\| \tag{8.10}$$

and

$$|\bar{y}_\lambda^{(l)}-y_\lambda|\leqq\frac{1}{n}(M_2^\nu-M_2^{\lambda-1})M_1^m\|\bar{y}^{(0)}-y^{(0)}\| \quad \text{for} \quad \nu\geqq\lambda\geqq 1 \tag{8.11}$$

whenever

$$\{(x^{(0)};y^{(0)}),\ (x^{(0)};\bar{y}^{(0)})\}\in S^{[l]} \qquad (l=mn+\nu).$$

(8.9) follows from (8.10) since $M_2^\nu<M_1+1$ by $n-1\geqq\nu\geqq 0$ and (8.5), (8.6).

For $l=0$, $T^{[l]}=T^{[0]}=$identity operator and $m=\nu=0$. Hence here (8.10) and (8.11) are trival.

Now we assume that (8.10), (8.11) are true for $l=l'=m'n+\nu'$ and let $\{(x^{(0)};y^{(0)}),(x^{(0)};\bar{y}^{(0)})\}\in S^{[l'+1]}(\subset S^{[l']})$. Then $\{(x^{(l')};y^{(l')}),(x^{(l')};\bar{y}^{(l')})\}\in S_{\nu'+1}$ since $T^{[l'+1]}=T_{\nu'+1}\cdot T^{[l']}$. Also $\{(x^{(l'+1)};y^{(l'+1)}),(x^{(l'+1)};\bar{y}^{(l'+1)})\}=T^{[l'+1]}\{(x^{(0)};y^{(0)}),\ (x^{(0)};\bar{y}^{(0)})\}=T_{\nu'+1}\{(x^{(l')};y^{(l')}),\ (x^{(l')};\bar{y}^{(l')})\}$.

Then by the definition of $T_{\nu'+1}$, (6.2), Lemma 3 (4.2) and (8.6), taking account of $|x^{(l'+1)}-x^{(l')}|<2L_1$, we have

$$\|\bar{y}^{(l'+1)}-y^{(l'+1)}\|\leqq\Big(\sum_{\mu=1}^{\nu'}|\bar{y}_\mu^{(l')}-y_\mu^{(l')}|+\sum_{\mu=\nu'+2}^{n}|\bar{y}_\mu^{(l')}-y_\mu^{(l')}|\Big)\times$$
$$\exp(nM_0|x^{(l'+1)}-x^{(l')}|)\leqq\|\bar{y}^{(l')}-y^{(l')}\|\exp(2nM_0L_1)=M_2\|\bar{y}^{(l')}-y^{(l')}\|$$

and so by (8.10) for $l=l'$,

$$\|\bar{y}^{(l'+1)}-y^{(l'+1)}\|\leqq M_2^{\nu'+1}M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|. \tag{8.12}$$

Also by the definition of $T_{\nu'+1}$, (6.2), Lemma 3 (4.3) and (8.6), we have

$$|\bar{y}_\lambda^{(l'+1)}-y_\lambda^{(l'+1)}|\leqq|\bar{y}_\lambda^{(l')}-y_\lambda^{(l')}|+\frac{1}{n}(\sum_{\mu=1}^{\nu'}|\bar{y}_\mu^{(l')}-y_\mu^{(l')}|$$
$$+\sum_{\mu=\nu'+2}^{n}|\bar{y}_\mu^{(l')}-y_\mu^{(l')}|)\{\exp(nM_0|x^{(l'+1)}-x^{(l')}|)-1\}\leqq|\bar{y}_\lambda^{(l')}-y_\lambda^{(l')}|$$
$$+\frac{1}{n}(M_2-1)\times\|\bar{y}^{(l')}-y^{(l')}\| \quad \text{for} \quad n\geqq\lambda\geqq1 \quad \lambda\neq\nu'+1,$$

and so by (8, 10) and (8.11) for $l=l'$,

$$|\bar{y}_\lambda^{(l'+1)}-y_\lambda^{(l'+1)}|\leqq\frac{1}{n}(M_2^{\nu'}-M_2^{\lambda-1})M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|$$
$$+\frac{1}{n}(M_2-1)\times M_2^{\nu'}M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|=\frac{1}{n}(M_2^{\nu'+1}-M_2^{\lambda-1})M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|$$
$$\text{for} \quad \nu'\geqq\lambda\geqq1. \tag{8.13}$$

Again by the definition of $T_{\nu'+1}$, (6.2) and Lemma 3 (4.3), we have

$$|\bar{y}_{\nu'+1}^{(l'+1)}-y_{\nu'+1}^{(l'+1)}|\leqq\frac{1}{n}(\sum_{\mu=1}^{\nu'}|\bar{y}_\mu^{(l')}-y_\mu^{(l')}|$$
$$+\sum_{\mu=\nu'+2}^{n}|\bar{y}_\mu^{(l')}-y_\mu^{(l')}|)\{\exp(nM_0|x^{(l'+1)}-x^{(l')}|)-1\}$$
$$\leqq\frac{1}{n}(M_2-1)\|\bar{y}^{(l')}-y^{(l')}\|,$$

and so by (8.10) for $l=l'$,

$$|\bar{y}_{\nu'+1}^{(l'+1)}-y_{\nu'+1}^{(l'+1)}|\leqq\frac{1}{n}(M_2^{\nu'+1}-M_2^{\nu'})M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|. \tag{8.14}$$

If $n-2\geqq\nu'\geqq0$, then (8.12), (8.13), (8.14) prove (8.10), (8.11) for $l=l'+1$ since $l'+1=nm'+\nu'+1$, $n-1\geqq\nu'+1\geqq1$ in this case.

If $\nu'=n-1$, then $l'+1=n(m'+1)$. In this case, by (8.13), (8.14), we obtain

$$\|\bar{y}^{(l'+1)}-y^{(l'+1)}\|=\sum_{\mu=1}^{n-1}|\bar{y}_\mu^{(l'+1)}-y_\mu^{(l'+1)}|+|\bar{y}_n^{(l'+1)}-y_n^{(l'+1)}|$$
$$\leqq\frac{1}{n}\{\sum_{\mu=1}^{n}(M_2^n-M_2^{\mu-1})\}M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|\leqq(M_2^n-1)M_1^{m'}\|\bar{y}^{(0)}-y^{(0)}\|$$
$$=M_1^{m'+1}\|\bar{y}^{(0)}-y^{(0)}\|,$$

since $M_2^{\mu-1}\geqq1$ for $n\geqq\mu\geqq1$ and $M_1=M_2^n-1$ by (8.8), (8.5), (8.6). This proves (8.10) for $l=l'+1$ in the case $\nu'=n-1$. In this case (8.11) for $l=l'+1$ is trivial, since then there is no λ for which it should be established.

Thus (8.10), (8.11) are proved for any non-negative integer l and so the proof of Proposition 6 is completed.

9. Further on the operation $T^{(l)}$.

In the following we put

$$M_3 = nM_2(1+M_1)(1-M_1)^{-1}. \tag{9.1}$$

By (8.7), (8.8), M_3 is positive.

Proposition 7. *If* $\{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} \in S^{[l]}$ *for a non-negative integer* l *and if we put*

$$\{(x^{(p)}; y^{(p)}), (x^{(p)}; \bar{y}^{(p)})\} = T^{[p]}\{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\},$$
$$\eta^{(p)} = \varphi(a, x^{(p)}; y^{(p)} | G_2), \quad \bar{\eta}^{(p)} = \varphi(a, x^{(p)}; \bar{y}^{(p)} | G_2)$$
$$\textit{for} \quad p = 0, 1, \cdots, l,$$

then

$$\| \bar{\eta}^{(l)} - \eta^{(0)} \| \leqq M_3 \| \bar{\eta}^{(0)} - \eta^{(0)} \|. \tag{9.2}$$

Proof. By the definition of $T^{[p]}$,

$$y^{(p)} = \varphi(x^{(p)}, x^{(p-1)}; \bar{y}^{(p-1)} | G_2) \quad \text{or} \quad \bar{y}^{(p)} = \varphi(x^{(p)}, x^{(p-1)}; \bar{y}^{(p-1)} | G_2)$$

for $p=1, \cdots, l$. Hence

$$\eta^{(p)} = \bar{\eta}^{(p-1)} \quad \text{or} \quad \bar{\eta}^{(p)} = \bar{\eta}^{(p-1)} \qquad p = 1, \cdots, l,$$

so that

$$\| \bar{\eta}^{(p)} - \bar{\eta}^{(p-1)} \| = \| \bar{\eta}^{(p)} - \eta^{(p)} \| \quad \text{or} \quad \| \bar{\eta}^{(p)} - \bar{\eta}^{(p-1)} \| = 0$$
$$\text{for} \quad p = 1, \cdots, l.$$

Therefore

$$\| \bar{\eta}^{(l)} - \eta^{(0)} \| \leqq \sum_{p=1}^{l} \| \bar{\eta}^{(p)} - \bar{\eta}^{(p-1)} \| + \| \bar{\eta}^{(0)} - \eta^{(0)} \| \leqq \sum_{p=0}^{l} \| \bar{\eta}^{(p)} - \eta^{(p)} \|. \tag{9.3}$$

By (6.2) and Lemma 3 (4.2), we obtain

$$\| \bar{\eta}^{(p)} - \eta^{(p)} \| \leqq \| \bar{y}^{(p)} - y^{(p)} \| \exp(nM_0 | a - x^{(p)} |)$$
$$\leqq \| \bar{y}^{(p)} - y^{(p)} \| \exp(nM_0 L_1) = \sqrt{M_2} \| \bar{y}^{(p)} - y^{(p)} \|$$
$$\text{for} \quad p = 0, \cdots, l, \tag{9.4}$$

since $|a - x^{(p)}| < L_1$ and $\sqrt{M_2} = \exp(nM_0 L_1)$ by (8.6). Similarly by (6.2) and Lemma 3 (4.2), we have

$$\| \bar{y}^{(0)} - y^{(0)} \| \leqq \| \bar{\eta}^{(0)} - \eta^{(0)} \| \exp(nM_0 | x^{(0)} - a |) \leqq \sqrt{M_2} \| \bar{\eta}^{(0)} - \eta^{(0)} \|. \tag{9.5}$$

On the other hand, by Proposition 6, if $p = qn + s$, $n-1 \geqq s \geqq 0$ q, $s =$ non-negative integer, then

$$\| \bar{y}^{(p)} - y^{(p)} \| \leqq (M_1 + 1) M_1^q \| \bar{y}^{(0)} - y^{(0)} \| \quad \text{for} \quad p = 0, \cdots, l. \tag{9.6}$$

By (9.4), (9.5), (9.6), we obtain

$$\| \bar{\eta}^{(p)} - \eta^{(p)} \| \leqq M_2 (M_1 + 1) M_1^q \| \bar{\eta}^{(0)} - \eta^{(0)} \| \quad \text{for} \quad p = 0, \cdots, l. \tag{9.7}$$

By (9.3), (9.7), taking account of (8.7) and (9.1), if $l = nm+\nu$, $n-1 \geqq \nu \geqq 0$ and $m, \nu =$ non-negative integer, then we get

$$\begin{aligned}
\| \bar{\eta}^{(l)} - \eta^{(l)} \| &\leqq \sum_{p=0}^{nm-1} \| \bar{\eta}^{(p)} - \eta^{(p)} \| + \sum_{p=nm}^{nm+\nu} \| \bar{\eta}^{(p)} - \eta^{(p)} \| \\
&\leqq nM_2(M_1+1)(\sum_{q=0}^{m-1} M_1^q) \| \bar{\eta}^{(0)} - \eta^{(0)} \| + (\nu+1)M_2(M_1+1)M_1^m \| \bar{\eta}^{(0)} - \eta^{(0)} \| \\
&\leqq nM_2(M_1+1)(\sum_{q=0}^{\infty} M_1^q) \| \bar{\eta}^{(0)} - \eta^{(0)} \| = nM_2(1+M_1)(1-M_1)^{-1} \| \bar{\eta}^{(0)} - \eta^{(0)} \| \\
&= M_3 \| \bar{\eta}^{(0)} - \eta^{(0)} \|, \text{ q.e.d.}
\end{aligned}$$

In the following we put

$$M_4 = nN(1+M_1)(1-M_1)^{-1} \tag{9.8}$$

By (8.7), M_4 is positive.

Proposition 8. *Let the premises and the notations be the same as in the proposition 7 and further let*

$$z^{(p)} = z(x^{(p)}; y^{(p)}) \quad \textit{and} \quad \bar{z}^{(p)} = z(x^{(p)}; \bar{y}^{(p)}) \qquad \textit{for} \quad p = 0, \cdots, l,$$

then

$$|\bar{z}^{(0)} - z^{(0)}| \leqq M_4 \| \bar{y}^{(0)} - y^{(0)} \| + |\bar{z}^{(l)} - z^{(l)}|. \tag{9.9}$$

Proof. We use the same notations as in the proof of Proposition 7.

If $l=0$, (9.9) is obvious by $M_4>0$. Hence we suppose $l>0$ in the following lines.

By Proposition 5,

$$|\bar{z}^{(p)} - z^{(p)}| - |\bar{z}^{(p+1)} - z^{(p+1)}| \leqq N \| \bar{y}^{(p)} - y^{(p)} \| \qquad \text{for} \quad p = 0, \cdots, l-1, \tag{9.10}$$

Adding the inequalities (9.10) for $p=0, \cdots, l-1$ side by side, we obtain

$$|\bar{z}^{(0)} - z^{(0)}| - |\bar{z}^{(l)} - z^{(l)}| \leqq N \sum_{p=0}^{l-1} \| \bar{y}^{(p)} - y^{(p)} \|. \tag{9.11}$$

On the other hand, by Proposition 6,

$$\| \bar{y}^{(p)} - y^{(p)} \| \leqq (M_1+1)M_1^q \| \bar{y}^{(0)} - y^{(0)} \| \qquad \text{for} \quad p=0, \cdots, l,$$

where q is determined for p as in Proposition 7 (9.6). Hence m, ν being determined for l as in the definition of $T^{[l]}$,

$$\begin{aligned}
\sum_{p=0}^{l-1} \| \bar{y}^{(p)} - y^{(p)} \| &\leqq n(M_1+1)(\sum_{q=0}^{m-1} M_1^q) \| \bar{y}^{(0)} - y^{(0)} \| \\
&\quad + \nu(M_1+1)M_1^m \| \bar{y}^{(0)} - y^{(0)} \| \leqq n(M_1+1)(\sum_{q=0}^{\infty} M_1^q) \| \bar{y}^{(0)} - y^{(0)} \| \\
&= n(1+M_1)(1-M_1)^{-1} \| \bar{y}^{(0)} - y^{(0)} \|,
\end{aligned} \tag{9.12}$$

since $n-1\geqq\nu\geqq 0$ and $0<M_1<1$ by (8.7).

By (9.11), (9.12), taking account of (9.8), we obtain finally

$$|\bar{z}^{(0)}-z^{(0)}|-|\bar{z}^{(l)}-z^{(l)}|\leqq nN(1+M_1)(1-M_1)^{-1}\,\|\bar{y}^{(0)}-y^{(0)}\|$$
$$\leqq M_4\,\|\bar{y}^{(0)}-y^{(0)}\|,$$

q.e.d.

10. Domain Ω_3, G_3. We put

$$M_5=1+2(M_1+1)M_2+2M_3. \tag{10.1}$$

By (8.7), (8.8), (9.1), $M_5>1$. Hence if we take a positive number L_3 such that

$$L_3M_5<L_2, \tag{10.2}$$

then

$$L_3<L_2. \tag{10.3}$$

Now we take a domain Ω_3 in R^n defined by

$$\eta:\ \|\eta-b^{(1)}\|<L_3. \tag{10.4}$$

Then by (7.1), (10.3), (10.4), (7.3),

$$\Omega_3\subset\Omega_2\subset\Omega_1 \tag{10.5}$$

We denote by G_3 the portion of Q_1 covered by the family of all the characteristic curves $C(a;\eta|Q_1)$ where $\eta\in\Omega_3$. In the same way as in the cases of G_1, G_2, we easily prove that G_3 is open in R^{n+1}, and $C(\xi;\eta|G_3)$ where $(\xi;\eta)\in G_3$, is defined just for the interval $|x-a|<L_1$. By (10.5) and the definitions of G_1, G_2, G_3,

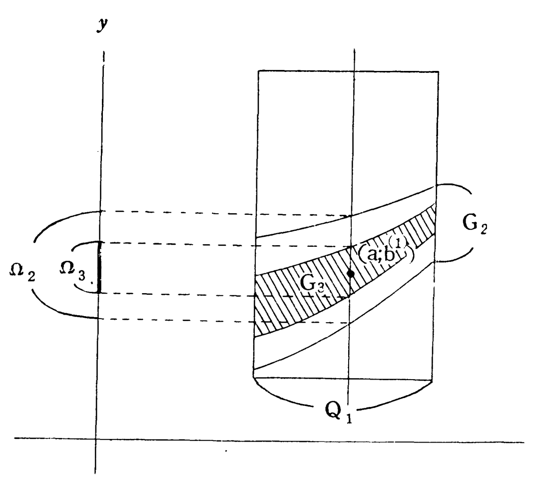

Fig. 5

$$G_3\subset G_2\subset G_1\subset Q_1.$$

Proposition 9. *If* $(x^{(0)};y^{(0)})\in F\cdot G_3$ *and* $(x^{(0)};\bar{y}^{(0)})\in G_3$, *then* $\{(x^{(0)};y^{(0)}),(x^{(0)};\bar{y}^{(0)})\}\in S^{[l]}$ *for any non-negative integer* l.

Proof. We use the same notations as in the Proposition 7 and 8.

We prove Proposition 9 by induction on l. For $l=0$, Proposition 9 is obvious by the definition of $S^{[0]}$ and $G_3\subset G_2$. We assume that

Proposition 9 is true for $l=l'$. Then if $(x^{(0)};y^{(0)})\in F\cdot G_3$ and $(x^{(0)};\bar{y}^{(0)})\in G_3$, the pair $\{(x^{(l')};y^{(l')}),\ (x^{(l')};\bar{y}^{(l')})\}=T^{[l']}\{(x^{(0)},y^{(0)}),\ (x^{(0)};\bar{y}^{(0)})\}$ is uniquely determined and $(x^{(l')};y^{(l')})\in F\cdot G_2$ and $(x^{(l')};\bar{y}^{(l')})\in G_2$ by the definition of $T^{[l']}$.

Let $l'=m'n+\nu'$, $n-1\geqq\nu'\geqq 0$ and $m',\nu'=$integer. Then we put $\tilde{y}^{(l')}=(y_1^{(l')},\cdots,y_{\nu'}^{(l')},\bar{y}_{\nu'+1}^{(l')},y_{\nu'+2}^{(l')},\cdots,y_n^{(l')})$. If $\tilde{y}^{(l')}\in G_2[x^{(l')}]$, that is, $(x^{(l')};\tilde{y}^{(l')})\in G_2$, then $\{(x^{(l')};y^{(l')}),\ (x^{(l')};\tilde{y}^{(l')})\}\in S_{\nu'+1}$ so that $\{(x^{(0)};y^{(0)}),\ (x^{(0)};\bar{y}^{(0)})\}\in S^{[l'+1]}$ and the proof of Proposition 9 is completed.

We suppose therefore, if possible, that $\tilde{y}^{(l')}\notin G_2[x^{(l')}]$. Then there is a point $y^*\in(G_2[x^{(l')}])^b$ on the segment of straight line which joins $\bar{y}^{(l')}$ and $\tilde{y}^{(l')}$ since $\bar{y}^{(l')}\in G_2[x^{(l')}]$ by $(x^{(l')};\bar{y}^{(l')})\in G_2$.

We can easily see that

$$\| y^*-\bar{y}^{(l')}\| \leqq \| \tilde{y}^{(l')}-\bar{y}^{(l')}\| \leqq \| y^{(l')}-\bar{y}^{(l')}\| ,$$

so that by Proposition 6 (8.9)

$$\| y^*-\bar{y}^{(l')}\| \leqq (M_1+1)M_1^{m'}\| \bar{y}^{(0)}-y^{(0)}\| . \tag{10.6}$$

By Proposition 4, since $y^*\in(G_2[x^{(l')}])^b$, $y^*\in G_1[x^{(l')}]$, that is, $(x^{(l')};y^*)\in G_1$ and if we put $\eta^*=\varphi(a,x^{(l')};y^*|G_1)$, $\eta^*\in\Omega_2^b$. Hence by (7.1)

$$\| \eta^*-b^{(1)}\| =L_2 . \tag{10.7}$$

By (6.2) and Lemma 3 (4.2), taking account of (8.6) and $|a-x^{(l')}|<L_1$, we have

$$\begin{aligned}\| \eta^*-\bar{\eta}^{(l')}\| &\leqq \| y^*-\bar{y}^{(l')}\| \exp(nM_0|a-x^{(l')}|)\\ &\leqq \| y^*-\bar{y}^{(l')}\| \exp(nM_0L_1)=\sqrt{M_2}\,\| y^*-\bar{y}^{(l')}\|\end{aligned}$$

and so by (10.6)

$$\| \eta^*-\bar{\eta}^{(l')}\| \leqq\sqrt{M_2}(M_1+1)M_1^{m'}\| \bar{y}^{(0)}-y^{(0)}\| . \tag{10.8}$$

On the other hand, by the definitions of $\bar{\eta}^{(0)}$, $\eta^{(0)}$, G_3 and by $(x^{(0)};y^{(0)})\in F\cdot G_3$, $(x^{(0)};\bar{y}^{(0)})\in G_3$, we have

$$\bar{\eta}^{(0)},\ \eta^{(0)}\in\Omega_3 ,$$

so that by (10.4),

$$\| \bar{\eta}^{(0)}-b^{(1)}\| <L_3 ,\quad \| \eta^{(0)}-b^{(1)}\| <L_3 . \tag{10.9}$$

Also, as we have seen in Proposition 7 (9.5),

$$\| \bar{y}^{(0)}-y^{(0)}\| \leqq\sqrt{M_2}\,\| \bar{\eta}^{(0)}-\eta^{(0)}\| .$$

Hence, by (10.9)

$$\| \bar{y}^{(0)}-y^{(0)} \| \leqq \sqrt{M_2}(\| \bar{\eta}^{(0)}-b^{(1)} \| + \| \eta^{(0)}-b^{(1)} \|) < 2L_3\sqrt{M_2} .$$

From this and (10.8) we get

$$\| \eta^* - \bar{\eta}^{(l')} \| < 2L_3M_2(M_1+1)M_1^{m'} \tag{10.10}$$

Also, by Proposition 7, (9.2) and (10.9), we have

$$\| \bar{\eta}^{(l')} - \eta^{(0)} \| \leqq M_3 \| \bar{\eta}^{(0)} - \eta^{(0)} \| < 2L_3M_3 \tag{10.11}$$

By (10.7), (10.9), (10.10), and (10.11), taking account of (8.7), (10.1), we obtain finally

$$L_2 = \| \eta^* - b^{(1)} \| \leqq \| \eta^* - \bar{\eta}^{(l')} \| + \| \bar{\eta}^{(l')} - \eta^{(0)} \| + \| \eta^{(0)} - b^{(1)} \|$$
$$< L_3\{2M_2(M_1+1)M_1^{m'}+2M_3+1\} \leqq L_3\{2M_2(M_1+1)+2M_3+1\} = L_3M_5 .$$

But this contradicts (10.2) and Proposition 9 is completely proved.

Proposition 10. *Let* $\{(x'\,;y'), (x'\,;\bar{y}')\}$ *be any pair of points of* G_3 *with the same x coordinate and let*

$$z' = z(x'\,;y'), \quad \bar{z}' = z(x'\,;\bar{y}') .$$

Then

$$| \bar{z}' - z' | \leqq M_2M_4 \| \bar{y}' - y' \| \tag{10.12}$$

Proof. There is the nearest x to x' in the interval $|x-a|<L_1$ such that either $(x, \varphi(x, x'\,;y'|G_3)) \in F \cdot G_3$ or $(x, \varphi(x, x'\,;\bar{y}'|G_3)) \in F \cdot G_3$, since by Proposition 3 and $G_3 \subset G_2$, each of the continuous curves $C(x'\,;y'|G_3)$ and $C(x'\,;\bar{y}'|G_3)$ which are defined just for the interval $|x-a|<L_1$ and are contained in G_3, has at least one point in common with $F \cdot G_3$ which is closed in G_3. We denote such x by $x^{(0)}$. If incidently two such x exist, we take as $x^{(0)}$ for example the one on the right side of x'.

Now we distinguish two cases.

i) If $(x^{(0)}, \varphi(x^{(0)}, x'\,;y'|G_3)) \in F \cdot G_3$, then we put

$$y^{(0)} = \varphi(x^{(0)}, x'\,;y'|G_3) , \quad \bar{y}^{(0)} = \varphi(x^{(0)}, x'\,;\bar{y}'|G_3)$$

ii) If $(x^{(0)}, \varphi(x^{(0)}, x'\,;y'|G_3)) \notin F \cdot G_3$ and so by the definition of $x^{(0)}$, $(x^{(0)}, \varphi(x^{(0)}, x'\,;\bar{y}'|G_3)) \in F \cdot G_3$, then we put

$$y^{(0)} = \varphi(x^{(0)}, x'\,;\bar{y}'|G_3) , \quad \bar{y}^{(0)} = \varphi(x^{(0)}, x'\,;y'|G_3) .$$

In any case $(x^{(0)}\,;y^{(0)}) \in F \cdot G_3$ and $(x^{(0)}\,;\bar{y}^{(0)}) \in G_3$.

By the definition of $x^{(0)}$, each of the characteristic curves $C(x'\,;y'|G_3)$

and $C(x'; \bar{y}'|G_3)$ has no point in common with F and so is contained in K for the interval $x'<x<x^{(0)}$ or $x^{(0)}<x<x'$. Hence by Lemma 1, if we put $z^{(0)}=z(x^{(0)}; y^{(0)})$ and $\bar{z}^{(0)}=z(x^{(0)}; \bar{y}^{(0)})$,

$$\begin{cases} z^{(0)}=z' \\ \bar{z}^{(0)}=\bar{z}' \end{cases} \quad \text{or} \quad \begin{cases} z^{(0)}=\bar{z}' \\ \bar{z}^{(0)}=z' . \end{cases}$$

Therefore in any case,

$$|\bar{z}'-z'|=|\bar{z}^{(0)}-z^{(0)}|. \tag{10.13}$$

By Lemma 3 (4.2) and (6.2), taking account of (8.6), we have

$$\begin{aligned} \|\bar{y}^{(0)}-y^{(0)}\| &\leqq \|\bar{y}'-y'\| \exp(nM_0|x^{(0)}-x'|) \\ &\leqq \|\bar{y}'-y'\| \exp(2nM_0L_1) \leqq M_2\|\bar{y}'-y'\| , \end{aligned} \tag{10.14}$$

since $|x^{(0)}-x'|<2L_1$.

Now, by Proposition 9, $\{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} \in S^{[l]}$ for any non-negative integer l, since $(x^{(0)}; y^{(0)}) \in F\cdot G_3$ and $(x^{(0)}; \bar{y}^{(0)}) \in G_3$. Hence we put

$$\{(x^{(l)}; y^{(l)}), (x^{(l)}; \bar{y}^{(l)})\} = T^{[l]}\{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} ,$$
$$z^{(l)}=z(x^{(l)}; y^{(l)}), \quad \bar{z}^{(l)}=z(x^{(l)}; \bar{y}^{(l)})$$

for any non-negative integer l. Then by Proposition 8

$$|\bar{z}^{(0)}-z^{(0)}| \leqq M_4\|\bar{y}^{(0)}-y^{(0)}\| + |\bar{z}^{(l)}-z^{(l)}| \tag{10.15}$$

On the other hand, by Proposition 6, if $l=nm+\nu$, $n-1\geqq\nu\geqq0$ and $m, \nu=$integer, then

$$\|\bar{y}^{(l)}-y^{(l)}\| \leqq (M_1+1)M_1^m\|\bar{y}^{(0)}-y^{(0)}\| .$$

Hence observing that $m\to\infty$ as $l\to\infty$ and $0<M_1<1$,

$$\|\bar{y}^{(l)}-y^{(l)}\| \to 0 \quad \text{as} \quad l\to\infty ,$$

Thus

$$|\bar{z}^{(l)}-z^{(l)}|=|z(x^{(l)}; \bar{y}^{(l)})-z(x^{(l)}; y^{(l)})| \to 0 \quad \text{as} \quad l\to\infty ,$$

since by the continuity of $z(x; y)$ on $\bar{Q}(\subset G)$, $z(x; y)$ is uniformly continuous on $\bar{Q}$ which is closed and bounded in R^{n+1} and by the definition of $T^{[l]}$, $(x^{(l)}; y^{(l)})$, $(x^{(l)}; \bar{y}^{(l)}) \in G_2 \subset Q$ for any non-negative integer l.

Therefore letting $l\to\infty$ on the right side of (10.15), we have

$$|\bar{z}^{(0)}-z^{(0)}| \leqq M_4\|\bar{y}^{(0)}-y^{(0)}\| . \tag{10.16}$$

By (10.13), (10.14) and (10.16), we obtain finally

$$|\bar{z}'-z'| \leqq M_2M_4 \|\bar{y}'-y'\| ,$$

q.e.d.

11. Domains Q', Q'', Ω_4, G_4 and Mapping $\mathfrak{A}$. Since G_3 is open in R^{n+1}, and $F\cdot G_3 \neq 0$ by the way of the construction of G_3 and Proposition 3, we can take a $(n+1)$-dimensional open parallelepiped Q':

$$|x-a'|<L_4, \quad |y_\lambda-b_\lambda'|<L_4(M_0+1) \qquad \lambda=1, \cdots, n \quad (L_4>0)$$

such that

$$(a'\,;\,b')=(a', b_1', \cdots, b_n')\in F\cdot G_3 \quad \text{and} \quad Q'\subset G_3 .$$

Evidently $Q'\subset G_3\subset Q$.

We denote by Ω_4 the n-dimensional open cube

$$\eta: |\eta_\lambda-b_\lambda'|<L_4 \qquad \lambda=1, \cdots, n.$$

Then if $\eta\in\Omega_4$,

$$\eta_\lambda+L_4M_0 \leqq b_\lambda'+(M_0+1)L_4, \quad \eta_\lambda-L_4M_0 \geqq b_\lambda'-(M_0+1)L_4 \qquad \lambda=1, \cdots, n.$$

Hence the characteristic curves $C(a'\,;\,\eta|Q')$ where $\eta\in\Omega_4$, are defined just for the interval $|x-a'|<L_4$ since $|f_\lambda|<M_0$ $\lambda=1, \cdots, n$ on $Q'(\subset Q)$ by (6.1).

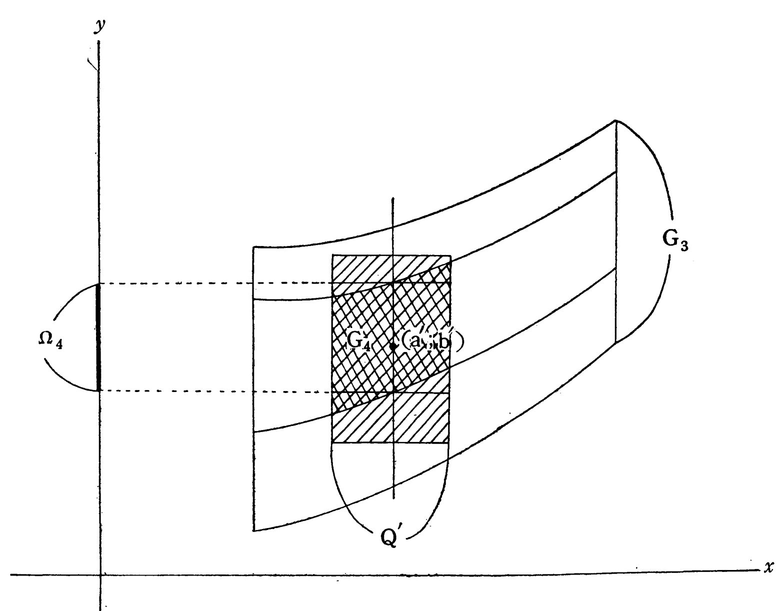

Fig. 6

We denote by G_4 the portion of Q' covered by the family of all the characteristic curves $C(a'; \eta | Q')$ where $\eta \in \Omega_4$. Evidently $G_4 \subset Q' \subset G_3 \subset Q$.

In the same way as in the cases of G_1, G_2 and G_3, we easily prove that G_4 is open in R^{n+1} and any characteristic curve $C(\xi; \eta | G_4)$ where $(\xi; \eta) \in G_4$ is defined just for $|x-a'| < L_4$.

We put

$$M_5 = N + nM_0M_2M_4 + M_2M_4. \tag{11.1}$$

Proposition 11. *Let* $(x^{(1)}; y^{(1)})$ *and* $(x^{(0)}; y^{(0)})$ *be any pair of points of* G_4 *and let*

$$z^{(1)} = z(x^{(1)}; y^{(1)}), \quad z^{(0)} = z(x^{(0)}; y^{(0)}).$$

Then

$$|z^{(1)} - z^{(0)}| \leq M_5(|x^{(1)} - x^{(0)}| + \| y^{(1)} - y^{(0)} \|) \tag{11.2}$$

Proof. We denote by $x^{(2)}$:

(Case I) the nearest x to $x^{(1)}$ in the interval $x^{(0)} \leqq x \leqq x^{(1)}$ or $x^{(1)} \leqq x \leqq x^{(0)}$ such that $(x; \varphi(x, x^{(1)}; y^{(1)} | G_4)) \in F$, if the portion of $C(x^{(1)}; y^{(1)} | G_4)$ for that interval has some points in common with F. Such x exists in this case since the continuous curve $C(x^{(1)}; y^{(1)} | G_4)$ which is defined for the interval $x^{(0)} \leqq x \leqq x^{(1)}$ or $x^{(1)} \leqq x \leqq x^{(0)}$, is contained in G_4 and $F \cdot G_4$ is closed in G_4.

(Case II) the number $x^{(0)}$, if the portion of $C(x^{(1)}; y^{(1)} | G_4)$ for the interval $x^{(0)} \leqq x \leqq x^{(1)}$ or $x^{(1)} \leqq x \leqq x^{(0)}$ has no point in common with F.

We put $y^{(2)} = \varphi(x^{(2)}, x^{(1)}; y^{(1)} | G_4)$, $z^{(2)} = z(x^{(2)}; y^{(2)})$. In Case I, $(x^{(2)}; y^{(2)}) \in F \cdot G_4$ and in Case II, $(x^{(2)}; y^{(2)}) = (x^{(0)}; \varphi(x^{(0)}, x^{(1)}; y^{(1)} | G_4)) \in G_4$.

Then in both Cases, by Lemma 1,

$$z^{(1)} = z^{(2)}, \tag{11.3}$$

since the portion of $C(x^{(1)}; y^{(1)} | G_4)$ for the open interval $x^{(1)} < x < x^{(2)}$ or $x^{(2)} < x < x^{(1)}$ has no point in common with F and so is contained in K by the definition of $x^{(2)}$.

Also by (6.1), (2.2), observing that $C(x^{(1)}; y^{(1)} | G_4)$ is contained in Q, we have

$$\begin{aligned} |y_\lambda^{(2)} - y_\lambda^{(1)}| &= |\varphi_\lambda(x^{(2)}, x^{(1)}; y^{(1)} | G_4) - \varphi_\lambda(x^{(1)}, x^{(1)}; y^{(1)} | G_4)| \\ &\leq M_0 |x^{(2)} - x^{(1)}| \qquad \lambda = 1, \cdots, n. \end{aligned}$$

Hence

$$\| y^{(2)} - y^{(1)} \| \leq \sum_{\mu=1}^{n} |y_\mu^{(2)} - y_\mu^{(1)}| \leq nM_0 |x^{(2)} - x^{(1)}| \leq nM_0 |x^{(1)} - x^{(0)}|,$$

since $x^{(0)} \leq x^{(2)} \leq x^{(1)}$ or $x^{(1)} \leq x^{(2)} \leq x^{(0)}$ by the definition of $x^{(2)}$.

Therefore we have

$$\| y^{(2)}-y^{(0)} \| \leqq \| y^{(2)}-y^{(1)} \| + \| y^{(1)}-y^{(0)} \|$$
$$\leqq nM_0 | x^{(1)}-x^{(0)} | + \| y^{(1)}-y^{(0)} \| . \tag{11.4}$$

Now

$$(x^{(0)} ; y^{(2)}) \in Q' \subset G_3 \subset Q ,$$

since $(x^{(0)} ; y^{(0)}) \in G_4 \subset Q'$ and $(x^{(2)} ; y^{(2)}) \in G_4 \subset Q'$ in both Cases. We put $z^{(3)} = z(x^{(0)} ; y^{(2)})$.

Then we have

$$| z^{(3)}-z^{(2)} | = | z(x^{(0)} ; y^{(2)}) - z(x^{(2)} ; y^{(2)}) | \leqq N | x^{(2)}-x^{(0)} |$$

in Case I, by Proposition 1 and $(x^{(0)} ; y^{(2)}) \in Q$, $(x^{(2)} ; y^{(2)}) \in F \cdot G_4 \subset F \cdot Q$, and in Case II, simply as $x^{(2)} = x^{(0)}$. Hence, by (11.3), we get

$$| z^{(3)}-z^{(1)} | = | z^{(3)}-z^{(2)} | \leqq N | x^{(2)}-x^{(0)} | \leqq N | x^{(1)}-x^{(0)} | , \tag{11.5}$$

since $x^{(0)} \leqq x^{(2)} \leqq x^{(1)}$ or $x^{(1)} \leqq x^{(2)} \leqq x^{(0)}$ by the definition of $x^{(2)}$.

Also, by Proposition 10 (10.12), since $(x^{(0)} ; y^{(2)}) \in G_3$, $(x^{(0)} ; y^{(0)}) \in G_3$ in both Cases, we have

$$| z^{(3)}-z^{(0)} | \leqq M_2 M_4 \| y^{(2)}-y^{(0)} \| .$$

Hence by (11.4), we get

$$| z^{(3)}-z^{(2)} | \leqq nM_0 M_2 M_4 | x^{(1)}-x^{(0)} | + M_2 M_4 \| y^{(1)}-y^{(0)} \| . \tag{11.6}$$

By (11.5), (11.6), taking account of (11.1), we obtain finally

$$| z^{(1)}-z^{(0)} | \leqq | z^{(3)}-z^{(1)} | + | z^{(3)}-z^{(0)} |$$
$$\leqq (N + nM_0 M_2 M_4) | x^{(1)}-x^{(0)} | + M_2 M_4 \| y^{(1)}-y^{(0)} \|$$
$$\leqq M_5 (| x^{(1)}-x^{(0)} | + \| y^{(1)}-y^{(0)} \|), \qquad \text{q.e.d.}$$

We denote by Q'' the $(n+1)$-dimensional open cube defined by

$$(x ; \eta) : \ | x-a' | < L_4 ,$$
$$| \eta_\lambda - b_\lambda' | < L_4 \quad \lambda = 1, \cdots , n.$$

We put $\chi_\lambda(x ; \eta) = \varphi_\lambda(x, a' ; \eta | G_4)$, $\lambda = 1, \cdots , n$. Then $\chi_\lambda(x ; \eta)$ are defined and continuous on Q'' and have continuous partial derivatives with respect to all their arguments on Q'', by the corresponding properties of $\varphi_\lambda(x, \xi ; \eta | G_4)$.

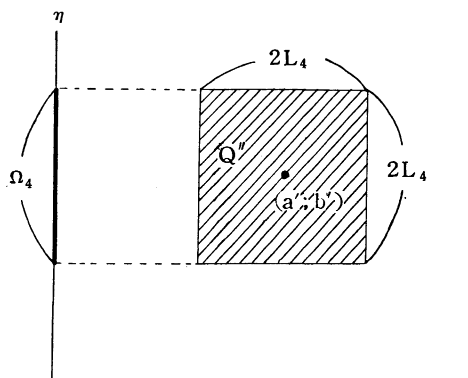

Fig. 7

We denote by $\mathfrak{A}$ the continuous mapping of Q'' onto G_4:

$$(x\,;\eta)\rightarrow(x\,;\chi(x\,;\eta))\,.$$

That $\mathfrak{A}$ maps Q'' onto G_4, follows from the definition of G_4.

By the properties of $C(\xi\,;\eta|G_4)$ and $\varphi_\lambda(x,\xi\,;\eta|G_4)$ as stated in §1.2, we easily see that $\mathfrak{A}$ is one to one and bicontinuous, and $\mathfrak{A}^{-1}$ is represented by

$$(x\,;y)\rightarrow(x\,;\gamma(x\,;y))\,,$$

if we put $\gamma_\lambda(x\,;y)=\varphi_\lambda(a',x\,;y|G_4)$ $\lambda=1,\cdots,n$ for $(x\,;y)\in G_4$. Further $\gamma_\lambda(x\,;y)$ have continuous partial derivatives with respect to all their arguments by the corresponding properties of $\varphi_\lambda(x,\xi\,;\eta|G_4)$.

From this, we can easily prove that $\mathfrak{A}^{-1}$ maps any null set in G_4 onto a null set in Q''[9].

Thus we have

Proposition 12. *The mapping $\mathfrak{A}$ of Q'' onto G_4 is one to one and bicontinuous, and $\mathfrak{A}^{-1}$ maps any null set in G_4 onto a null set in Q''.*

12. Completion of the proof. By Proposition 11, we have

$$\limsup_{(x\,;y)\rightarrow(x^{(0)};y^{(0)})}\frac{|z(x\,;y)-z(x^{(0)};y^{(0)})|}{|x-x^{(0)}|+\|y-y^{(0)}\|}\leqq M_5\,, \tag{12.1}$$

whenever $(x^{(0)};y^{(0)})\in G_4$. Hence $z(x\,;y)$ is totally differentiable almost everywhere in G_4, by a theorem of Rademacher on almost everywhere total differentiability[10]. Also, by Proposition 12, $\mathfrak{A}^{-1}$ maps any null set in G_4 onto a null set in Q''. Therefore, if we write $\zeta(x\,;\eta)=z(x\,;\chi(x\,;\eta))$, and $y_\lambda=\chi_\lambda(x\,;\eta)$ $\lambda=1,\cdots,n$ for $(x\,;\eta)\in Q''$, we obtain

$$\frac{\partial}{\partial x}\zeta(x\,;\eta)=\frac{\partial z}{\partial x}(x\,;y)+\sum_{\mu=1}^{n}\frac{\partial z}{\partial y_\mu}(x\,;y)\frac{\partial\chi_\mu}{\partial x}(x\,;\eta) \tag{12.2}$$

for almost all $(x\,;\eta)$ of Q''.

Since $\chi_\lambda(x\,;\eta)=\varphi_\lambda(x,a'\,;\eta|G_4)$ for $(x\,;\eta)\in Q''$, we obtain by (2.2),

$$\frac{\partial}{\partial x}\chi_\lambda(x\,;\eta)=f_\lambda(x\,;\chi(x\,;\eta))=f_\lambda(x\,;y)\qquad\lambda=1,\cdots,n \tag{12.3}$$

for $(x\,;\eta)\in Q''$. Substituting this into (12.2), we get

$$\frac{\partial}{\partial x}\zeta(x\,;\eta)=\frac{\partial z}{\partial x}(x\,;y)+\sum_{\mu=1}^{n}f_\mu(x\,;y)\frac{\partial z}{\partial y_\mu}(x\,;y) \tag{12.4}$$

9) Cf. Tsuji [9], pp. 49–50. Also Cf. Rademacher [7], pp. 354–355.

10) Cf. Rademacher [7], pp. 341–347. Also Cf. Saks [8], p. 311.

for almost all $(x;\eta)$ of Q''.

Since by assumption, $z(x;y)$ satisfies (2.1) almost everywhere in $G(\supset G_4)$ and by Proposition 12, $\mathfrak{A}^{-1}$ maps any null set in G_4 onto null set in Q'', the right side of (12.4) regarded as a function of $(x;\eta)$, vanishes almost everywhere in Q''. Therefore

$$\frac{\partial}{\partial x}\zeta(x;\eta)=0 \tag{12.5}$$

almost everywhere in Q''.

On the other hand, if we write $y_\lambda^{(0)}=\chi_\lambda(x^{(0)};\eta^{(0)})$ $\lambda=1,\cdots,n$ for a point $(x^{(0)};\eta^{(0)})\in Q''$, we have $(x^{(0)};y^{(0)})\in G_4$ and

$$\limsup_{x\to x^{(0)}}\frac{|\zeta(x;\eta^{(0)})-\zeta(x^{(0)};\eta^{(0)})|}{|x-x^{(0)}|}$$
$$\leqq \limsup_{x\to x^{(0)}}\frac{|\zeta(x;\eta^{(0)})-\zeta(x^{(0)};\eta^{(0)})|}{|x-x^{(0)}|+\|\chi(x;\eta^{(0)})-\chi(x^{(0)};\eta^{(0)})\|}$$
$$\times\limsup_{x\to x^{(0)}}\frac{|x-x^{(0)}|+\|\chi(x;\eta^{(0)})-\chi(x^{(0)};\eta^{(0)})\|}{|x-x^{(0)}|}$$
$$\leqq\left(\limsup_{(x;y)\to(x^{(0)};y^{(0)})}\frac{|z(x;y)-z(x^{(0)};y^{(0)})|}{|x-x^{(0)}|+\|y-y^{(0)}\|}\right)\left(1+\sum_{\mu=1}^{n}\left|\frac{\partial\chi_\mu}{\partial x}(x^{(0)};\eta^{(0)})\right|\right).$$

Hence by (12.1) and (12.3), observing that $|f_\lambda(x;y)|<M_0$ on $G_4(\subset Q)$ by (6.1), we obtain

$$\limsup_{x\to x^{(0)}}\frac{|\zeta(x;\eta^{(0)})-\zeta(x^{(0)};\eta^{(0)})|}{|x-x^{(0)}|}\leqq M_5\Big(1+\sum_{\mu=1}^{n}|f_\mu(x^{(0)};y^{(0)})|\Big)$$
$$\leqq M_5(1+nM_0) \tag{12.6}$$

whenever $(x^{(0)};\eta^{(0)})\in Q''$.

By Fubini's theorem, $\zeta(x;\eta)$ as a function of x, satisfies (12.5) almost everywhere in the interval $|x-a'|<L_4$, for almost every η in the domain Ω_4 and by (12.6), $\zeta(x;\eta)$ as a function of x, is absolutely continuous in the interval $|x-a'|<L_4$ for any η in the domain Ω_4.

Therefore by Lebesgue's theorem, $\zeta(x;\eta)$ as a function of x, is constant in the interval $|x-a'|<L_4$ for almost every η in the domain Ω_4. Hence, by the continuity of $z(x;y)$ and $\chi_\lambda(x;\eta)$, accordingly of $\zeta(x;\eta)$, it follows that $\zeta(x;\eta)$ as a function of x, is constant in the interval $|x-a'|<L_4$ for any η in the domain Ω_4.

From this, by the definition of $\zeta(x;\eta)$, we easily see that $z(x;y)$ is constant on any characteristic curve of (2.1) in G_4. Hence, by the definition of K, we have $G_4\subset K$ and so observing that $G_4(\subset G)$ is open in R^{n+1},

$$F=\overline{G-K}\cdot G\subset\overline{G-G_4}\cdot G=G-G_4\,.$$

This is however excluded, since $(a';b')\in F\cdot G_4$. Thus we arrive at a contradiction and this completes the proof of Theorem 1.

§ 4. Proof of Theorem 2 and Theorem 3

In this §, the notations are the same as in § 1 and § 2.

13. Proof of Theorem 2. By the assumption on G and Theorem 1, if we put $\omega_\lambda(x;y)=\varphi_\lambda(\xi^{(0)},x;y|G)$ $\lambda=1,\cdots,n$ for $(x;y)\in G$, then we have $\omega(x;y)\in G[\xi^{(0)}]$ for $(x;y)\in G$ and

$$z(x;y)=\psi(\omega(x;y))\quad \text{on } G \tag{13.1}$$

for any quasi-solution $z(x;y)$ of (2.1) on G such that $z(\xi^{(0)};\eta)=\psi(\eta)$ on $G[\xi^{(0)}]$. Hence there is at most only one such quasi-solution.

Conversely if we define a function $z(x;y)$ by the right side of (13.1) on G, then by the total differentiability of $\psi(\eta)$ on $G[\xi^{(0)}]$ and of $\omega_\lambda(x;y)$ on G, $z(x;y)$ is totally differentiable on G and

$$\frac{\partial z}{\partial x}=\sum_{\mu=1}^{n}\frac{\partial\psi}{\partial\eta_\mu}\frac{\partial\omega_\mu}{\partial x},\quad \frac{\partial z}{\partial y_\lambda}=\sum_{\mu=1}^{n}\frac{\partial\psi}{\partial\eta_\mu}\frac{\partial\omega_\mu}{\partial y_\lambda}\qquad \lambda=1,\cdots,n$$

on G. Hence

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n}f_\mu(x;y)\frac{\partial z}{\partial y}=\sum_{\lambda=1}^{n}\frac{\partial\psi}{\partial\eta_\lambda}\left(\frac{\partial\omega_\lambda}{\partial x}+\sum_{\mu=1}^{n}\frac{\partial\omega_\lambda}{\partial y_\mu}f_\mu(x;y)\right)\quad \text{on } G. \tag{13.2}$$

But for $\omega_\lambda(x;y)$ $(=\varphi_\lambda(\xi^{(0)},x;y|G))$, we have[11]

$$\frac{\partial\omega_\lambda}{\partial x}+\sum_{\mu=1}^{n}f_\mu(x;y)\frac{\partial\omega_\lambda}{\partial y_\mu}=0\qquad \lambda=1,\cdots,n\quad \text{on } G.$$

Therefore by (13.2), for $z(x;y)$ defined by (13.1)

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n}f_\mu(x;y)\frac{\partial z}{\partial y_\mu}=0\quad \text{on } G.$$

Also for $z(x;y)$ defined by (13.1), we have

$$z(\xi^{(0)};\eta)=\psi(\eta)\quad \text{on } G[\xi^{(0)}],$$

since $\omega_\lambda(\xi^{(0)};\eta)=\varphi_\lambda(\xi^{(0)},\xi^{(0)};\eta|G)=\eta$.

Thus there is at least one quasi-solution $z(x;y)$ of (2.1) on G such that $z(\xi^{(0)};\eta)=\psi(\eta)$ on $G[\xi^{(0)}]$ and this quasi-solution is also a solution

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11) Cf. Kamke [3], §18, Nr. 87, Satz 1.

of (2.1) on G in the ordinary sense. This completes the proof of Theorem 2.

14. Proof of Theorem 3. For the special case $n=1$, we write (2.1) in the form

$$\frac{\partial z}{\partial x}+f(x, y)\frac{\partial z}{\partial y}=0 \tag{14.1}$$

and the characteristic curve of (14.1) in G which passes through the point (ξ, η) of G, in the form

$$y=\varphi(x, \xi, \eta\,|\,G) \quad \alpha(\xi, \eta\,|\,G)<x<\beta(\xi, \eta\,|\,G)\,.$$

Let $z(x, y)$ be any quasi-solution of (14.1) on G and $(x^{(0)}, y^{(0)})$ be any point of G. Then there is at least one point $(\xi^{(0)}, \eta^{(0)})$ on $C(x^{(0)}, y^{(0)}\,|\,G)$ where $z(x, y)$ has $\partial z/\partial y$, since $z(x, y)$ has $\partial z/\partial y$ except at most at the points of an enumerable set in G.

If we put $\omega(x, y)=\varphi(\xi^{(0)}, x, y\,|\,G)$ and for $\eta\in G[\xi^{(0)}]$, $\psi(\eta)=z(\xi^{(0)}, \eta)$, then by the properties of the family of the characteristic curves as stated in § 1.2, $\omega(x, y)$ is defined and $\omega(x, y)\in G[\xi^{(0)}]$ for (x, y) in some neighbourhood of $(x^{(0)}, y^{(0)})$ and by Theorem 1

$$z(x, y)=\psi(\omega(x, y)) \tag{14.2}$$

in that neighbourhood. Evidently $\omega(x^{(0)}, y^{(0)})=\varphi(\xi^{(0)}, x^{(0)}, y^{(0)}\,|\,G)$ $=\eta^{(0)}\in G[\xi^{(0)}]$. Also $\psi(\eta)$ $(=z(\xi^{(0)}, \eta))$ is differentiable at $\eta^{(0)}$ since $z(x, y)$ has $\partial z/\partial y$ at $(\xi^{(0)}, \eta^{(0)})$.

Since $\psi(\eta)$ is differentiable at $\eta^{(0)}=\omega(x^{(0)}, y^{(0)})$ and $\omega(x, y)$ $(=\varphi(\xi^{(0)}, x, y\,|\,G))$ is totally differentiable at $(x^{(0)}, y^{(0)})$, by (14.2) $z(x, y)$ is totally differentiable at $(x^{(0)}, y^{(0)})$ and

$$\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)})=\psi'(\eta^{(0)})\frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)})\,,$$

$$\frac{\partial z}{\partial y}(x^{(0)}, y^{(0)})=\psi'(\eta^{(0)})\frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)})\,.$$

Hence

$$\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)})+f(x^{(0)}, y^{(0)})\frac{\partial z}{\partial y}(x^{(0)}, y^{(0)})$$

$$=\psi'(\eta^{(0)})\left\{\frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)})+f(x^{(0)}, y^{(0)})\frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)})\right\}\,. \tag{14.3}$$

But for $\omega(x, y)=\varphi(\xi^{(0)}, x, y)$, we have[12]

12) Cf. Kamke [3], §18, Nr. 87, Satz 1.

$$\frac{\partial\omega}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)})\frac{\partial\omega}{\partial y}(x^{(0)}, y^{(0)}) = 0 .$$

Hence, by (14. 3)

$$\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)})\frac{\partial z}{\partial y}(x^{(0)}, y^{(0)}) = 0 .$$

Therefore $z(x, y)$ is totally differentiable and satisfies (14. 1) at any point $(x^{(0)}, y^{(0)})$ of G, q.e.d.

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References

[1] R. Baire; Sur les fonctions de variables réelles, Ann. Mat. Pura e Appl. (3), **3**, (1899), 1–122.
[2] F. Hausdorff: Mengenlehre, 2. Aufl. (1927).
[3] E. Kamke: Differentialgleichungen reeller Funktionen, (1930).
[4] T. Kasuga: Generalisation of R. Baire's theorem on differential equation, Proc. Japan. Acad., **27** (1951), 117–121.
[5] T. Kusuga: On the linear partial differential equation of the first order, Osaka Math. J. **5** (1953), 211–225.
[6] P. Montel: Sur les suites infinies de fonctions, Ann. Ecole Norm. (3), **24** (1907), 233–334.
[7] H. Rademacher: Ueber partielle und totale Differentierbarkeit I, Math. Ann. **79** (1919), 340–359.
[8] S. Saks: Theory of integral, 2nd ed. (1937).
[9] M. Tsuji: Change of variables in the multiple Lebesgue integrals, J. Math. Soc. Japan **2** (1950), 48–56.