Osaka Mathematical Journal
Vol. 6, No. 1, June, 1954

Note on Generalised Uniserial Algebras, I

By Tensho YOSHII

An associative algebra with unit element is called *generalized uniserial* if every primitive left ideal[1] as well as every primitive right ideal possesses only one composition series, and it is called *uniserial* if it is generalized uniserial and moreover it is primarily decomposable.

The structure of absolutely uniserial algebras i. e. of those uniserial algebras which remain so after any coefficient field extension, was studied by T. Nakayama and G. Azumaya[2]. In the present note we shall consider absolutely generalized uniserial algebras i. e. those generalized uniserial algebras which remain so after any coefficient field extension.

Let A be an (associative and finite dimensional) algebra with unit element over a field K, and let N be its radical. Let

$$\bar{A} = A/N = \bar{A}_1 + \cdots + \bar{A}_k$$

be the direct decomposition of the semi simple residue class algebra $\bar{A} = A/N$ into simple components. Let $\bar{E}_\kappa$ be the unit element of $\bar{A}_\kappa$. Then $\bar{E}_\kappa$ is a sum of mutually orthogonal primitive idempotent elements $\bar{e}_{\kappa 1}, \bar{e}_{\kappa 2}, \dots, \bar{e}_{\kappa f(\kappa)}$. There are then mutually orthogonal primitive idempotent elements $e_{\kappa i} (\kappa = 1, \dots, k; i = 1, \dots, f(\kappa))$ in A such that $e_{\kappa i} \pmod N = \bar{e}_{\kappa i}$ and $\sum_{\kappa i} e_{\kappa i} =$ unit element in A. We put $E_\kappa = \sum_i e_{\kappa i}$ for each $\kappa = 1, \dots, k$. $A^0 = E'AE'$ is called a basic algebra of A, were $E' = \sum_\kappa e_{\kappa 1}$. It is clear that the radical of A^0 is $N^0 = E'NE'$.

The following lemma is easily seen.[3]

Lemma 1. *A is a generalized uniserial algebra if and only if $N^0 = CA^0 = A^0C$, $C \in A^0$.*

Lemma 2. *If A is generalized uniserial and A/N is separable, then A is absolutely generalized uniserial.*

---

1) A primitive left (right) ideal is a left (right) ideal generated by a primitive idempotent element. Cf. M. Thrall [1].

2) See G. Azumaya and T. Nakayama [2].

3) See K. Morita [3], Theorem 1.

PROOF. If A is generalized uniserial, the radical N^0 of A^0 is expressible as a principal ideal by Lemma 1. We put $N^0 = CA^0 = A^0C$.

Since the radical of A_L is N_L from the assumption, the radical of A_L^0 is $N_L^0 = CA_L^0 = A_L^0C$. Therefore A_L^0 is generalized uniserial.

Now let A^* be a basic algebra of A_L^0. Then A^* is also a basic algebra of A_L. On the other hand the radical of A^* is expressible as a principal ideal. Therefore A_L is generalized uniserial. Thus our lemma is proved.

From this lemma, we get the following theorem.

Theorem. *Let A be an absolutely generalized uniserial algebra. Then A is a direct sum of two subalgebras A_1 and A_2, where A_1 is a generalized uniserial algebra with the separable residue class algebra over its radical and A_2 is an absolutely uniserial algebra. The converse is also true.*

PROOF. (i) Suppose that A is an absolutely generalized uniserial algebra and, in the direct decomposition of $\bar{A}$, $\bar{A}_i$ $(i = 1, \dots, r)$ is a separable simple algebra and $\bar{A}_j$ $(j = r + 1, \dots, k)$ is an insepbrable simple algebra.

Now Let Γ be an algebraically closed field and let M be a radical of A_Γ. Then $\bar{A}_\Gamma = A_\Gamma / N_\Gamma$ is the direct sum of some matric algebras over Γ and some primary algebras with non-zero radicals. Here the former ones are obtained from $\bar{A}_1, \dots, \bar{A}_r$ and the latter ones are obtained from $\bar{A}_{r+1}, \dots, \bar{A}_k$. We put $\bar{A}_\Gamma = A_\Gamma / N_\Gamma = \bar{A}_1' + \cdots + \bar{A}_s' + \bar{A}_{s+1}' + \cdots + A_t'$, where $\bar{A}_1', \dots, \bar{A}_s'$ are matric algebras over Γ and $\bar{A}_{s+1}', \dots, \bar{A}_t'$ are primary algebras. Further let $\bar{E}_\kappa^*$ be the unit element of $\bar{A}_\kappa'$ $(\kappa = 1, \dots, s)$ and let $\bar{E}_\lambda^*$ be the unit element of $\bar{A}_\lambda'$ $(\lambda = s + 1, \dots, t)$. Then $\bar{E}_\kappa^*$ is a sum of mutually orthogonal idempotent elements $\bar{e}_{\kappa 1}^*, \dots, \bar{e}_{\kappa, \varphi(\kappa)}^*$ and $\bar{E}_\lambda^*$ is a sum of mutually orthogonal primitive idempotent elements $\bar{f}_{\lambda 1}^*, \dots, \bar{f}_{\lambda \varphi(\lambda)}^*$. There are then mutually orthogonal primitive idempotent elements $e_{\kappa i}^*$ $(\kappa = 1, \dots, s, i = 1, \dots, \varphi(\kappa))$ and $f_{\lambda j}^*$ $(\lambda = s + 1, \dots, t, j = 1, \dots, \varphi(\lambda))$ in A_Γ such that $e_{\kappa i}^* \pmod{N_\Gamma} = \bar{e}_{\kappa i}^*$, $f_{\lambda j}^* \pmod{N_\Gamma} = \bar{f}_{\lambda j}^*$ and $\sum_{\kappa, i} e_{\kappa i}^* + \sum_{\lambda, j} f_{\lambda j}^* =$ unit element in A_Γ. Furthermore we put $E^* = \sum_{\kappa, i} e_{\kappa i}^*$, $F^* = \sum_{\lambda, j} f_{\lambda j}^*$.

From the assumption of $\bar{A}_j'$ $(j = s + 1, \dots, t)$,

$$\bar{f}_{\lambda j}^* \bar{M} \neq 0, \quad \bar{f}_{\lambda j}^* \bar{M} / \bar{f}_{\lambda j}^* \bar{M}^2 \cong \bar{f}_{\lambda j}^* \bar{A}_\Gamma / \bar{f}_{\lambda j}^* \bar{M}, \quad \text{where} \quad \bar{f}_{\lambda j}^* \bar{A}_\Gamma = \bar{f}_{\lambda j}^* \bar{A}_\lambda'.$$

Now the right hand side is simple from our assumption. Therefore the left hand side is also simple. On the other hand, $\bar{f}_{\lambda j}^* A_\Gamma$ has a unique composition series from the assumption of A_Γ. Therefore there exists an n such that $f^*_{\lambda j}N_\Gamma = f^*_{\lambda j}M^n$ $(n \geq 2)$ and

$$\bar{f}^*_{\lambda j}\bar{M} / \bar{f}^*_{\lambda j}\bar{M}^2 \cong f^*_{\lambda j}M / f^*_{\lambda j}M^2 \cong f^*_{\lambda j}A_\Gamma / f^*_{\lambda j}M ,$$

consequently

$$f^*_{\lambda j}M^l / f^*_{\lambda j}M^{l+1} \cong f^*_{\lambda j}A_\Gamma / f^*_{\lambda j}M \quad (l = 1, \ldots, \rho - 1, f^*_{\lambda j}M^\rho = 0) .$$

Therefore $e^*_{\kappa i}A_\Gamma f^*_{\lambda j} = 0$ for each $e^*_{\kappa i}$. In the same way $f^*_{\lambda j}A_\Gamma e^*_{\kappa i} = 0$ for each $e^*_{\kappa i}$. Therefore $A_\Gamma = (E^* + F^*)A_\Gamma(E^* + F^*) = E^*A_\Gamma E^* + F^*A_\Gamma F^*$ and $F^*A_\Gamma F^*$ is a uniserial algebra from the above proof and $E^*A_\Gamma E^*$ is the direct sum of matric algebras over Γ. If we put $A = A_1 + A_2$, where $A_{1\Gamma} = E^*A_\Gamma E^*$, $A_{2\Gamma} = F^*A_\Gamma F^*$, it is clear that A is the direct sum of A_1 and A_2 which satisfy the conditions of this theorem.

(ii) Conversely suppose that A is generalized uniserial and $A = A_1 + A_2$ such that A_1 is a generalized uniserial algebra with the separable residue class algebra over its radical and A_2 is an absolutely uniserial algebra. Then $A_\Gamma = A_{1\Gamma} + A_{2\Gamma}$ and since A_2 is absolutely uniserial, $A_{2\Gamma}$ is uniserial and $A_{1\Gamma}$ is generalized uniserial from Lemma 2. Therefore A_Γ is generalized uniserial and A is absolutely generalized uniserial.

From this theorem, it follows readily

Corollary. *If A is an algebra such that the radical of A and that of A_Γ are expressible as principal ideals for any coefficient field extension, then A is a direct sum of A_1 and A_2, where A_1 has a radical expressible as a principal ideal with the separable residue class algebra over its radical and A_2 is an absolutely uniserial algebra. The converse is also true.*

(Received March 24, 1954)

Bibliography

[1] M. Thrall: Some generalizations of Quasi-Frobeniusean algebras. Trans. Amer. Math. Soc. **64** (1948), 173–183.

[2] G. Azumaya and T. Nakayama: On absolutely uniserial algebras, Japan J. of Math. **19** (1948), 263–273.

[3] K. Morita: On group rings over a modular field which possesses radicals expressible as principal ideals, Sci. Rep. Tokyo Bunri Daigaku, **4** (1951), 177–194.