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On the Existence of Harmonic Functions on Riemann Surfaces

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H. L. Royden[1)] proved by the use of the theory of Banach algebra that the class of O_{HD} or O_{HDN}[2)] is invariant under a quasi-conformal mapping whose dilatation quotient is bounded. The purpose of this article is to give a function-theoretic proof of the theorem.

Let F be an abstract Riemann surface, let $\{F_n\}$ be its exhaustion and let G be a non-compact subdomain whose relative boundary ∂G[3)] consists of at most an enumerably infinite number of analytic curves clustering nowhere in F.

Theorem. (EXTENSION OF L. MYRBERG'S THEOREM[4)]). *Let $U(p): p \in F$ be a harmonic function on an abstract Riemann surface F such that $D_F(U(p)) < \infty$ and suppose that the universal covering surface F^∞ of F is mapped onto the unit-circle: $|z| < 1$. Then $U(p)$ is represented by Poisson's integral.*

Lemma. *Let $V(z)$ be a continuous sub-harmonic function on $|z| \leqq 1$ such that $\int |V(e^{i\theta})| d\theta \leqq M$ and let G be a simply connected domain in $|z| < 1$ with a rectifiable boundary ∂G. Then $\int_{\partial G} |V(z)| d\omega \leqq M$, where ω is the harmonic measure of ∂G with respect to G.*

Proof of the lemma. Denote by $V^*(z)$ the upper envelope of sub-harmonic functions $\{V^i(z)\}$ such that $O \leqq V^i(z) \leqq |V(z)|$ on the complementary set of G. Then $V^*(z) = |V(z)|$ on ∂G and sub-harmonic in $|z| < 1$ and is harmonic in G. Let $V^{**}(z)$ be a harmonic function in $|z| < 1$ with the boundary value $|V(e^{i\theta})|$. Then

$$M \geqq \int |V(e^{i\theta})| d\theta = \int V^{**}(e^{i\theta}) d\theta = V^{**}(O) \geqq V^*(O) = \int_{\partial G} V^*(z) d\omega.$$

Map the universal covering surface F^∞ of F onto $|z| < 1$. Then the circle: $|z| < \rho$ $(\rho < 1)$ is contained in the image of F_n^∞ for sufficiently large number n. Denote by $G_n(p, p_0)$ the Green's function of F_n and denote by $h_n(p, p_0)$ its conjugate function, where p_0 is the image of $z = 0$. Put $e^{-G_n - ih_n} = r_n e^{i\varphi_n}$. Then

$$D_{F_n}(U(p)) = \frac{1}{2}\int_{\partial F_n}\frac{\partial U^2(r_n e^{i\varphi_n})}{\partial r_n} r_n d\varphi_n \leqq M'$$

and

$$\int_{\partial F_n} U^2(r_n e^{i\varphi_n}) d\varphi_n = 2\pi U^2(O) + \iint_{F_n} 2 \text{ grad } U^2(p) \log r_n dr_n d\varphi_n$$

$$+ \frac{\partial U^2(r_n e^{i\varphi_n})}{\partial r_n} \log r_n r_n d\varphi_n \leqq M',$$

whence

$$\int_{\partial F_n} U^2(r_n e^{i\varphi_n}) d\varphi_n \leqq M'',$$

where M'' is independent of n.

Map F_n^∞ onto $|\xi|<1$ and let G_ρ be the image of $|z|<\rho$ by this mapping. Since the connectivity of F_n is finite, ∂F_n is mapped in $|\xi|=1$ except possibly a set of linear measure zero, and further $d\varphi_n$ corresponds to ds on $|\xi|=1$, where ds consists of at most an enumerably infinite of arcs. Since $U^2(\xi)$ is sub-harmonic, we have by the lemma

$$M'' \geqq \int U^2(r_n e^{i\varphi_n}) d\varphi_n = \int_{|\xi|=1} U^2(\xi) d\theta \geqq \int_{\partial G_\rho} U^2(\xi) d\omega = \int_{|z|=\rho_n} U(\rho_n e^{i\theta_n}) d\theta_n$$

Let $\rho_n \to 1$. Then by Fatou's theorem $U(p)$ is represented by Poisson's integral in $|z|<1$.

Let $\tilde{U}(p): p \in F$ be a harmonic function of Dirichlet bounded. Then there exist subdomains[5] G_i $(i=1,2)$ in F with the property as follows: there exist harmonic functions of Dirichlet bounded such that $\tilde{U}_i(p) \geqq O$, $\tilde{U}_i(p) = O$ on ∂G_i and $G_1 \cap G_2 = O$.

Let G be one of them and let $U(p)$ be one of $\tilde{U}_i(p)$ in the sequel. Map the universal covering surface G^∞ of G onto $|z|<1$. Then there exists a constant δ $(\delta>0)$ and a set E_δ of positive measure on $|z|=1$ such that $U(p)$ has angular limits larger than δ, because $U(p)$ is represented by Poisson's integral on $|z|<1$. Let G' be the subdomain where $U(p) > \frac{\delta}{2}$. Then G' determines a set $B_{\frac{\delta}{2}}$[6] of the ideal boundary. Let $V_{n,n+i}(p)$ be a harmonic function in $(F_{n+i} \cap G) - (G' \cap (F_{n+i} - F_n)) = H_{n,n+i}$ such that $V_{n,n+i}(p) = 0$ on $\partial G' + (\partial F_{n-i} \cap (G-G')$ and $V_{n,n+i}(p) = 1$ on $\partial F_n \cap G'$. Then $V(p) = \lim_n \lim_i V_{n,n+i}(p) \geqq \omega_\delta(z)$, where $\omega_\delta(z)$ is the harmonic measure of E_δ, with respect to G^∞. Hence $V(p)$ is non-constant. Let $U_{n,n+i}(p)$ be a harmonic function in $H_{n,n+i}$ such that $U_{n,n+i}(p)=O$ on ∂G, $U_{n,n+i}(p)=1$ on $(\partial G'\frown(F_{n+i}-F_n))+(\partial F_n\frown G')$ and $\frac{\partial U_{n,n+i}}{\partial n}=O$ on $\partial F_{n+i}\frown(G-G')$. It is clear

$$D_G(U(p))\geqq D_{Hn,n+i}(U_{n,n+i}(p))$$

and

$$D_{Hn,n+i+j}(U_{n,n+i+j}(p))\geqq D_{Hn,n+i}(U_{n,n+i}(p)).$$

Hence $U_{n,n+i}(p)$ converges to $U_n(p)$ in mean. Since $U_{n,n+i}(p)\geqq V_{n,n+i}(p)$, $U(p)$ is non-constant.
Let C_δ' be the niveau curve of $U_n(p)$ with height δ' $(0<\delta'<1)$ and put

$${}_0G_{\delta'}=\varepsilon_p[0<U_n(p)<\delta'] \text{ and } {}_1G_{\delta'}=\varepsilon_p[\delta'<U(p)<1]$$

respectively.

Let $U_{n,n+i}(p)$ be a harmonic function in ${}_0G_{\delta'}\frown F_{n+i}$ such that $U'_{n,n+i}(p)=O$ on ∂G, $U'_{n,n+i}(p)=\delta'$ on ${}_0G_{\delta'}\frown(F_{n+i}-F_n)+(\partial F_n\frown{}_0G_{\delta'})$ and $\frac{\partial U'_{n,n+i}}{\partial n}=O$ on ∂F_{n+i}. Then we can prove as the previous manner[7] the following

Lemma. *$U'_{n,n+i}(p)\to U_n(p)$ in mean and*

$$\lim_i \underset{{}_0G_{\delta'}\frown F_{n+i}}{D}(U'_{n,n+i}(p))=\underset{{}_0G_{\delta'}}{D}(U_n(p)).$$

On the other hand, for given number ε and i, we see easily that there exists a number j_0 such that

$$\underset{{}_0G_{\delta'}\frown F_{n+i}}{D}(U'_{n,n+i}(p))\leqq\underset{{}_0G_{\delta'}\frown F_{n+i+j}}{D}(U'_{n,n+i+j}(p))+\varepsilon<\delta' D(U_n(p))+\varepsilon \quad \text{for } j\geqq j_0.$$

Let $\varepsilon\to 0$ and then $i\to\infty$. Then we have

$$\underset{{}_0G_{\delta'}}{D}(U'_n(p))\leqq\delta'\underset{G-(G'\frown F_n)}{D}(U_n(p)).$$

Let $U''_{n,n+i}(p)$ be a harmonic function in ${}_{\delta'}G_1\frown F_{n+i}$ such that $U''_{n,n+i}(p)=1$ on $\partial G'$, $U''_{n,n+i}(p)=\delta'$ on $(\partial_1G_{\delta'}\frown(F_{n+i}-F_n))+(\partial F_n\frown{}_1G_{\delta'})$ and $\frac{\partial U''_{n,n+i}}{\partial n}=0$ on ∂F_{n+i}. Then by the same manner we have

$$\lim_i U''_{n,n+i}(p)=U_n''(p)=U_n(p) \text{ and } \underset{{}_1G_{\delta'}}{D}(U_n(p))\leqq(1-\delta')\underset{G-(G\frown(F-F_n))}{D}(U_n(p)),$$

whence

$$\underset{{}_0G_{\delta'}}{D}(U_n(p))=\delta'\underset{G-(G\frown(F-F_n))}{D}(U_n(p)).$$

Next we have by the same manner as the previous[8] the following

Lemma.

$$\int_{\partial_0 G\varepsilon \cap F} \frac{\partial U_n}{\partial n}\, ds = D_{G-(G'\cap F_n)}\, U_n(p))\,,$$

for every number ε *except for at most two numbers* ε_1 *and* ε_2.

We say that the sequence $\{G'\wedge(F-F_n)\}$ $(n=1,2,\cdots)$ determines a set B_{G}' of the ideal boundary and call $\lim_i D_{G-G'}(U_n(p))$ the capacity of B_{G}' with respect to G. Then by the above lemmas, we can prove[9] as the previous the following

Theorem. *$U_n(p)$ converges to $U^0(p)$ in mean and*

$$D_{G-((F-F_n)\cap G')}(U_n(p)) \downarrow D_G(U^0(p)).$$

Since $U^0(p) \geqq V(p)$, $U^0(p)$ is non-constant and since $D_G(U(p)) \geqq D_G(U^0(p))$,

$$\infty > \mathrm{Cap}(B_{G'}) > 0.$$

Proof of the Royden's theorem.

Let F^* be another Riemann surface such that $F^* \ni p^* : p^* = T(p) : p \in F$, where $T(p)$ is a quasi-conformal mapping whose dilatation quotient is bounded $\leq K$. Put $U_n{}^Q(p) = U_n(T(p))$. Then $U_n{}^Q(p)$ is not necessarily harmonic and

$$\frac{1}{K} D_{Hn,n}(U_{n,n+i}(p)) \leqq D_{T(Hn,n+i)}(U^Q_{n,n+i}(p^*)) \leqq KD_{Hn,n+i}(U_{n,n+i}(p)),$$

where $T(G)$ is the image of G by the mapping $p^* = T(p)$.
Let $U_{n,n+i}(p^*)$ be a harmonic function in $T(H_{r,n+i})$ such that $U_{n,n+i}(p^*) = 0$ on $\partial(T(G))$, $U_{n,n+i}(p^*) = 1$ on $\partial(T(F_{n+i})) + T(\partial G' \wedge (F_{n+i} - (F_n))$ and $\dfrac{\partial U_{n,n+i}(p^*)}{\partial n} = 0$ on $\partial(T(F_{n+i}))$. Then by the Dirichlet principle

$$D_{T(Hn,n+i)}(U_{n,n+i}(p^*)) \leqq D_{T(Hn,n+i)}(U^Q_{n,n+i}(p^*)) \leqq KD_{Hn,n+i}(U_{n,n+i}(p)).$$

Since the inverse mapping $T^{-1}(p)$ of $T(p)$ is also a quasi-conformal mapping,

$$D_{Hn,n+i}(U^Q_{n,n+i}(T^{-1}(p^*)) \leqq K_{T(Hn,n+i}D(U_{n,n+i}(p^*)) \leqq KD_{Hn,n+i}(U_{n,n+i}(p)),$$

where $U^Q_{n,n+i}(T^{-1}(p^*)) = U^Q_{n,n+i}(p)$. On the other hand, by the Dirichlet priniple, we have

$$D_{H_n, n+i}(U_{n, n+i}(p)) \leqq D_{H_n, n+i}(U^Q_{n, n+i}(T^{-1}(p^*)).$$

Hence

$$D_{T(H_n, n+i)}(U_{n, n+i}(p^*)) \geqq \frac{1}{K} D_{H_n, n+i}(U^Q_{n, n+i}(T^{-1}(p^*)) \geqq \frac{1}{K} D_{H_n, n+i}$$
$$(U_{n, n+i}(p)).$$

$T(G')$ also determines a set $B_{T(G')}$ of the ideal boudary of $T(F)$. Let $i \to \infty$ and then $n \to \infty$. Then

$$KD_G(U^0(p)) \geqq D_{T(G)}(U^0(p^*)) \geqq \frac{1}{K} D_G(U^0(p)).$$

Hence $B_{T(G')}$ is a set of positive capacity with respect to $T(G)$ and

$$\frac{1}{K} \text{ Cap } B_{G'} \leqq \text{Cap } B_{T(G')} \leqq K \text{ Cap } B_{G'} .$$

Therefore on $T(G)$, there exists a Dirichlet bounded harmonic function with boundary value O on $T(\partial G)$. Thus we obtain our theorem by the method M. Parreau and A. Mori[10].

We construct an open Riemann surface $[(G-G') \cap \hat{F_n}]$ by the process of the symmetrization with respect to ∂F_n. Then $((G-G') \cap F_n) + [(G-G') \cap \hat{F_n}]$ is a ring domain. Let $n \to \infty$. Then $(G-G') \cap F$ is a generalized semi-ring domain. Thus our theorem is an extension of the well known "Modulsatz".

Let F_0 be a compact set of F and let $N_n(p, p_0)$ be a harmonic function in $(F-F_0) \cap F_n$ such that $N_n(p, p_0) = 0$ on ∂F_0 and $\frac{\partial N_n}{\partial n} = 0$ on ∂F_n and has one logarithmic singularity at p_0. We can prove that $N_n(p, p_0)$ converges to $N(p, p_0)$, when $n \to \infty$. Let $\{p_i\}$ be a sequence of points tending to the boundary. If a subsequence of $\{N(p, p_i)\}$ converges to $N(p, \{p_{i'}\})$, we say that $\{p_i'\}$ determines an ideal point[11]. We can introduce a topology of the ideal boundary by the use of $N(p, \{p_{i'}\})$ and prove that the above topology is invariant under the quasi-conformal mapping.

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References

1) H. L. Royden: A property of quasi-conformal mapping. Proc. Amer. Math. Soc, 5, pp-266–269.

2) O_{HD} and O_{HDN} are the class of Riemann surfaces on which no harmonic function of Dirichlet bounded exists or the dimension of harmonic function of Dirichlet bounded is N respectively.

3) In this article, we denote the relative boundary of G by ∂G.

4) L. Myrberg: Bemerkungen zur Theorie der harmonischen Funktionen. Annales Acad. Fenn. 1952.

5) M. Parreau: Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann. Annales de l'institut Fourier. 1952.

A. Mori: On the existance of harmonic functions on a Riemann surface: Journal. Fac. Sci. Univ. Tokyo, **6**, 1951.

6) Z. Kuramochi: Harmonic measures and capacity of sets of the ideal boundary. I and II. Proc. Jap. Acad. **30**. 1954 and **31**. 1955.

7), 8) and 9) See 6).

10) See 5).

11) A note on this topology will appear in this Journal.