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On the Pseudo-Analytic Functions

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Introduction. Various extensions of the analytic functions have been studied as 'pseudo-analytic functions', the definitions of which differ more or less from one another (cf. Grötzsch [1], Lavrentieff [3], Teichmüller [10]). In the present paper we shall define and investigate a kind of pseudo-analytic function which seems to us the fittest in order to preserve the validity of some qualitative theorems in the theory of functions.

In § 1, it is shown that a known theorem (cf. Pompeiu [6], [7]) holds for our pseudo-regular function. In § 2, families of pseudo-regular functions are studied. Finally in § 3, theorems on the analytic functions are extended to our class of functions.

Definition. A complex-valued function $f(z)=u+iv$ defined in a domain D of the $z(=x+iy)$-plane is *pseudo-regular*, if it has the following property:

1) $f(z)$ is one-valued and continuous in D;
2) $f(z)$ satisfies the following conditions a), b) except for the set which is at most enumerable and closed with respect to D:
   a) continuous partial derivatives u_x, u_y, v_x, v_y exist,
   b) $J(z)\equiv\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}>0$.

Let $f(z)$ be pseudo-regular in a domain D and let E be the set on which the condition a) or b) is not satisfied. Then a point of D is said to be *pseudo-conformal* or *critical*, according as it belongs to $D-E$ or to E. If $f(z)$ is pseudo-regular in some neighbourhood of a point z_0, we say simply that it is pseudo-regular at z_0. We agree also to say that $f(z)$ is pseudo-regular on a closed domain $\bar{D}$, if it is so in an appropriate domain containing $\bar{D}$. Let $f(z)$ be pseudo-regular in a neighbourhood of z_0 except for z_0 and $\lim_{z\to z_0} f(z)=\infty$. Then the point z_0 is a *pole* of the function $f(z)$. A function which is pseudo-regular at every point of a domain D except for poles is called *pseudo-meromorphic* in D. A function is called *pseudo-analytic*, when it is pseudo-regular, pseudo-meromorphic or a constant.

§ **1.** Let W and W_0 be two orientable surfaces and $p_0=S(p)$ a transformation from W to W_0. Then $S(p)$ is called an *interior transformation* in Stoïlow's sense, if and only if it satisfies the following conditions:

1) $S(p)$ is one-valued and continuous on W;
2) It transforms each open set on W to an open set on W_0;
3) It never transforms any continuum on W to a point on W_0.

Theorem 1. *A pseudo-regular function $f(z)$ in D is an interior transformation of D.*

PROOF. Since $f(z)$ is univalent in an appropriate neighbourhood of a pseudo-conformal point, it is obviously an interior transformation in D except for critical points.

Let $\{D_n\}$ $(n=1,2,\cdots)$ be an interior exhaustion of the domain D, each of which is enclosed by a finite number of Jordan curves C_n passing no critical points of $f(z)$. Since the set of critical points in D_n is closed and at most enumerable, $f(z)$ is an interior transformation in D_n, therefore we see easily that it is so in the whole domain D (cf. Stoïlow [8]).

Let z_0 be a pseudo-conformal point. Then $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ varies generally with the direction, in which z approaches z_0. We denote this directional derivative by $\left.\frac{df(z)}{dz}\right|_\theta$, where θ is the angle between the x-axis and the curve of approach. The following is an immediate consequence:

$$\left.\frac{df(z)}{dz}\right|_\theta=M[f(z)]+e^{-2i\theta}P[f(z)],$$

where

$$M[f(z)]=\frac{1}{2}[f_x(z)-if_y(z)]=\frac{1}{2}[(u_x+v_y)+i(v_x-u_y)],$$

$$P[f(z)]=\frac{1}{2}[f_x(z)+if_y(z)]=\frac{1}{2}[(u_x-v_y)+i(v_x+u_y)].$$

$M[f(z)]$ and $P[f(z)]$ are called respectively the *mean* and the *Pompeiu's derivative* of $f(z)$.

It is well-known that the infinitesimal circle with centre at any pseudo-conformal point z_0 is transformed by $f(z)$ to the infinitesimal ellipse with centre $f(z_0)$, whose major and minor axes are of length a and b respectively. *Dilatation-quotient* $Q[f(z_0)]$ of $f(z)$ at z_0 is defined by ratio a/b and we have the expression

$$Q[f(z)] = \frac{g_{11}+g_{22}+\sqrt{(g_{11}+g_{22})^2-4(g_{11}g_{22}-g_{12}^2)}}{2\sqrt{g_{11}g_{22}-g_{12}^2}},$$

where

$$g_{11} = u_x^2+v_x^2, \quad g_{12} = u_x u_y + v_x v_y, \quad g_{22} = u_y^2+v_y^2.$$

Dilatation-quotient is conformally invariant and for the inverse function $z = f^{-1}(w)$ of $w = f(z)$

$$(1) \qquad Q[f(z)] = Q[f^{-1}(w)].$$

Let ds be the line-element corresponding to $|dz|$ by $f(z)$. Then we have

$$ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$$

and

$$(2) \qquad \frac{ds^2}{|dz|^2} \leq Q[f(z)]\cdot J[f(z)].$$

It is possible to choose a disc $|z-z_0| \leq r$ in D, so that $f(z)$ has no critical point on its periphery $|z-z_0| = r$. Suppose the disc is mapped by $f(z)$ onto a region, whose area is $A(r)$ and whose boundary curve has length $L(r)$. Then we obtain by Schwarz's inequality and (2)

$$(L(r))^2 = \left(\int_0^{2\pi} \frac{ds}{|dz|} r d\theta\right)^2 \leq \int_0^{2\pi} r d\theta \int_0^{2\pi} \frac{ds^2}{|dz|^2} r d\theta \leq 2\pi r Q[f(z)]\cdot \frac{dA(r)}{dr},$$

that is,

$$(3) \qquad \frac{dr}{r} \leq 2\pi Q[f(z)] \frac{dA(r)}{(L(r))^2}.$$

On the behaviour of $f(z)$ in the neighbourhood of critical points various cases may be considered. We shall give some examples, in which the origin is an isolated critical point:

Example 1.

$$w = f(re^{i\theta}) = ar^K \cos n\theta + ibr^K \sin n\theta \quad (a, b, K > 0;\ n = 1, 2, \cdots).$$

This is pseudo-regular in $0 \leq r < \infty$ and all the points except for the origin are its pseudo-conformal points.

In case $K > 1$ and $n = 1$, all the partial derivatives u_x, u_y, v_x, v_y are continuous and vanish at the origin. But $w = 0$ is no branch-point.

In case $K = 1$ and $n > 1$, non-vanishing partial derivative exists at the origin. But $w = 0$ is a branch-point.

In case $K<1$, no finite partial derivative exists at the origin. $w=0$ is a branch-point or not, according as $n>1$ or $n=1$.

Example 2. $w=f(re^{i\theta})=r\cos\frac{1}{r}\cos\theta+ir\sin\frac{1}{r}\sin\theta$.

This function, pseudo-regular in $0\leq r<\infty$, has no partial derivative at the origin and has no branch-point.

Lemma 1. *Let C be a rectifiable Jordan curve and D its interior. Let $f(z)$ be a pseudo-regular function on $D+C$. Then*

$$\int_C f(z)dz=2i\iint_{D-E} P[f(z)]\,d\sigma \qquad (d\sigma=dx\,dy),$$

where E is the set of critical points of $f(z)$ in D.

PROOF. We can choose a finite number of disjoint smooth Jordan curves C' in D, so that C' encloses E and its total length is less than arbitrary positive number ε. By Green's formula

$$\int_C f(z)dz+\int_{C'} f(z)dz=\int_C (u+iv)(dx+idy)+\int_{C'}(u+iv)(dx+idy)$$
$$=2i\iint_{D'} P[f(z)]\,d\sigma,$$

where D' is the domain bounded by C and C'. Since

$$\left|\int_{C'} f(z)dz\right|\leq\varepsilon\cdot\max_{D+C}|f(z)|,$$

the left-hand side tends to zero with ε. Therefore

$$\int_C f(z)\,dz=2i\iint_{D-E} P[f(z)]\,d\sigma.$$

Hereafter we shall write simply

$$\iint_{D-E} P[f(z)]\,d\sigma\equiv\iint_D P[f(z)]\,d\sigma.$$

Lemma 2. *Let $f(z)$ be pseudo-regular and let $\psi(z)$ be regular. Then*

$$P[f(z)\cdot\psi(z)]=\psi(z)\cdot P[f(z)].$$

PROOF.

$$P[f(z)\cdot\psi(z)]=\frac{1}{2}\left[\frac{\partial}{\partial x}\{f(z)\cdot\psi(z)\}+i\frac{\partial}{\partial y}\{f(z)\cdot\psi(z)\}\right]$$
$$=\frac{1}{2}\left[\psi(z)\left\{\frac{\partial}{\partial x}f(z)+i\frac{\partial}{\partial y}f(z)\right\}+f(z)\left\{\frac{\partial}{\partial x}\psi(z)+i\frac{\partial}{\partial y}\psi(z)\right\}\right]$$

$$= \psi_r(z)\cdot P[f(z)] + f(z)\cdot P[\psi_r(z)]$$
$$= \psi_r(z)\cdot P[f(z)]\,.$$

Theorem 2. *Let D be a domain bounded by a finite number of rectifiable Jordan curves C and let $f(z)$ be a pseudo-regular function on $D+C$. Then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{\pi}\iint_D \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma\,.$$

PROOF. Let C' be a circle $|\zeta - z| = r$ in D with centre at a pseudo-conformal point z. Then by Lemma 1

$$\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{C'} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 2i\iint_{D'} P\left[\frac{f(\zeta)}{\zeta - z}\right]d\sigma\,,$$

where D' is the domain bounded by C and C'. Applying Lemma 2 to the right-hand side, we have

$$\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta - i\int_0^{2\pi} f(z+re^{i\theta})\,d\theta = 2i\iint_{D'} \frac{P[f(\zeta)]}{\zeta - z}\,d\sigma\,.$$

Let r tend to zero. Then

$$(4) \qquad f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{\pi}\iint_D \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma\,.$$

Next we shall prove that the above relation (4) is also valid at any critical point. For an arbitrary critical point z' in D we can choose a positive number r (<1), a pseudo-conformal point z and a circle C'': $|\zeta - z| = r$, so that $|z'-z| = \frac{r^3}{2}$ and the disc $|\zeta - z| \leq r$ is contained in D.

Then

$$(5) \quad \left|\iint_D \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma - \iint_{D''} \frac{P[f(\zeta)]}{z'-\zeta}\,d\sigma\right| \leq \left|\iint_D \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma - \iint_{D''} \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma\right| + \left|\iint_{D''} \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma - \iint_{D''} \frac{P[f(\zeta)]}{z'-\zeta}\,d\sigma\right|,$$

where D'' is the domain bounded by C and C''. By (4)

$$\left|\iint_D \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma - \iint_{D''} \frac{P[f(\zeta)]}{z-\zeta}\,d\sigma\right| \leq \frac{1}{2}\int_{C''}\left|\frac{f(\zeta)-f(z)}{\zeta - z}\right||d\zeta|$$

$$= \frac{1}{2}\int_0^{2\pi} |f(z+re^{i\theta}) - f(z)|\,d\theta \leq \pi \max_{0\leq\theta<2\pi} |f(z+re^{i\theta}) - f(z)|\,,$$

and by Lemma 1 and 2

$$\left|\iint_{D''}\frac{P[f(\zeta)]}{z'-\zeta}d\sigma-\iint_{D''}\frac{P[f(\zeta)]}{z-\zeta}d\sigma\right|=\left|\iint_{D''}\left(\frac{1}{z'-\zeta}-\frac{1}{z-\zeta}\right)P[f(\zeta)]\,d\sigma\right|$$

$$=|z-z'|\cdot\left|\iint_{D}\frac{P[f(\zeta)]}{(z'-z)(z-\zeta)}d\sigma\right|$$

$$=|z-z'|\cdot\left|\iint_{D''}P\left[\frac{f(\zeta)}{(z'-\zeta)(z-\zeta)}\right]d\sigma\right|$$

$$=\frac{|z-z'|}{2}\left|\int_{C+C''}\frac{f(\zeta)}{(z'-\zeta)(z-\zeta)}d\zeta\right|\leq\frac{r}{2}\int_{C+C''}|f(\zeta)|\,|d\zeta|$$

$$\leq\frac{r}{2}\max_{D+C}|f(\zeta)|\cdot(L+2\pi r),$$

where L is the length of C. Therefore the left-hand side of (5) tends to zero with r. Since z tends to z' as $r\to 0$, it follows that

$$\lim_{z\to z'}f(z)=f(z'),$$

$$\lim_{z\to z'}\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z'}d\zeta,$$

$$\lim_{z\to z'}\frac{1}{\pi}\iint_{D''}\frac{P[f(\zeta)]}{z-\zeta}d\sigma=\frac{1}{\pi}\iint_{D}\frac{P[f(\zeta)]}{z'-\zeta}d\sigma.$$

Therefore we have

$$f(z')=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z'}d\zeta+\frac{1}{\pi}\iint_D\frac{P[f(\zeta)]}{z'-\zeta}d\sigma.$$

We can easily extend this result to obtain the following:

Theorem 3. *Let D be the domain bounded by a finite number of rectifiable Jordan curves C. Let $f(z)$ be pseudo-regular in D and continuous on $D+C$. Then we have*

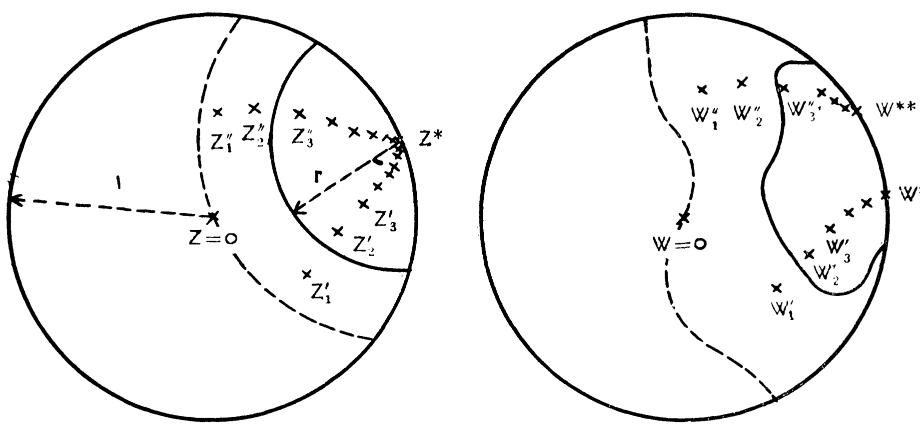
$$f(z)=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z}d\zeta+\frac{1}{\pi}\iint_D\frac{P[f(\zeta)]}{z-\zeta}d\sigma.$$

§ **2.** From this § on, we deal with a little more restricted class of the pseudo-regular functions, that is, of bounded dilatation-quotient.

Theorem 4. *Let $w=f(z)$ be a pseudo-regular function of bounded dilatation-quotient $(Q[f(z)]\leq K)$, which maps $|z|<1$ one-to-one to $|w|<1$. Then $f(z)$ is continuously prolongable up to the circumference.*

PROOF. We may assume $f(0)=0$ without loss of generality. We show first that the boundary values of $f(z)$ is uniquely determined. In fact, otherwise, there would exist two sequences of points $\{z_n'\}$ and $\{z_n''\}$ $(n=1,2,\cdots)$ both converging to z^* on $|z|=1$, such that $\{f(z_n')\}$ and $\{f(z_n'')\}$ $(n=1,2,\cdots)$ converge to different points w^* and w^{**} on $|w|=1$ respectively. Suppose the circular arc of $|z-z^*|=r$ $(\varepsilon \leq r < 1)$ inside of $|z|<1$ is mapped by this function onto an arc in $|w|<1$, the length of which is denoted by $L(r)$. Then the arc necessarily divides the origin from both w^* and w^{**}, whence

$$L(r) \geq |w^*-w^{**}| .$$



$|z|<1$ $|w|<1$

Fig. 1

The common part of $r<|z-z^*|<1$ with $|z|<1$ will then be mapped onto some portion of $|w|<1$, the area of which is denoted by $A(r)$. So by (3)

$$\int_{\varepsilon}^{1} \frac{dr}{r} \leq 2\pi K \int_{A(1)}^{A(\varepsilon)} \frac{dA(r)}{|w^*-w^{**}|^2}$$

or

$$\log \frac{1}{\varepsilon} < \frac{2\pi K}{|w^*-w^{**}|} A(\varepsilon) ,$$

while $A(\varepsilon) < \pi$ must always hold. We have a contradiction when ε tends to zero.

It is the same with the inverse function $z=f^{-1}(w)$ of $w=f(z)$, since $Q[f^{-1}(w)]=Q[f(z)] \leq K$ by (1).

Thus the boundary correspondence is biunique and continuous.

Theorem 5. *Let* $w=f_n(z)$ $(n=1,2,\cdots)$ *be the pseudo-regular func-*

tions of uniformly bounded dilatation-quotient ($Q[f_n(z)]\leq K$) *with the condition* $f_n(0)=0$, *each of which is a topological mapping from* $|z|<1$ *to* $|w|<1$. *If the sequence* $\{f_n(z)\}$ *converges to a function* $f(z)$ *uniformly in* $|z|<1$, *then* $f(z)$ *is also a topological mapping from* $|z|<1$ *to* $|w|<1$.

PROOF. It is clear that $f(z)$ is one-valued, continuous and $|f(z)|\leq 1$ in $|z|<1$.

i) We shall show $|f(z)|<1$ for $|z|<1$. If it were not true, there would exist a point z_0, such that $|z_0|<1$, $|f(z_0)|=1$. However small ε (>0) may be preassigned, $f_n(z_0)$ is contained in $|w-w_0|<\varepsilon$ for sufficiently large n, where $w_0=f(z_0)$. Suppose the circular arc $|w-w_0|=r$ ($\varepsilon\leq r<1$) inside of $|w|<1$ is mapped by the inverse function of $w=f_n(z)$, say $z=f_n^{-1}(w)$, onto an arc in $|z|<1$. Then

$$L(r)>1-|z_0|,$$

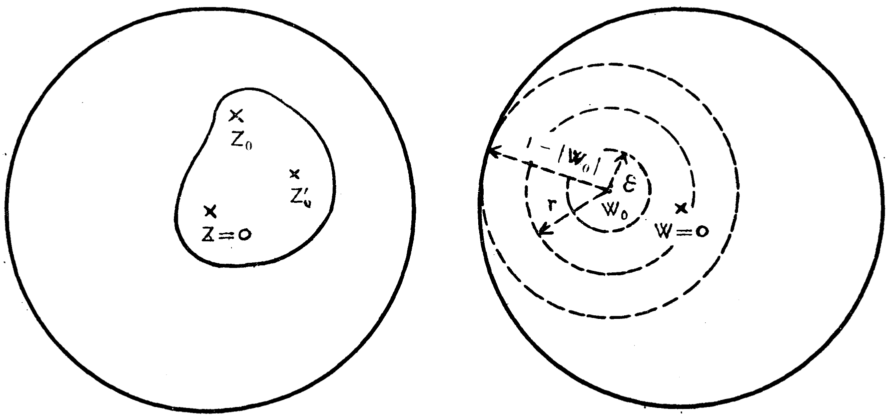
where $L(r)$ is the length of the arc. The common part of $r<|w-w_0|<1$ with $|w|<1$ will be mapped by the same function onto some portion of $|z|<1$, the area of which is denoted by $A(r)$. Then by (3)

$$\int_\varepsilon^1\frac{dr}{r}<2\pi K\int_{A(1)}^{A(\varepsilon)}\frac{dA(r)}{(1-|z_0|)^2}<\frac{2\pi K}{(1-|z_0|)^2}\cdot A(\varepsilon)<\frac{2\pi^2K}{(1-|z_0|)^2},$$

which is a contradiction.

ii) Let $\{z_m\}$ ($m=1,2,\cdots$) be an arbitrary sequence converging to a periphery point z_0. Let w_0 be one of the accumulating points of $\{f(z_m)\}$. Then an appropriate subsequence, say again $\{f(z_m)\}$, converges to w_0. We shall show $|w_0|=1$. For otherwise, for arbitrarily preassigned ε there would exist a number N, such that all the images of $|z-z_0|<\varepsilon$ by $w=f_n(z)$ have points in common with $|w-w_0|<\frac{1}{2}(1-|w_0|)$ so long as $n\geq N$. Then the image of $|z-z_0|=r$ ($\varepsilon\leq r<1$) by $w=f_n(z)$ would have length greater than $1-|w_0|$. On the other hand the common part of $r<|z-z_0|<1$ with $|z|<1$ is mapped onto some portion of $|w|<1$, the area of which is obviously less than π. We can thus extract a contradiction in the same way as in i).

iii) $f(z)$ is univalent. For, otherwise, there would exist $z_0\neq z_0'$ such that $f(z_0)=f(z_0')=w_0$. On account of i), $|w_0|<1$. For any ε all $f_n(z_0)$ and $f_n(z_0')$ fall within $|w-w_0|<\varepsilon$ so long as $n\geq N$. The length of the image, onto which $|w-w_0|=r$ ($\varepsilon<r<1-|w_0|$) is mapped by $f_n^{-1}(w)$, would be greater than $|z_0 - z_0'|$. The common part of $r < |w - w_0| < 1 - |w_0|$ with $|w| < 1$ evidently has an image confined in $|z| < 1$. Thus we have a contradiction as above.



$|z|<1$ $|w|>1$

Fig. 2

Lemma 3. *If $f(z)$ is a pseudo-regular function of bounded dilatation-quotient $Q[f(z)] \leq K$, then we have*

$$|P[f(z)]|^2 \leq \frac{(K-1)^2}{4K} \cdot J[f(z)].$$

PROOF.

$$\frac{g_{11}+g_{22}}{2J} = \frac{Q^2-1}{2Q} \quad \text{since} \quad Q = \frac{g_{11}+g_{22}+\sqrt{(g_{11}+g_{22})^2-4J^2}}{2J}.$$

Hence

$$|P|^2 = \frac{1}{4}\left(g_{11}+g_{22}-2J\right) = \frac{J}{2}\left(\frac{g_{11}+g_{22}}{2J}-1\right) = \frac{J}{2}\left(\frac{Q^2+1}{2Q}-1\right)$$
$$= \frac{(Q-1)^2}{4Q} \cdot J \leq \frac{(K-1)^2}{4K} \cdot J.$$

Theorem 6. *Let each term of a sequence $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ be the pseudo-regular function which furnishes a topological mapping from $|z| < 1$ to $|w| < 1$ with the condition $f_n(0) = 0$ and the sequence be uniformly convergent in $|z| < 1$. If further $Q[f_n(z)] \leq K_n$, $\lim_{n\to\infty} K_n = 1$, then $\lim_{n\to\infty} f_n(z) = e^{i\theta} z$ $(0 \leq \theta < 2\pi)$.*

PROOF. Let z_0 be a point in $|z| < 1$ and let C be a smooth Jordan curve enclosing it. If we put $r_n = \sqrt{\dfrac{K_n - 1}{2\sqrt{K_n}}}$, then $\lim_{n\to\infty} r_n = 0$. Hence, for sufficiently large n, the circle $C_n: |z-z_0|=r_n$ is containd in the interior of $[C]$, where $[C]$ is the domain bounded by C. Application of Theorem 2 to $f_n(z)$ and $[C]$ yields

$$\frac{1}{\pi}\iint_{[C]}\frac{P[f_n(\zeta)]}{z_0-\zeta}d\sigma=f_n(z_0)-\frac{1}{2\pi i}\int_C\frac{f_n(\zeta)}{\zeta-z_0}d\zeta\,.$$

Denoting the interior of C_n by $[C_n]$, we have by Lemma 3 and Schwarz's inquality

$$\left|\frac{1}{\pi}\iint_{[C]-[C_n]}\frac{P[f_n(\zeta)]}{z_0-\zeta}d\sigma\right|\leq\frac{1}{\pi}\iint_{[C]-[C_n]}\frac{r_n^2\sqrt{J[f_n(\zeta)]}}{r_n}d\sigma\leq r_n\,,$$

while by Theorem 2

$$\left|\frac{1}{\pi}\iint_{[C_n]}\frac{P[f_n(\zeta)]}{z_0-\zeta}d\sigma\right|=\left|f_n(z_0)-\frac{1}{2\pi i}\int_{C_n}\frac{f_n(\zeta)}{\zeta-z}d\zeta\right|=\frac{1}{2\pi}\left|\int_{C_n}\frac{f_n(\zeta)-f_n(z_0)}{\zeta-z_0}d\zeta\right|$$

$$\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{|f_n(z_0+r_ne^{i\theta})-f_n(z_0)|}{r_n}\cdot d\theta=\frac{1}{2\pi}\int_0^{2\pi}|f_n(z_0+r_ne^{i\theta})-f_n(z_0)|\,d\theta\,.$$

Hence it follows that

$$\lim_{n\to\infty}\left(f_n(z_0)-\frac{1}{2\pi i}\int_C\frac{f_n(\zeta)}{\zeta-z_0}d\zeta\right)=0\,.$$

If we put $\lim_{n\to\infty}f_n(z)=f(z)$, it is regular, since the above integral is regular and the convergence is uniform. Moreover, $f(0)=0$, and $w=f(z)$ supplies a homeomorphism between $|z|<1$ and $|w|<1$ by Theorem 4. Consequently we obtain $f(z)=e^{i\theta}z$ $(0\leq\theta<2\pi)$.

Lemma 4. *A family* $\{f_\lambda(z)\}$ $(\lambda\in\Lambda)$ *of the pseudo-regular functions of uniformly bounded dilatation-quotient* $(Q[f_\lambda(z)]\leq K)$, *each of which is a topological mapping from* $|z|<1$ *to* $|w|<1$, *is normal in* $|z|<1$.

PROOF. Since uniform boundedness of $f_\lambda(z)$ is evident, we shall show that $\{f_\lambda(z)\}$ is equicontinuous in $|z|<1$. For, otherwise, there would exist a positive number α, such that the relations $|f_\lambda(z')-f_\lambda(z'')|\geq\alpha>0$ and $|z'-z''|<\varepsilon$ simultaneously hold for appropriate $f_\lambda\in\{f_\lambda\}$ and z', z'' in any $|z|\leq\rho<1$, however small ε may be chosen. Consider the mapping by $w=f_\lambda(z)$. The image of the circle $|z-z'|=r$ $(\varepsilon<r<1-\rho)$ would have length greater than α. The circular ring $\varepsilon<|z-z'|<r$ is mapped onto some ring-domain contained entirely in $|w|<1$. We would have

$$\int_{\varepsilon}^{1-\varepsilon} \frac{dr}{r} \leq \frac{2\pi K}{\alpha^2} \int_{A(\varepsilon)}^{A(1-\varepsilon)} dA(r) ,$$

then the same reasoning as in the proof of Theorem 4 leads to a contradiction.

Lemma 5. *Let D_z and D_w be domains in the z- and w-plane respcctively. Let $w = f_n(z)$ be a topological mapping from D_z to D_w. If the sequence $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ converges to a topological mapping $f(z)$ from D_z to D_w uniformly in D_z, then the sequence $\{f_n^{-1}(w)\}$ $(n = 1, 2, \cdots)$ of their inverse functions converges to the inverse function $f^{-1}(w)$ of the limit function $f(z)$ uniformly in D_w.*

PROOF. Let D_w^* be an arbitrary closed domain contained in D_w and $\{w_n\}$ $(n = 1, 2, \cdots)$ be a sequence such that $w_n \in D_w^*$, $\lim_{n\to\infty} w_n = w_0$. Then the sequence $\{z_n\}$ satisfying $w_n = f_n(z_n)$ $(n = 1, 2, \cdots)$ has its accumulating points in D_z, one of which we denote by z_0. An appropriate subsequence $\{z_{n_\nu}\}$ will converge to z_0. Uniform convergence of $\{f_n(z)\}$ yields equicontinuity of $f_n(z)$, and so for sufficiently large ν

$$|f_{n_\nu}(z_{n_\nu}) - f(z_0)| \leq |f_{n_\nu}(z_{n_\nu}) - f_{n_\nu}(z_0)| + |f_{n_\nu}(z_0) - f(z_0)| < \varepsilon ,$$

that is

$$\lim_{\nu\to\infty} f_{n_\nu}(z_{n_\nu}) = f(z_0) = w_0 .$$

This implies that the original sequence $\{z_n\}$ accumulates only at a single point $f^{-1}(w_0)$, since $f(z)$ is assumed to be univalent.

Hence

$$\lim_{n\to\infty} f_n^{-1}(w_n) = f^{-1}(w_0) ,$$

and in particular

$$\lim_{n\to\infty} f_n^{-1}(w_0) = f^{-1}(w_0) .$$

If the last convergence were not uniform, there would exist a positive number ε, a subsequence $\{f_{n_\mu}^{-1}(w)\}$ of $\{f_n^{-1}(w)\}$ and a sequence $\{w_\mu'\}$ $(\mu = 1, 2, \cdots)$ in D_w^* converging to w_0, such that

$$|f_{n_\mu}^{-1}(w_\mu') - f^{-1}(w_0')| \geq \varepsilon > 0 ,$$

which contradicts the above relation

$$\lim_{\mu\to\infty} f_{n_\mu}^{-1}(w_\mu') = f^{-1}(w_0) .$$

Let $w = f(z)$ be a pseudo-analytic function of bounded dilatation-quotient in $|z| < 1$. Then it is easily seen that the Riemann surface

W of its inverse function is of hyperbolic type (cf. Kakutani [2], Teichmüller [10]). Let $\zeta = F^{-1}(w)$ be the function which maps W conformally onto $|\zeta| < 1$. Then its inverse function $w = F(\zeta)$ is analytic in $|\zeta| < 1$. Put $\zeta = F^{-1}(w) = F^{-1}(f(z)) \equiv \varphi(z)$. Then $\zeta = \varphi(z)$ is a pseudo-regular function which maps $|z| < 1$ one-to-one to $|\zeta| < 1$. Since $w = f(z) = F(\varphi(z))$, it can be considered as an analytic function in $|z| < 1$, if we define the metric by $\zeta = \varphi(z)$. In particular, if we normalize it so that $\varphi(0) = 0$, $\lim_{z \to 1} \varphi(z) = 1$, then $\varphi(z)$ is a function uniquely determined by $f(z)$. It is called the *uniformizer for* $f(z)$.

Theorem 7. *If a sequence* $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ *of pseudo-regular functions of uniformly bounded dilatation-quotient is uniformly convergent in* $|z| < 1$, *then the limit function* $f(z)$ *is an interior transformation in* $|z| < 1$ *unless it reduces to a constant.*

PROOF. Let $\zeta = \varphi_n(z)$ be the uniformizer for $f_n(z)$ and put $f_n(z) \equiv F_n(\varphi_n(z))$. Then the family $\{\varphi_n(z)\} (n=1, 2, \cdots)$ is normal in $|z| < 1$ by Lemma 4. Hence we can choose a subsequence $\{\varphi_{n_\nu}(z)\}$ $(\nu = 1, 2, \cdots)$ out of it, which is uniformly convergent in $|z| < 1$. Let $\zeta = \varphi(z)$ be the limit function. Then it is a topological mapping from $|z| < 1$ to $|\zeta| < 1$ by Theorem 4. Hence by Lemma 5 $\{\varphi_{n_\nu}^{-1}(\zeta)\}$ converges uniformly to $\varphi^{-1}(\zeta)$ in $|\zeta| < 1$. $F_{n_\nu}(\zeta)$ is also uniformly convergent in $|\zeta| < 1$, since $F_{n_\nu}(\zeta) = f_{n_\nu}(\varphi_{n_\nu}^{-1}(\zeta))$. Let $F(\zeta)$ be the limit function. Then it is regular in $|\zeta| < 1$. Consequently $f(z)$ is an interior transformation in $|z| < 1$, since $f(z) = F(\varphi(z))$.

Theorem 8. *Let* $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ *be a sequence of pseudo-regular functions of uniformly bounded dilatation-quotient, which is uniformly convergent in* $|z| < 1$. *Let* $\zeta = \varphi_n(z)$ *be the uniformizer for* $f_n(z)$ *and* $f_n(z) \equiv F_n(\varphi_n(z))$. *Then* $\{\varphi_n(z)\}$ *and* $\{F_n(\zeta)\}$ $(n = 1, 2, \cdots)$ *are uniformly convergent in* $|z| < 1$ *and* $|\zeta| < 1$ *respectively, and further* $f(z) = F(\varphi(z))$, *where* $f(z) = \lim_{n\to\infty} f_n(z)$, $\varphi(z) = \lim_{n\to\infty} \varphi_n(z)$, *and* $F(\zeta) = \lim_{n\to\infty} F_n(\zeta)$.

PROOF. Since $w = f(z)$ is an interior transformation by Theorem 7, there exists the well-determined inverse function $z = f^{-1}(w)$. If $\{\varphi_n(z)\}$ $(n = 1, 2, \cdots)$ were not uniformly convergent, there would exist at least two functions $\varphi(z)$ and $\tilde{\varphi}(z)$, to which some subsequences of it uniformly converge respectively in $|z| < 1$. Put $f(z) \equiv F(\varphi(z)) \equiv \tilde{F}(\tilde{\varphi}(z))$. Then both $F(\zeta)$ and $\tilde{F}(\zeta)$ are regular in $|\zeta| < 1$. If we denote each inverse function of them by $F^{-1}(w)$ and $\tilde{F}^{-1}(w)$ respectively, we have

$$F^{-1}(w) = \varphi(f^{-1}(w)), \quad \tilde{F}^{-1}(w) = \tilde{\varphi}(f^{-1}(w)).$$

Put $f(0)=w_0$. Then

$$F^{-1}(w_0)=\varphi(f^{-1}(w_0))=0=\tilde{\varphi}(f^{-1}(w_0))=\widetilde{F}^{-1}(w_0).$$

Hence

$$F^{-1}(w)=e^{i\theta}\widetilde{F}^{-1}(w) \qquad (0\leq\theta<2\pi),$$

while $\theta=0$ on account of the normalizing condition. Thus we have

$$F^{-1}(w)=\widetilde{F}^{-1}(w),$$

that is,

$$F(\zeta)=\widetilde{F}(\zeta), \quad \varphi(z)=\tilde{\varphi}(z).$$

It follows that

$$\lim_{n\to\infty}\varphi_n(z)=\varphi(z), \quad \lim_{n\to\infty}F_n(\zeta)=F(\zeta),$$

and consequently

$$f(z)=F(\varphi(z)).$$

Let $f(z)$ be a pseudo-analytic function of bounded dilatation-quotient in a domain D. For each point z_0 in D consider a sufficiently small circular neighbourhood V_{z_0} entirely contained in D. Let W be the Riemann surface of the inverse function $z=f^{-1}(w)$ and let $\zeta=F_{z_0}^{-1}(w)$ be the function which maps conformally the image of V_{z_0} on W onto $|\zeta|<1$. Put $\zeta=F_{z_0}^{-1}(f(z))\equiv\varphi_{z_0}(z)$. Then $\varphi_{z_0}(z)$ is a pseudo-regular function which supplies a homeomorphism between V_{z_0} and $|\zeta|<1$. It will be uniquely determined by $f(z)$ and z_0, if we normalize it as follows:

i) $F_{z_0}^{-1}(f(z_0))=0$;

ii) $\lim_{z\to z'}F_{z_0}^{-1}(f(z))=1$,

where z' is the point at which the radius of V_{z_0} parallel to the positive real axis intersects the circumference of V_{z_0}. We shall call it *local uniformizer for $f(z)$ at z_0*. The analogous statements to Theorem 6, 7 and 8 will be obtained if we consider V_{z_0} in place of $|z|<1$.

Theorem 9. *If a sequence $\{f_n(z)\}$ $(n=1,2,\cdots)$ of pseudo-regular functions of uniformly bounded dilatation-quotient is uniformly convergent in a domain D, then the limit function is an interior transformation of D.*

PROOF. It is evident by Theorem 7 if we consider the local uniformizer for $f_n(z)$ at each point of D.

Theorem 10. *If a sequence $\{f_n(z)\}$ $(n=1,2,\cdots)$ of pseudo-regular functions is uniformly convergent in D and further $Q[f_n(z)]\leq K_n$, $\lim_{n\to\infty}K_n=1$, then the limit function is regular in D.*

PROOF. Let $\varphi_{z_0,n}(z)$ be the local uniformizer for $f_n(z)$ at $z_0 \in D$. Then by Theorem 8 the sequence $\{\varphi_{z_0,n}(z)\}$ $(n=1, 2, \ldots)$ converges uniformly to a function $\varphi_{z_0}(z)$, which is regular by Theorem 6. Again by Theorem 8 we see that $\lim_{n\to\infty} f_n(z)$ is regular.

§ 3. A pseudo-regular function in a domain D is an interior transformation of D by Theorem 1. Therefore by Stoïlow's theorem [9] we obtain immediately the following two:

Theorem 11. *Let $\{z_n\}$ $(z_n \neq z_m,\ n \neq m;\ n=1,2,\cdots)$ be a sequence of points in a domain D accumulating at a point in D. If $f(z)$ is a pseudo-analytic function in D such that $f(z_n)=0$ $(n=1,2,\cdots)$, then $f(z)\equiv 0$.*

Theorem 12. (MAXIMUM-MODULUS PRINCIPLE) *If $f(z)$ is pseudo-regular in a domain D, then $|f(z)|$ cannot attain its maximum in D at any interior point of D.*

A point z_0, at which $f(z)$ is not pseudo-analytic, is called a singular point of the pseudo-analytic function $f(z)$.

If $f(z)$ is pseudo-analytic in D except an interior point z_0 in D and $\lim_{z\to z_0} f(z)$ does not exist, then the point z_0 is called an isolated essential singularity of the pseudo-analytic function $f(z)$.

Theorem 13. *If $w=f(z)$ is a pseudo-analytic function of bounded dilatation-quotient in $0<|z|<1$, then the Riemann surface of its inverse function $z=f^{-1}(w)$ can be mapped one-to-one and conformally to $0<|\zeta|<1$.*

PROOF. This Riemann surface can be mapped one-to-one and conformally to a ring-domain $0\leq r<|\zeta|<1$ by an analytic function $\zeta=g(w)$. Then $\varphi(z)\equiv g(f(z))$ is a univalent pseudo-regular function of bounded dilatation-quotient in $0<|z|<1$. By Teichmüller's theorem [10] we conclude $r=0$.

Put $\varphi(0)=0$ in the above proof. Then $z=0$ must be a pseudo-conformal point or a critical point of $\varphi(z)$. Hence $\zeta=\varphi(z)$ is pseudo-regular in $|z|<1$. Then $\zeta=\varphi(z)$ can be considered as the uniformizer for $f(z)$ at the isolated singularity. The local uniformizer at the isolated singularity is defined in the same way. The following theorem is well-known (cf. Grötzsch [1], Lavrentieff [3]):

Theorem 14. (EXTENSION OF PICARD'S THEOREM) *A pseudo-analytic function of bounded dilatation-quotient takes every value infinitely often, with two possible exceptions, in any neighbourhood of an isolated essential singularity of it.*

PROOF. By the local uniformizer for $f(z)$ at the singularity we can reduce our theorem to the Picard's theorem on the analytic functions.

Theorem 15. *If a pseudo-regular function $f(z)$ of bounded dilatation-quotient in $0<|z|<1$ is bounded, it is pseudo-regular in $|z|<1$.*

PROOF. By Theorem 14 $z=0$ cannot be an essential singularity of $f(z)$. Hence $\lim f(z)=a$ exists and is finite. Put $f(0)=a$. Then $f(z)$ is continuous in $0\leq|z|<1$ and consequently is pseudo-regular there.

Theorem 16. (EXTENSION OF LIOUVILLE'S THEOREM) *A pseudo-analytic function of boundnd dilatation-quotient cannot be bounded at all finite points of the plane unless it reduces to a constant.*

PROOF. Let $f(z)$ be pseudo-regular and bounded in $|z|<\infty$. Suppose it were not a constant. Put $z=\frac{1}{\zeta}$. Then $f\left(\frac{1}{\zeta}\right)$ is bounded in a neighbourhood of $\zeta=0$. Hence by Theorem 15 $\zeta=0$ is a removable singularity. Therefore $|f(z)|$ must take its maximum at a finite point, which contradicts Theorem 12.

Let D_0 be the domain after extracting a closed set of capacity zero from $|z|<1$. The following two facts are already known:

Let $w=f(z)$ be the univalent pseudo-regular function of bounded dilatation-quotient in D_0 such that it establishes the correspondence between $|z|=1$ and $|w|=1$ and that $|f(z)|<1$ for $|z|<1$. Then the boundary of this image in $|w|<1$ is a closed set of capacity zero (cf. Pfluger [5], Yosida [11]).

If an analytic function $f(z)$ in D_0 is bounded, then it is regular in D (cf. Nevanlinna [4]).

Therefore we obtain immediately the following:

Theorem 17. *If a pseudo-analytic function of bounded dilatation-quotient in a domain D except for a closed set consisting of an enumerable number of points is bounded, then it is pseudo-regular in D.*

We say that a sequence $\{f_n(z)\}$ $(n=1,2,\cdots)$ of functions is spherically convergent if and only if, for an arbitrary positive number ε we can find a number N such that for $n, m\geq N$ we have $d(f_n(z), f_m(z))<\varepsilon$, where d is the distance on the Riemann sphere.

We shall extend the concept of the normal family by replacing the planer uniform convergence with the spherical one.

Theorem 18. *Let $\mathfrak{F}$ be a family of pseudo-analytic functions of uniformly bounded dilatation-quotient in D. If all functions of $\mathfrak{F}$ do* *not take three fixed values in D, then* $\mathfrak{F}$ *is a normal family in D.*

PROOF. Let D^* be an arbitrary closed domain in D. If $\mathfrak{F}$ were not spherically equicontinuous, there would exist a positive number α, such that $d(f_n(z_n), f_n(z_n')) \geq \alpha > 0$ for appropriate sequences $\{z_n\}$, $\{z_n'\}$ and $\{f_n(z)\}$ $(n=1,2,\cdots)$, where $z_n \in D^*$, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_n' = z_0 \neq z_n$ and $f_n \in \mathfrak{F}$. We denote by $\zeta = \varphi_n(z)$ the local uniformizer for $f_n(z)$ at z_0 and put $F_n(\zeta) \equiv f_n(\varphi_n^{-1}(\zeta))$. Since $\{\varphi_n(z)\}$ $(n=1,2,\cdots)$ is normal by Lemma 4, an appropriate subsequence $\{\varphi_{n_\nu}(z)\}$ $(\nu=1,2,\cdots)$ of it will converge to a function $\varphi(z)$. Denote $\zeta_{n_\nu} = \varphi_{n_\nu}(z_{n_\nu})$, $\zeta_{n_\nu}' = \varphi_{n_\nu}(z_{n_\nu}')$ and $\zeta_0 = \varphi(z_0)$. Then we have $\lim_{\nu\to\infty} \zeta_{n_\nu} = \lim_{\nu\to\infty} \zeta_{n_\nu}' = \zeta_0$ as in the proof of Lemma 5, while $d(F_{n_\nu}(\zeta_{n_\nu}), F_{n_\nu}(\zeta_{n_\nu}')) \geq \alpha > 0$. This contradicts the fact that $\{F_{n_\nu}(\zeta)\}$ is spherically equicontinuous, for the functions $F_{n_\nu}(\zeta)$ are analytic and do not take three fixed values in $|\zeta| < 1$.

Theorem 19. (EXTENSION OF SCHOTTKY'S THEOREM). *If* $f(z)$ *is a pseudo-regular function of bounded dilatation-quotient* $(Q[f(z)] \leq K)$ *in* $|z| < R$ *with the conditions* $f(z) \neq 0$, $f(z) \neq 1$ *and* $f(0) = a_0$, *then we have*

$$|f(z)| < S(a_0, \theta, K) \quad in \quad |z| \leq \theta R \quad (0 \leq \theta < 1),$$

where $S(a_0, \theta, K)$ *depends on* a_0, θ *and* K *only.*

PROOF. If it were not true, there would exist a sequence $\{f_n(z)\}$ of pseudo-regular functions of bounded dilatation-quotient and a sequence $\{z_n\}$ in $|z| \leq \theta R$ $(n=1,2,\cdots)$, such that $\lim_{n\to\infty} f_n(z_n) = \infty$. But by Theorem 18 $\{f_n(z)\}$ is normal in $|z| < R$, so an appropriate subsequence $\{f_{n_\nu}(z)\}$ $(\nu=1,2,\cdots)$ of it would converge to an interior transformation $g(z)$ there. Since $\lim_{\nu\to\infty} z_{n_\nu} = z_0$, $|z_0| \leq \theta R$, we would have $g(z_0) = \infty$. This contradicts the maximum-modulus principle.

Let $f(z)$ be a pseudo-analytic function and let $\zeta = \varphi_t(z)$ be the local uniformizer for $f(z)$ at $z = t$. If we put $f(z) = f(\varphi_t^{-1}(\zeta)) \equiv F(\zeta)$, then $F(\zeta)$ is alalytic. Let V_t and $V_{t'}$ be the circular neighbourhoods with centres at t and t' respectively. If $V_t \cdot V_t' \neq 0$, then for $z \in V_t \cdot V_{t'}$ the correspondence between $\varphi_t(z)$ and $\varphi_{t'}(z)$ is conformal. Hence

$$\frac{df(z)}{d\varphi_t(z)} \cdot d\varphi_t(z) = \frac{df(z)}{d\varphi_{t'}(z)} \cdot d\varphi_{t'}(z) .$$

We have immediately:

Theorem 20. *Let* $f(z)$ *be a pseudo-meromorphic function in a domain*

D, and $N(0)$ *and* $N(\infty)$ *be respectively the number of zeros and poles of* $f(z)$ *within a closed contour C in D. Then*

$$\frac{1}{2\pi i}\int_C \frac{\frac{df(z)}{d\varphi(z)}}{f(z)}\, d\varphi(z) = N(0)-N(\infty)\,,$$

where $\varphi(z)$ *is the local uniformizer for* $f(z)$ *at* z.

Corollary.

$$\frac{1}{2\pi}\int_C d\arg f(z) = N(0)-N(\infty)\,.$$

Suppose a sequence of pseudo-analytic (pseudo-regular) functions of uniformly bounded dilatation-quotient ($Q \leq K$) converges uniformly in a domain D. Then the limit function of it is an interior transformation on the Riemann sphere, but is not pseudo-analytic (pseudo-regular) in general. The class of all such functions contains all the pseudo-analytic (pseudo-regular) functions of bounded dilatation-quotient ($Q \leq K$). We shall call it *PAK-class* (*PRK-class*) in D. It will be easily seen that almost all theorems after § 2 remain valid for functions belonging to *PAK*-class. In addition we have for this class the following:

Theorem 21. *The PAK-class in a domain D is complete.*

PROOF. Let $\{f_n(z)\}$ $(n=1, 2, \cdots)$ be a uniformly covergent sequence of functions of *PAK*-class in D. Then for each term $f_n(z)$ we can choose a sequence $\{g_{n,m}(z)\}$ $(m=1, 2, \cdots)$ of pseudo-analytic functions of uniformly bounded dilatation-quotient ($Q[g_{n,m}(z)] \leq K$) converging uniformly to $f_n(z)$ in D. Hence for sufficiently large N, we have

$$|f(z)-f_n(z)| < \frac{\varepsilon}{2}\,, \quad |f_n(z)-g_{n,m}(z)| < \frac{\varepsilon}{2}\,,$$

provided that $n, m \geq N$, where $f(z) = \lim_{n\to\infty} f_n(z)$. Therefore

$$|f(z)-g_{n,m}(z)| < \varepsilon\,,$$

that is, $f(z)$ is the limit function of uniformly convergent sequence of uniformly bounded dilatation-quotient ($Q \leq K$) in D.

In the proof of Theorem 9 let $\zeta = \varphi_{n,t}(z)$ be the local uniformizer for $f_n(z)$ at t. Then the sequence $\{\varphi_{n,t}(z)\}$ $(n=1, 2, \cdots)$ is uniformly convergent in D. If we denote this limit function by $\zeta = \varphi_t(z)$, then $f(\varphi_t^{-1}(\zeta))$ is analytic in $|\zeta| < 1$. In the analogous manner as for Theorem

20 and Corollary we obtain the following:

Theorem 22. *Let $f(z)$ be a function of PAK-class in a domain D, and let $N(0)$ and $N(\infty)$ be respectively the number of zeros and poles of $f(z)$ within a closed contour C in D. Then*

$$\frac{1}{2\pi i}\int_C \frac{df(z)}{f(z)} = N(0)-N(\infty)\,.$$

Corollary.

$$\frac{1}{2\pi}\int_C d\arg f(z) = N(0)-N(\infty)\,.$$

Theorem 23. (EXTENSION OF ROUCHÉ'S THEOREM). *If $f(z)$ and $f(z)+g(z)$ are functions of PRK-class inside and on a closed contour C, and $|g(z)|<|f(z)|$ on C, then $f(z)+g(z)$ has exactly as many zeros inside C as $f(z)$.*

PROOF. If z is on C

$$\log\,(f(z)+g(z)) = \log f(z)+\log\left(1+\frac{g(z)}{f(z)}\right),$$

whence

$$\arg\,(f(z)+g(z)) = \arg f(z)+\arg\left(1+\frac{g(z)}{f(z)}\right).$$

On C, we have further $\left|\frac{g(z)}{f(z)}\right|<1$, and it follows therefore that the points $w=1+\frac{g(z)}{f(z)}$ are all situated in the interior of the circle $|1-w|<1$. Hence

$$\int_C d\arg\,(f(z)+g(z)) = \int_C d\arg f(z)\,.$$

By Corollary of Theorem 22 we complete the proof.

Theorem 24. (EXTENSION OF HURWITZ'S THEOREM). *Let a sequence $\{f_n(z)\}$ $(n=1,2,\cdots)$ of pseudo-regular functions of uniformly bounded dilatation quotient $(Q[f_n(z)]\leq K)$ be uniformly convergent in a domain D. Then the limit function $f(z)$ has exactly as many zeros in D^* as the function $f_n(z)$ for sufficiently large n, where D^* is an arbitrary subdomain bounded by a closed contour C in D.*

PROOF. $f(z)$ is a function of *PRK*-class in D. We take ε small enough so that all points of the circle $|z-z_0|=\varepsilon$ are in the interior of D and, moreover, $f(z)$ does not vanish in $|z-z_0|\leq\varepsilon$ except at z_0. Since $f(z)$ is continuous on $|z-z_0|=\varepsilon$, there exists a positive number

m such that $|f(z)|<m$ on this circumference. The sequence $\{f_n(z)\}$ converges uniformly on $|z-z_0|=\varepsilon$ and we shall therefore have $|f(z)-f_n(z)|<m$ for $|z-z_0|=\varepsilon$, provided n is taken large enough. Hence

$$|f(z)-f_n(z)|<m<|f(z)|, \quad \text{on} \quad |z-z_0|=\varepsilon .$$

By Theorem 23 the function

$$f_n(z) = f(z)+(f_n(z)-f(z))$$

will therefore have the same number of zeros in $|z-z_0|<\varepsilon$ as $f(z)$. Thus our theorem is proved.

Theorem 25. *If the terms of a sequence* $\{f_n(z)\}$ $(n=1, 2, \cdots)$ *are univalent pseoudo-analytic functions of uniformly bounded dilatation-quotient* $(Q[f_n(z)]\leq K)$ *in a domain D and the sequence converges uniformly to a non-constant function* $f(z)$ *in D, then* $f(z)$ *is also univalent in D.*

PROOF. $f(z)$ is of a function of *PAK*-class in *D*. Suppose $f(z_1)=f(z_2)$ $(z_1\neq z_2, z_1, z_2\in D)$ and consider the sequence of functions

$$g_n(z)=f_n(z)-f_n(z_1) \quad (n=1, 2, \ldots).$$

Since $f_n(z)$ is univalent, we shall have $g_n(z)\neq 0$ except at $z=z_1$. The limit function $g(z)=f(z)-f(z_1)$ vanishes at $z=z_2$. By Theorem 24 $g_n(z)$ must therefore vanish within an arbitrary small neighbourhood of z_2, provided n is large enough. However, since $g_n(z)$ does not vanish in D except at z_1, this is impossible. Our assumption that $f(z_1)=f(z_2)$ thus leads to a contradiction.

Theorem 26. (EXTENSION OF BLOCH'S THEOREM). *If* $f(z)$ *is a pseudo-regular function of bounded dilatation-quotient* $(Q[f(z)]\leq K)$ *in* $|z|<1$ *and* $\max\limits_{|z|\leq\frac{1}{2}}|f(z)-f(0)|\geq 1$, *then the Riemann surface of its inverse function always contains a schlicht circular disc with radius* β, *where* β *is a positive constant independent of the function* $f(z)$.

PROOF. Let $\zeta=\varphi(z)$ be the uniformizer for $f(z)$. Then $F(\zeta)\equiv f(\varphi^{-1}(\zeta))$ is regular in $|\zeta|<1$. Then there exists a positive number α independent of $f(z)$ such that

$$\max_{|\varphi^{-1}(\zeta)|\leq\frac{1}{2}}|F'(\zeta)|\geq\alpha>0 .$$

For, otherwise, we could find a sequence $\{f_n(z)\}$ $(n=1, 2, \cdots)$ satisfying the conditions of the theorem, such that

$$\lim_{n\to\infty} \max_{|\varphi_n^{-1}(\zeta)|\leq\frac{1}{2}} |F_n'(\zeta)| = 0,$$

where $\zeta = \varphi_n(z)$ is the uniformizer for $f_n(z)$ and $F_n(\zeta) \equiv f_n(\varphi_n^{-1}(\zeta))$. Then we would have $\max_{|z|\leq\frac{1}{2}} |f_n(z) - f_n(0)| < 1$, provided n is taken large enough. This is contrary to our assumption.

Therefore there exists a point ζ_0 in $|\varphi^{-1}(\zeta)| \leq \frac{1}{2}$, such that $|F'(\zeta_0)| = \alpha' \geq \alpha$. Put

$$\Phi(t) \equiv \frac{F(\zeta)}{\alpha'(1-|\zeta_0|)^2}$$

with $t = \dfrac{\zeta-\zeta_0}{1-\bar{\zeta}_0\zeta}$. Then $\Phi(t)$ is regular in $|t|<1$ and $|\Phi'(0)| = 1$. By Bloch's theorem the Riemann surface of the inverse function of $\Phi(t)$ contains a schlicht circular disc with radius B. Hence our Riemann surface contains a schlicht circular disc with radius β.

Theorem 27. *If $f(z)$ and $g(z)$ are the pseudo-analytic functions with common local uniformizer $\{\varphi_t(z)\}$ at every point of a domain D, and if $f(z_n) = g(z_n)$ for $z_n \in D$ ($z_n \neq z_m$, $n = 1, 2, \cdots$; $\lim_{n\to\infty} z_n = z_0 \in D$), then we have*

$$f(z) \equiv g(z)$$

in D.

PROOF. Put $f(z) - g(z) = f(\varphi_t^{-1}(\zeta)) - g(\varphi_t^{-1}(\zeta)) \equiv F(\zeta)$. Then $F(\zeta) = F(\varphi_t(z))$ is an analytic function of ζ. Hence $f(z) - g(z)$ is pseudo-analytic in D. By Theorem 11 we complete the proof.

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