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On a Topological Characterization of the Dilatation in E^3

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Introduction

A topological characterization of the dilatation in E^2 has been given by B. v. Kerékjártó [5][1] and recently in another form by us [2]. The purpose of this paper is to give a topological characterization of the dilatation in E^3. In fact we shall prove the following

Theorem. *Let h be a homeomorphism of E^3 onto itself satisfying the following conditions:*

(i) *for each $x \in E^3$ the sequence $h^n(x)$ converges to the origin o when $n \to \infty$ and*

(ii) *for each $x \in E^3$ except for o the sequence $h^n(x)$ converges to the point at infinity when $n \to -\infty$.*

Then if h is sense preserving, h is topologically equivalent to the transformation

$$x' = \frac{1}{2}x,\ y' = \frac{1}{2}y,\ z' = \frac{1}{2}z$$

and if h is sense reversing, h is topologically equivalent to the transformation

$$x' = \frac{1}{2}x,\ y' = \frac{1}{2}y,\ z' = -\frac{1}{2}z$$

in Cartesian coordinates.

§1.

1. NOTATIONS. Throughout this paper h is a given homeomorphism of the 3-dimensional Euclidean space E^3 onto itself given by the assumption of our Theorem.

Following notations will be used:

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1) The numbers in the brackets refer to the references at the end of this paper.

$B(T)=$ the boundary of T.
$\mathrm{Int}(T)=T-B(T)$.
$U_\varepsilon(T)=\{x\mid d(x, T)<\varepsilon\}$.
$[a, b]=\{x\mid a\leqq x\leqq b\}$.
$S_r=\{x\mid |x|=r\}$.

Let M and M' be two 2-manifolds in E^3. $M'\ll M$ means that M' is contained in the bounded component of the complementary domain of M. As an exceptional case we shall write $o\ll M$ which means also that o is contained in the bounded component of the complementary domain of M.

2. **Lemma 1.** *If T is a compact subset of E^3, then the sequence $h^n(T)$ converges to o when $n\to\infty$ and if T is a compact subset which does not contain o, then the sequence $h^n(T)$ converges to the point at infinity when $n\to-\infty$.*

This is a consequence of Lemmas 5 and 6 of [2].

§2.

3. Now we shall prove the following

Lemma 2. *Let T be a compact subset of E^m and g a homeomorphism of E^m onto itself such that*

(i) $g(T)\subset T$,
(ii) $B(g^n(T))\frown B(T)\neq 0$,
(iii) $g^{n+1}(T)\subset Int\,(T)$,

where n is a natural number. Let ε be a positive real number. Then there exists a compact subset T' such that

(i) $T\subset T'\subset U_\varepsilon(T)$,
(ii) $g(T')\subset T'$,
(iii) $g^n(T)\subset Int\,(T')$.

And if T is a continuum, then T' is also a continuum.

Proof. Put $B(g^n(T))\frown B(T)=C\neq 0$. Since C is compact and

$$g(C)\subset g(B(g^n(T)))=B(g^{n+1}(T))\subset \mathrm{Int}(T),$$

there exists a positive real number δ_0 such that

$$g(\overline{U_{\delta_0}(C)})\subset \mathrm{Int}(T).$$

Let $\delta<\mathrm{Min}(\varepsilon, \delta_0)$ and put

$$T'=T\smile \overline{U_\delta(C)}.$$

It is easy to see that $T \subset T' \subset U_\varepsilon(T)$ and that $g(T') \subset T'$. Now we prove that $g^n(T') \subset \text{Int}(T')$. If $x \in T$, then

$$g^n(x) \in g^n(T) \subset \text{Int}(T) \cup (B(T) \cap g^n(T))$$
$$\subset \text{Int}(T) \cup C \subset \text{Int}(T) \cup U_\delta(C) \subset \text{Int}(T').$$

If $x \in \overline{U_\delta(C)}$, then

$$g(x) \in g(\overline{U_\delta(C)}) \subset \text{Int}(T) \subset \text{Int}(T').$$

Therefore $g^n(x) \in \text{Int}(T')$. Then we have $g^n(T') \subset \text{Int}(T')$.

From the above construction of T' it follows that if T is a continuum, then T' is also a continuum. Thus the proof of Lemma 2 is complete.

It follows from Lemma 2 the following

Lemma 3. *Let T be a compact subset of E^m and g a homeomophism of E^m onto itself such that*

(i) $g(T) \subset T$,

(ii) *there exists a natural number N such that* $g^N(T) \subset \text{Int}(T)$.

Let ε be a positive real number. Then there exists a compact subset T' such that

(i) $T \subset T' \subset U_\varepsilon(T)$,

(ii) $g(T') \subset \text{Int}(T')$.

And if T is a continuum, then T' is also a continuum.

4. Now we put

$$V = \{x \mid |x| \leqq 1\}.$$

By Lemma 1 there exists a natural number N such that $h^N(V) \subset \text{Int}(V)$. Put

$$\bigcup_{n=0}^{N-1} h^n(V) = T.$$

Clearly $h(T) \subset T$, $h^N(T) \subset \text{Int}(T)$ and T is a continuum. Then by Lemma 3 there exists a continuum T' such that

(i) $V \subset T'$,

(ii) $h(T') \subset \text{Int}(T')$.

From this fact it follows that there exists a polyhedral 2-manifold M such that $o \ll h(M) \ll M$.

REMARKS. It is to be remarked that the existence of M can also be proved by the method used by Prof. H. Terasaka [9].

§ 3.

5. In this paragraph we shall construct a piecewise linear ap-

proximation h_o of h with suitable properties.

Let M_0 be a polyhedral 2-manifold homeomorphic to the polyhedral 2-manifold M given in §2. Let φ be a piecewise linear homeomorphism of the product space $M_o \times [-1, 1]$ into E^3 such that

(i) $\varphi(M_0 \times 0) = M$,

(ii) $\varphi(M_0 \times t) \gg M$, where $0 < t \leq 1$,

(iii) $\varphi(M_0 \times t) \ll M$, where $-1 \leq t < 0$.

Then there exists a positive real number η such that

$$h\varphi(M_0 \times [-\eta, \eta]) \cap \varphi(M_0 \times [-\eta, \eta]) = 0$$

and that

$$h^{-1}\varphi(M_0 \times [-\eta, \eta] \cap \varphi(M_0 \times [-\eta, \eta]) = 0 .$$

Now let ψ be a homeomorphism of $M_0 \times [-\eta, \eta]$ onto itself such that

(i) if $0 \leq t \leq 1$, then $\psi(m \times t\eta) = (m \times (\frac{1}{2}t + \frac{1}{2})\eta)$,

(ii) if $-1 \leq t \leq 0$, then $\psi(m \times t\eta) = (m \times (\frac{3}{2}t + \frac{1}{2})\eta)$,

where $m \in M_0$.

Let h_1 be a homeomorphism of E^3 onto itself such that

(i) for each $x \in E^3 - h^{-1}\varphi(M_0 \times [-\eta, \eta]) - \varphi(M_0 \times [-\eta, \eta])$
$$h_1(x) = h(x),$$

(ii) for each $x \in \varphi(M_0 \times [-\eta, \eta])$
$$h_1(x) = h\varphi\psi\varphi^{-1}(x),$$

(iii) for each $x \in h^{-1}\varphi(M_0 \times [-\eta, \eta])$
$$h_1(x) = \varphi\psi\varphi^{-1}h(x).$$

By the construction of h_1 clearly

$$h(M) \ll h_1(M) \ll M \ll h_1^{-1}(M) \ll h^{-1}(M).$$

Now let ε be a positive real number such that

$$\varepsilon < \mathrm{Min}\,(d(M, h_1(M)), d(h_1(M), h(M)), d(M, h_1^{-1}(M)), d(h_1^{-1}(M), h^{-1}(M)))$$

and let h_0 be a piecewise linear homeomorphism of E^3 onto itself such that

$$d(h_0(x), h_1(x)) < \varepsilon .$$

The existence of such a homeomorphism h_0 is proved by E. E. Moise [7]. Clearly

(i) $h(M) \ll h_0(M) \ll M \ll h_0^{-1}(M) \ll h^{-1}(M)$,

(ii) all $M, h_0(M), h_0^2(M), \ldots$ and $h_0^{-1}(M), h_0^{-2}(M), \ldots$ are polyhedral.

§4.

6. In this paragraph we shall define two modifications which will be used in §5.

The modification m_1. Let M and M_0 be two polyhedral 2-manifolds in E^3 such that $M_0 \ll M$. Suppose that M_0 is not a polyhedral 2-sphere. Suppose further that there exist an (E^3-M)-unknotted polygon P on M and one of the associated disk[2], say $D(P)$, such that $D(P) \cap M_0$ is the union of a (non-zero) finite number of mutually disjoint simple closed polygons Q_i. Let $D(Q_i)$ be the polyhedral disk bounded by Q_i in $D(P)$.

Under the above assumption we shall define the modification m_1 as follows: For each Q_i homotopic to 0 in M_0 there exists one and only one polyhedral disk $D[Q_i]$ on M_0 whose boundary-polygon is Q_i. Put $Q_i < Q_j$, if $D[Q_i] \subset D[Q_j]$. Let Q_0 be one of the minimal elements (homotopic to 0 in M_0) with respect to the above ordering. Let Q_0' be a simple closed polygon in $D(P)$ sufficiently near to Q_0 without intersecting $D(Q_0)$. Then there exists a polyhedral disk $D'[Q_0']$ whose boundary-polygon is Q_0' such that $D'[Q_0']$ is sufficiently near to $D[Q_0]$ and that

$$D'[Q_0'] \cap D(P) = Q_0' \quad \text{and} \quad D'[Q_0'] \cap M_0 = 0 .$$

Put

$$m_1'(D(P)) = (D(P) - D(Q_0')) \cup D'[Q_0'] .$$

This is a modification of $D(P)$. If we repeat this modification step by step as long as possible, then we have an associated disk $m_1(D(P))$, which will be called the associated disk deduced from $D(P)$ by the modification m_1.

It should be pointed out that the added part to $D(P)$ by the modification m_1 is sufficiently near to M_0.

If $Q_i \subset D[Q_j]$ for some Q_j homotopic to 0 in M_0, then Q_i is homotopic to 0 in M. From this fact it follows that $m_1(D(P)) \cap M_0$ consists of only a finite number of simple closed polygons not homotopic to 0 in M.

7. *The modification* m_2. Let M be a polyhedral 2-manifold in E^3 with genus p. Let P be an $(E^3\text{-}M)$-unknotted polygon on M and $D(P)$ one of the associated disks.. Then there exist a simple closed polygon P' on M sufficiently near to P without intersecting P and a polyhedral disk $D'(P')$ whose boundary-polygon is P' such that $D'(P')$ is sufficiently near to $D(P)$ and that

$$D'(P') \cap D(P) = 0 \quad \text{and} \quad D'(P') \cap M = P .$$

2) Let P be an $(E^3\text{-}N)$-unknotted polygon in M and $D(P)$ one of the associated disks. Hereafter it is always assumed that $D(P)$ is a polyhedral disk, where the boundary-polygon of $D(P)$ is P, such that $D(P) \cap M = P$ and that $N \cap (D(P)-P) = 0$. (See [3]).

Let R be the ring bounded by P and P' in M. Put

$$m_2(M) = (M - R) \cup D(P) \cup D'(P') .$$

This modification will be called the modification m_2 of M along $D(P)$.

If P is not homologous to 0 in M, then $m_2(M)$ is a polyhedral 2-manifold with genus $p-1$. If P is homologous to 0 in M, then $m_2(M)$ consists of two polyhedral 2-manifolds M' and M'' with genus p' and p'', where $p'+p''=p$. And if P is homologous to 0 but not homotopic to 0 in M, then $p'<p$ and $p''<p$ hold.

§ 5.

8. In this paragraph we shall obtain by modifying the polyhedral 2-manifold M a polyhedral 2-sphere S such that $o \ll h(S) \ll S$. If the genus p of M is equal to 0, then we have already the required 2-sphere. If $p>0$, we are only to prove that there exists a polyhedral 2-manifold M' of genus p' smaller than p such that $0 \ll h(M') \ll M'$. Assume therefore that the genus of M is different from 0.

By a theorem of T. Homma [3] there exists at least one (E^3-M)-unknotted polygon on M not homotopic to 0 in M. It is easy to see that for each (E^3-M)-unknotted polygon P on M there exists one of the associated disks say $D(P)$ such that $D(P) \cap o = 0$. Therefore if h_0 is a piecewise linear approximation sufficiently near to h, then there is a natural number N such that there exists an (E^3-M)-unknotted polygon P_1 on M not homotopic to 0 in M and one of the associated disks, say $D_1(P_1)$, satisfying the conditions

$$D_1(P_1) \cap h_0^N(M) = 0 \quad \text{and} \quad D_1(P_1) \cap h_0^{-N}(M) = 0 .$$

Hereafter we assume that h_0 is such a piecewise linear approximation sufficiently near to h.

Let M_0 be a polyhedral 2-manifold such that $o \ll M_0 \ll M$. Let P be an (E^3-M)-unknotted polygon on M and $D(P)$ one of the associated disks such that $D(P) \cap M_0 \neq 0$. Let ε be a positive real number. It is also easy to see that there exists one of the associated disks say $D_2(P)$ such that $D_2(P) \cap M_0$ is the union of a finite number of mutually disjoint simple closed polygons and that $D_2(P) \subset U_\varepsilon(D(P))$.

The similar statement holds for a polyhedral 2-manifold M_0' such that $M \ll M_0'$.

9. Now we shall prove the following proposition.

(*) *Suppose that there exist an (E^3-M)-unknotted polygon P on M*

not homotopic to 0 *in M and one of the associated disks, say* $D(P)$, *such that*

$$D(P) \cap h_0^n(M) \neq 0 \quad and \quad D(P) \cap h_0^{n+1}(M) = 0 \,,$$

where n is a natural number. Then there exists an $(E^3 - M - h_0^n(M) - h_0^{-n}(M))$*-unknotted polygon on M not homotopic to* 0 *in M.*

PROOF. By the above arguments there exists one of the associated disks, say $D_0(P)$, such that $D_0(P) \cap h^n(M)$ is the union of a finite number of mutually disjoint simple closed polygons and that $D_0(P) \cap h_0^{n+1}(M) = 0$.

If $D_0(P) \cap h_0^n(M) = 0$, then P itself is the required polygon. Now suppose that $D_0(P) \cap h_0^n(M) \neq 0$. Using the modification m_1, we have one of the associated disks, say $m_1(D_0(P))$, such that $m_1(D_0(P)) \cap h_0^n(M)$ consists of only a finite number s of mutually disjoint simple closed polygons not homotopic to 0 in $h_0^n(M)$ and that $m_1(D_0(P)) \cap h_0^{n+1}(M) = 0$.

If $s = 0$, then $m_1(D_0(P)) \cap h_0^n(M) = 0$. Therefore P is again the required polygon. Now we assume that $s > 0$. Let Q be one of the innermost simple closed polygons in the associated disk $m_1(D_0(P))$. Then it is easy to see that Q is an $(E^3 - h_0^n(M) - h_0^{n+1}(M) - M)$-unknotted polygon on $h_0^n(M)$ not homotopic to 0 in $h_0^n(M)$. Put $P_0 = h_0^{-n}(Q)$. Then P_0 is an $(E^3 - M - h_0(M) - h_0^{-n}(M))$-unknotted polygon on M not homotopic to 0 in M. Therefore P is the required polygon and the proof of (*) is complete.

Similarly we have the following proposition.

(**) *Suppose that there exist an* $(E^3 - M)$*-unknotted polygon P on M not homotopic to* 0 *in M and one of the associated disks say* $D(P)$ *such that*

$$D(P) \cap h_0^{-n}(M) \neq 0 \quad and \quad D(P) \cap h_0^{-(n+1)}(M) = 0 \,.$$

Then there exists an $(E^3 - M - h_0^n(M) - h_0^{-n}(M))$*-unknotted polygon on M not homotopic to* 0 *in M.*

By the arguments in Nr. 8 and propositions (*) and (**) we see immediately that there exists an $(E^3 - M - h_0(M) - h_0^{-1}(M))$-unknotted polygon P on M not homotopic to 0 in M.

Since $h(M) \ll h_0(M) \ll M \ll h_0^{-1}(M) \ll h^{-1}(M)$, we see also that there exists an $(E^3 - M - h(M) - h^{-1}(M))$-unknotted polygon P on M not homotopic to 0 in M.

10. If above given P is not homologous to 0 in M, then by the modification m_2 of M along an associated disk we have a polyhedral 2-manifold M' with genus $p' = p - 1$ such that $o \ll h(M') \ll M'$.

If P is homologous to 0 in M, then by the modification m_2 of M along an associated disk we have two polyhedral 2-manifolds M_1'' and M_2'' with genus $p_1'' < p$ and $p_2'' < p$, where $p_1'' + p_2'' = p$. It is easy to see that one of M_1'' and M_2'' say M' has the property $o \ll h(M') \ll M'$.

Then by the arguments in Nr. 8 we have a polyhedral 2-sphere S such that $o \ll h(S) \ll S$.

§ 6.

11. Since S is a polyhedral 2-sphere and $h(S) \frown S = 0$, it is easy to see that $S \smile h(S)$ is semi-locally tamely imbedded in E^3. Then by a theorem of E. E. Moise [8] there exists a homeomorphism g_1 of E^3 onto itself such that

(i) $g_1(o) = o$,

(ii) $g_1(x) = x$ for every $x \in S$,

(iii) $g_1 h(S)$ is polyhedral,

(iv) $d(x, g_1(x)) < d(S, h(S))$ for every $x \in E^3$.

Since $g_1 h(S) \ll S$, by a theorem of Alexander-Moise [1] [6] there exists a homeomorphism g_2 of E^3 onto itself such that

(i) $g_2 g_1(o) = o$,

(ii) $g_2 g_1(S) = S_2$,

(iii) $g_2 g_1 h(S) = S_1$.

Using the polar coordinates in E^3, for each $x = (\varphi, \psi, 2) \in S_2$ put

$$f(x) = f(\varphi, \psi, 2) = (\varphi', \psi', 1) = g_2 g_1 h g_1^{-1} g_2^{-1}(x)$$

and put

$$f'(\varphi, \psi, 2) = (\varphi', \psi', 2).$$

Then f' is a homeomorphism of S_2 onto itself.

12. Now we assume that h is sense preserving. Then it is easy to see that f' is a sense preserving homeomorphism of S_2 onto itself. Therefore by the deformation theorem of Tietze (See for instance [4]) there exists a family of homeomorphisms $f_t(\varphi, \psi, 2) = (\varphi_t, \psi_t, 2)$, where $0 \leq t \leq 1$, such that $f_0 = f'$ and that f_1 is the identity mapping of S_2. Now we define a homeomorphism F_0 of the closure of the domain bounded by S_1 and S_2 onto itself as follows:

$$F_0(\varphi, \psi, 1 + t) = (\varphi_t, \psi_t, 1 + t),$$

where $0 \leq t \leq 1$. This homeomorphism F_0 can be extended to a homeomorphism F of E^3 onto itself as follows: If $x = (\varphi, \psi, r)$, where $r \neq 0$, then there exists one and only one integer n such that $1 < 2^n r \leq 2$.

Put

$$F(x)=F(\varphi,\ \psi,\ r)=g_2g_1h^ng_1^{-1}g_2^{-1}F_0(\varphi,\ \psi,\ 2^nr)$$

and

$$F(\varphi,\ \psi,\ 0)=(\varphi,\ \psi,\ 0)\,.$$

Now let H be the transformation

$$H(\varphi,\ \psi,\ r)=(\varphi,\ \psi,\ \tfrac{1}{2}r)\,.$$

Then it will be seen that

$$H(x)=F^{-1}g_2g_1hg_1^{-1}g_2^{-1}F(x)$$

for every $x\in E^3$. For if $x=o$, then

$$F^{-1}g_2g_1hg_1^{-1}g_2^{-1}F(o)=H(o)$$

is evident. If $x=(\varphi,\ \psi,\ r)$, where $1<2^nr\leqq 2$, then

$$\begin{aligned}&F^{-1}g_2g_1hg_1^{-1}g_2^{-1}F(\varphi,\ \psi,\ r)\\&\quad=F^{-1}g_2g_1hg_1^{-1}g_2^{-1}g_2g_1h^ng_1^{-1}g_2^{-1}F_0(\varphi,\ \psi,\ 2^nr)\\&\quad=F^{-1}g_2g_1h^{n+1}g_1^{-1}g_2^{-1}F_0(\varphi,\ \psi,\ 2^nr)\\&\quad=F^{-1}g_2g_1h^{n+1}g_1^{-1}g_2^{-1}F_0(\varphi,\ \psi,\ 2^{n+1}\cdot\tfrac{1}{2}r)\\&\quad=F^{-1}F(\varphi,\ \psi,\ \tfrac{1}{2}r)=(\varphi,\ \psi,\ \tfrac{1}{2}r)=H(\varphi,\ \psi,\ r)\,.\end{aligned}$$

Thus h is topologically equivalent to H and the proof of the first part of our Theorem is complete.

The second part of our Theorem, where h is sense reversing, can be proved similarly.

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