

On Covering Property of Abstract Riemann Surfaces

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Let R be an abstract Riemann surface of finite genus belonging to the class O_{AB} , then it is well known that any covering surface on the w -plane, defined by a non-constant analytic function on R covers any point except at most a null-set, that is, the boundary of the surface of O_{AB} on the w -plane. In this paper we shall study Iversen's and Gross's property, but at present what we can prove is only that a subclass of O_{AB} has Iversen's property, thus the validity of Iversen's property of O_{AB} is an open problem.

1) We suppose a conformal metric is given on R , of which a line element is given by the local parameter $ds = \lambda(t)|dt|$, and let O be a fixed point of R . Denote by D_ρ the domain bounded by the point set having a distance $\rho : \rho < \infty$ from O , and suppose for $\rho < \infty$ that the domain D_ρ is compact, $\lim_{\rho \rightarrow \infty} D_\rho = R$, the boundary Γ_ρ of D_ρ is composed of $n(\rho)$ components, r_1, r_2, \dots, r_n , and that $\Lambda(\rho)$ is the largest length of r_k ($k = 1, 2, \dots, n$):

$$l_k = \int_{r_k} ds, \quad \Lambda(\rho) = \max_k l_k .$$

Put
$$N(\rho) = \max_{\rho' \leq \rho} n(\rho') .$$

Pfuger proved⁽¹⁾ that if

$$\limsup_{\rho \rightarrow \infty} \left[4\pi \int_{\rho_0}^{\rho} \frac{d\rho}{\Lambda(\rho)} - \log N(\rho) \right] = \infty ,$$

then $R \in O_{AB}$.

Theorem 1. *If*

$$\limsup_{\rho \rightarrow \infty} \left[\pi \int_{\rho_0}^{\rho} \frac{d\rho}{\Lambda(\rho)} - \log N(\rho) \right] = \infty \quad (\text{genus of } R \leq \infty) ,$$

1) A. Pfuger: Sur l'existence de fonctions non constantes, analytiques, uniformes et bornées sur une surface de Riemann ouverte, C. R. Acad. Sci. Paris, 230, 1950, pp. 166-168,

then every connected piece of R over $|w-w_0| < S$, covers every points except at most the null-set ${}^{(2)}E_{AB}$, which is the boundary set of a domain of O_{AB} on the w -plane.

Proof. If there exists a lacunary set E , which is being clearly closed, not contained in E_{AB} in $|w-w_0| < S$, we can construct a bounded analytic function $A(w)$ in the w -plane except E and regular on $|w-w_0| = S$. Define a harmonic function $U(w)$ on $|w-w_0| \leq S$ such that $U(w) = \text{real part of } A(w)$ on $|w-w_0| = S$, then it is clear that the conjugate function $V(w)$ of $U(w)$ is bounded on $|w-w_0| \leq S$ and $A(w) - U(w) - iV(w) = B(w)$ is bounded on $|w-w_0| \leq S$ and further $B(w) \neq \text{constant}$. Consider the closed domain \bar{G} such that ${}^{(3)}\text{Re}(B(w)) \geq 0: |w-w_0| \leq S$, and let V be the image of \bar{G} on R , then V has relative boundaries $l_1, l_2, \dots, l_p, \dots$, on which the $\text{Re } B(w)$ vanishes.

Each l_i is non compact, since otherwise $\Im_m B(w)$ is not one valued on account of $\int_{l_i} \frac{\partial \text{Re } B(w)}{\partial n} ds > 0$.

Every l_i converges to the boundary of R . Let $B(p)$ be the function $B(w)$ considered as the function on $R \cap V$, $p \in R \cap V$.

Since $B(p): p \in (R \cap V)$ is bounded, we can suppose that V is mapped on the semi-circle $|\xi| < 1$ $\text{Re } \xi \geq 0$ and every l_i is mapped on the imaginary axis. After Pfluger we introduce in $|\xi| < 1$ the hyperbolic metric by the line element defined by $ds = \frac{|d\xi|}{1-|\xi|^2}$. Consider V in D_p and put $D_p' = D_p \cap V$. The boundary of D_p' is composed of l_i and $\sum_{i=1}^{n(p)} \sum_{j=1}^{i(p)} r_i^j$, where r_i^j is an arc contained in r_i . Let L_i^j be a segment on imaginary axis connecting two end-points of the image r_i^j lying on the imaginary axis, and \tilde{L}_i^j be image of r_i^j . Then

$$\tilde{L}_i^j = \int_{r_i^j} ds \geq \text{length of } L_i^j.$$

Let A_i^j be the area bounded by \tilde{L}_i^j and L_i^j . Then by the isoperimetric problem

$$4A_i^j(A_i^j + \pi) \leq (\tilde{L}_i^j + L_i^j)^2 \leq 4(\tilde{L}_i^j)^2,$$

$$4A_i^j(A_i^j + \pi) \leq 4\tilde{L}_i^j{}^2,$$

2) In this article, we denote by E_{AB} the null set of O_{AB} on the plane.

3) Without loss of generality we may assume that there exists a point w_0 satisfying the real part of $B(w_0)$ is positive.

where

$$A_i = \sum_j A_i^j, \quad \tilde{L}_i = \sum_j \tilde{L}_i^j.$$

If r_j has no common point with any l_i , then we have

$$4A_j(A_j + \pi) \leq \tilde{L}_j^2.$$

Thus

$$4A_i(A_i + \pi) \leq 4\tilde{L}_i^2, \quad \text{for every } i.$$

Denote by A_p the area of image of D_p' . Then $A_p \leq \sum A_i$, and in the same manner as used by Pfluger, we have

$$4A_p\left(\pi + \frac{A_p}{n}\right) \leq 4\pi(\sum A_i + \sum A_i^2) \leq 4 \sum \tilde{L}_i^2.$$

On the other hand

$$\begin{aligned} \tilde{L}_i^2 &\leq l_i \int_{r_i} \frac{\left|\frac{d\xi}{dz}\right|^2}{(1-|\xi|^2)^2} dz, \\ \sum^n \tilde{L}_i^2 &\leq \Lambda(\rho) \frac{dA_p}{d\rho}, \\ A_p\left(\pi + \frac{A_p}{n}\right) &\leq \Lambda(\rho) \frac{dA_p}{d\rho}, \end{aligned}$$

hence

$$\frac{A_{\rho_0}}{\pi + A_{\rho_0}} \leq N(\rho) \exp\left(-\pi \int_{\rho_0}^{\rho} \frac{d\rho}{\Lambda(\rho)}\right).$$

Thus by assumption A_{ρ_0} must be zero, from which the conclusion follows.

Denote by $n(w)$ the number of sheets of connected piece of R on $|w-w_0| < S$ over a point w .

Theorem 2. *Let R be a Riemann surface belonging to $O_{AB}(O_{AD})$ of finite genus and let V be a connected piece on $|w-w_0| < \rho$ such that $n(w) \leq N < \infty$. Then V covers every point except at most a null-set $E_{AB}(E_{AD})$.*

Denote by D_N set of points of projection of V such that $n(w) = N$. Then from the lower semi-continuity of $n(w)$, it is clear that D_N is an open set and the boundary B_N of D_N is closed.

1) B_N is a totally disconnected set. If it were not so, take a continuum-component $B_{N'}$ of B_N and a point p such that $n(p) = \max n(w)$

$= S : w \in B_N$, and let $v(p)$ be a neighbourhood of p with boundary l such that l has at least one component $l' (\in D_N)$ of $(l - B_N')$ and $v(p) \cap (B_N') \not\subset D_{S'+1}$. Since p is covered S times by V , there exists at most S discs $k_{S'}, \dots, k_{S'} (S' \leq S)$ on v and at least another disc k_0 on v , and V on k_0 has at least a connected piece with lacunary of a continuum, larger than $v(p) \cap B'_N$, and at most $(N-S)$ number of relative boundary components $L_1, L_2, \dots, L_{N'-S'}$ lying on l' ($N'-S' \leq N-S$). We denote such a connected piece by \tilde{V} . Since the genus of R is finite, it can be mapped by $w = f(p)$ onto a sub-Riemann surface R in the other closed surface R^* . $R^* - R$ is a totally disconnected set. Consider the image of \tilde{V} in R^* . Then we can see easily that every image of $L_i (i = 1, 2, \dots, N'-S')$ converge to a point of R^* , because $R^* - R$ is totally disconnected and $p = f^{-1}(w) ; p \in R^*$ is continuous. Denote by \tilde{V} the domain on R^* bounded by the image L_i and by a finite number of points of a subset of $R^* - R$. On the other hand by assumption $v(p)$ has a continuum boundary except the projection of L_i , thus we can define a bounded (Dirichlet bounded) analytic function $\varphi(w(p))$ on $v(p)$ with vanishing real part on L_i . If $\varphi(w(p))$ is analytic in \tilde{V} , it must be a constant, therefore there exists in \tilde{V} , a closed set E where $\varphi(p)$ is not regular. Therefore by Neumann's⁽⁴⁾ method and by Abel's integral, we can construct a bounded analytic (Dirichlet bounded) function on R , which contradicts the fact that $R \in O_{AB}(O_{AD})$.

2) Since B_N is a totally disconnected closed set, we can take a neighbourhood $V'(p)$ such that the boundary⁽⁵⁾ of $V'(p)$ is completely contained in D_N and enclosing a lacunary set E of the connected piece. Thus by the same method as above, we can conclude that $R \in O_{AB}(O_{AD})$.

Remark 1) If $R \in O_{AB}(O_{AD})$ covers the w -plane a bounded number of times, then we can see easily that the mapping function is regular throughout R^* , and the function must be an algebraic function.

4) Since $E(\subset(R^* - R))$ is a closed and totally disconnected set, we can find a domain D , with relative boundary ∂D , in \tilde{V} such that $D \supset E' (E \supset E')$, distance $(\partial D, \text{relative boundary of } \tilde{V}) > 0$, and distance $(E', \partial D) > 0$. Then by Neumann's method, we can construct a non constant harmonic function $U_1(p)$ such that $(\text{Re } \varphi(p) - U_1(p))$ is harmonic in \tilde{V} , $U_1(p)$ is harmonic in $R^* - D$, and the conjugate of $U_1(p)$ is single valued in D , therefore we can construct a bounded (Dirichlet bounded) function with a linear form of Abel's first kind of integral.

5) \tilde{V} in R^* , above defined, of every connected piece on $V'(p)$ has at most N number of analytic curves as its relative boundary.

Remark 2) We conjecture that every Riemann surface belonging to O_{AB} of finite genus has Iversen's property but the present author did not succeed to prove it.

Theorem 3. *A Riemann surface belonging to O_{AB} of finite genus has not necessarily the Gross's property.*

Example. Let F_0 be the unit-circle $|z| < 1$ with slits $S_i^n : n = 1, 2, 3, \dots ; i = 1, 2, \dots, q_n$ such that (Fig. 1)

$$S_i^n : 1 - \frac{1}{p_n} \leq |z| < 1, \quad \arg z = \frac{2\pi i}{q_n},$$

$$i = 1, 2, 3, \dots, q_n, \quad p_n = \bar{a}^n : \bar{a} > 4.$$

Lemma. *Let F_i^n be the unit-circle with slits S_i^n , and connect F_0 with every F_i^n on corresponding slits S_i^n crosswise, then we have infinitely many sheeted covering surface on the unit-circle. If we take q_n sufficiently large, then we have $\omega(p) \equiv 0$, where $\omega(p)$ is the harmonic measure of the boundary of F_0 on $|z| = 1$.*

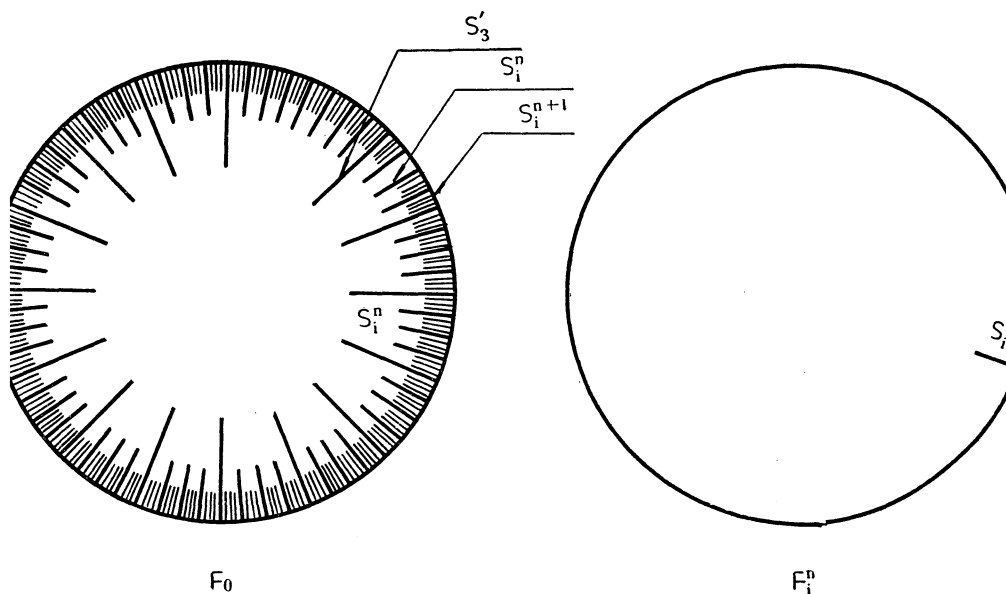


Fig. 1 $F_i^n : i = 1, 2, \dots, i_0(n) . n = 1, 2, 3, \dots$

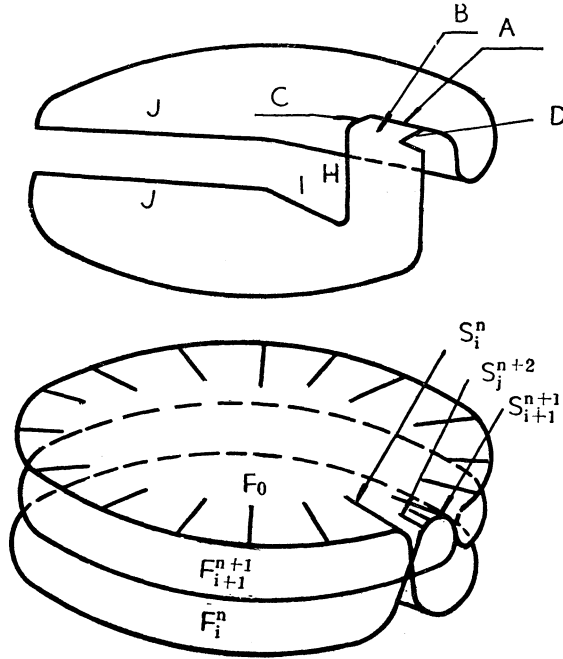


Fig. 2

We denote by $G_{i,i+1}^{m,n}$ ($n \geq m$) the domain of F_0 enclosed by straight lines A, B and circular arcs C, D such that (Fig. 2)

$$\begin{aligned}
 A: & \quad 1 - \frac{1}{p_n} \leq |z| \leq 1 - \frac{1}{p_{n+1}}; \quad \arg A = \arg S_i^m = \arg S_i^n, \\
 B: & \quad 1 - \frac{1}{p_n} \leq |z| \leq 1 - \frac{1}{p_{n+1}}; \quad \arg B = \arg S_{i+1}^n, \\
 C: & \quad |z| = 1 - \frac{1}{p_{n+1}}; \quad \arg S_{i+1}^n \leq \arg z \leq \arg S_i^n, \\
 D: & \quad |z| = 1 - \frac{1}{p_n}; \quad \arg S_{i+1}^n \leq \arg z \leq \arg S_{i+1}^n.
 \end{aligned}$$

$F_i^m(F_{i+1}^n)$ has a slit $S_i^m(S_{i+1}^n)$ with edges $+S_i^m, -S_i^m(+S_{i+1}^n, -S_{i+1}^n)$ (S_i^m has two edges). We consider $\omega(p)$ in the surface $F_i^m + G_{i,i+1}^{m,n} + F_{i+1}^n$, where $G_{i,i+1}^{m,n}$ is connected with F_i^m on A by $+S_i^m$, with F_{i+1}^n on B by $-S_{i+1}^n$. $F_i^m + G_{i,i+1}^{m,n} + F_{i+1}^n$ has boundaries $C, D, -S_i^m, +S_{i+1}^n, (+S_i^m - A), (-S_{i+1}^n - B)$ and the boundary on $|z|=1$.

Let $\omega^*(p)$ be harmonic measure of $-S_i^m + +S_{i+1}^n + (+S_i^m - A) + (-S_{i+1}^n - B)$ with respect to $F_i^m + G_{i,i+1}^{m,n} + F_{i+1}^n$. Then it is clear

$$\omega^*(p) \geq \omega(p).$$

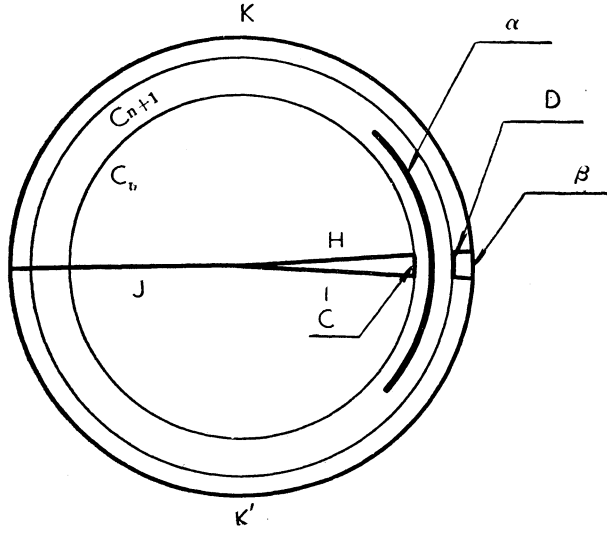


Fig. 3

Denote by $(F_i^n + G_{i,i+1}^{m,n} + F_{i+1}^m)^*$ the simply connected domain with boundaries such that (Fig. 3)

$$\begin{aligned}
 H : 0 \leq |z| \leq 1 - \frac{1}{p_n}, \quad \arg z = \arg S_i, \\
 E : 1 - \frac{1}{p_{n+1}} \leq |z| \leq 1, \quad \arg z = \arg S_i, \\
 I : 0 \leq |z| \leq 1 - \frac{1}{p_n}, \quad \arg z = \arg S_{i+1}, \\
 F : 1 - \frac{1}{p_{n+1}} \leq |z| \leq 1, \quad \arg z = \arg S_{i+1}, \\
 J : 0 \leq |z| \leq 1, \quad \arg z = \pi + \frac{\arg S_i + \arg S_{i+1}}{2}, \\
 K : |z| = 1, \quad 0 \leq \arg z \leq \pi, \\
 K' : |z| = 1, \quad \pi \leq \arg z \leq 2\pi, \\
 \text{and } C+D.
 \end{aligned}$$

Let $\omega^{**}(p)$ be the harmonic measure of $E+D+H+I+J+F+C$. Then

$0 \leq \omega(p) \leq \omega^*(p) \leq \omega^{**}(p)$. Let α be a half of the semi-circle passing through the point $|z| = \frac{1}{2} \left(2 - \frac{1}{p_n} - \frac{1}{p_{n+1}} \right)$, $\arg z = \frac{1}{2} (\arg S_i + \arg S_{i+1})$.

1°) The value of $\omega^{**}(p) : p \in \alpha$.

Suppose $(\arg S_i + \arg S_{i+1}) = 0$, and let S be a point on $\alpha : S = \left(1 - \frac{p_{n+1} - p_n}{2p_n p_{n+1}} \right) e^{i\theta}$, $|\theta| \leq \frac{\pi}{4}$. To investigate the behaviour of $\omega^{**}(p)$,

we transform these figures by a linear function $w = \frac{z-S}{1-\bar{S}z}$.

Let C_n, C_{n+1} be circles such that

$$C_n: |z| = 1 - \frac{1}{p_n}, \quad C_{n+1}: |z| = 1 - \frac{1}{p_{n+1}}.$$

Then C_n, C_{n+1} will be mapped on to circles C'_n, C'_{n+1} such that

$$C'_n: \left| w - \frac{a^2 r - r}{1 - a^2 r} e^{i\theta} \right| = \frac{a - r^2 a}{1 - a^2 r^2}, \quad C'_{n+1}: \left| w - \frac{r^2 - r b^2}{1 - b^2 r} e^{i\theta} \right| = \frac{b - r^2 b}{1 - b^2 r},$$

where $a = 1 - \frac{1}{p_n}, \quad b = 1 - \frac{1}{p_{n+1}}, \quad r = 1 - \frac{p_{n+1} - p_n}{2p_n p_{n+1}} \dots$ (A)

$$\begin{aligned} 1) \text{ distance } (C'_n, 0) &= \frac{p_n(2p_n p_{n+1} - p_{n+1} - p_n)}{4p_n^2 p_{n+1} - 3p_n p_{n+1} + p_n^2 + p_{n+1} - p_n} \\ &= \frac{2 - \frac{1}{\tilde{a}^n} - \frac{1}{\tilde{a}^{n+1}}}{4 - \frac{3}{\tilde{a}^n} + \frac{1}{\tilde{a}^{n+1}} + \frac{1}{\tilde{a}^{2n}} + \frac{1}{\tilde{a}^{2n+1}}}. \end{aligned}$$

$$2) \text{ distance } (C'_{n+1}, 0) = \frac{(p_{n+1} + 3p_n)p_{n+1}}{p_{n+1}^2 + 3p_n p_{n+1} + p_{n+1} - p_n} = \frac{1 - \frac{3}{\tilde{a}}}{1 + \frac{1}{\tilde{a}} + \frac{1}{\tilde{a}^{n+1}} - \frac{1}{\tilde{a}^{n+2}}}.$$

$$3) \quad z = S \longrightarrow w = 0.$$

$$4) \quad z = 0 \longrightarrow w = -S.$$

$$5) \quad z = e \longrightarrow w = e^{i\theta_1'} : \theta_1' = \arg S_i.$$

$$6) \quad z = e \longrightarrow w = e^{i\theta_2'} : \theta_2' = \arg S_{i+1}.$$

$$7) \text{ the radius } z=0, \quad z=e^{i\theta_1} \longrightarrow \text{orthogonal circle } \overbrace{e^{i\theta_1'}}^{-}, -S.$$

$$8) \text{ the radius } 0, e^{i\theta_2} \longrightarrow \text{orthogonal circle } \overbrace{e^{i\theta_2'}}^{-}, -S.$$

$$9) \text{ the radius } 0, -1 \longrightarrow \text{orthogonal circle } -re^{i\theta}, \overbrace{\frac{-1-re^{i\theta}}{1+re^{i\theta}}}^{-},$$

(this circle tends to $e^{-i\theta}$ when $r \rightarrow 1$.)

$$z = 1 \longrightarrow w = \frac{1-re^{i\theta}}{1-re^{-i\theta}} : \theta \leq \frac{\pi}{2}.$$

Let $\omega^\beta(z)$ be the harmonic measure of β (β lies on $|z|=1$ and $\arg S_i \leq \arg z \leq \arg S_{i+1}$) with respect to the unit-circle. Then $\omega^\beta(z) : z \in \alpha$ attains its maximum when $\arg z = \frac{1}{2}(\arg S_i + \arg S_{i+1})$, which implies that the length of the image of β is largest when $\arg S = 0$, in which case the mapping function is reduced to $w = \frac{z-r}{1-rz}$. If we denote by β_1^* and β_2^* the end-points of the image β^* of β , then we have

$$\arg \beta_1^* = \tan^{-1} \left[\frac{\sin \theta (1-r^2)}{(-2r) + \cos (1+r^2)} \right].$$

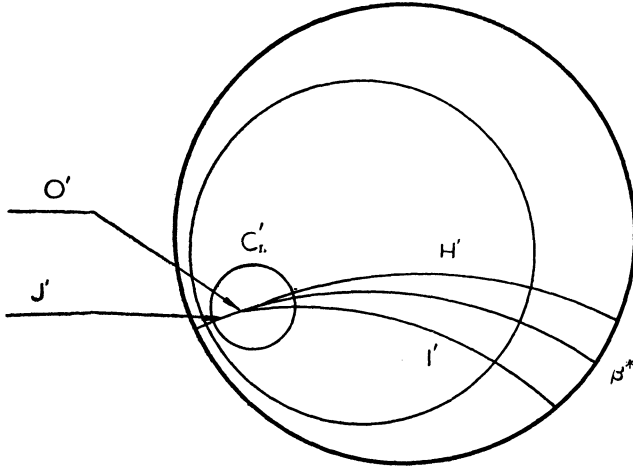


Fig. 4

If we take θ so small that

$$(B) \quad \cos \theta > \frac{2r}{1+r^2},$$

then the length of β^* is smaller than π .

Elementary calculation yields from (A) and (B)

$$\theta < \sqrt{\frac{\tilde{a}-1}{2\tilde{a}^{n+1}}},$$

i.e.,

$$q_n > 4\pi \sqrt{\frac{2\tilde{a}^{n+1}}{\tilde{a}-1}}.$$

If we consider in 9) the radius $0, \overleftrightarrow{-1}$, then the argument of its image $\arg \left(\left(\frac{-1-re^{i\theta}}{1+re^{-i\theta}} \right) \right)$ is smallest when $\theta = \frac{\pi}{4}$: thus

$$\arg \frac{-1-re^{i\theta}}{1+re^{-i\theta}} \geq 2 \left(\tan^{-1} \frac{1+\sqrt{2}}{\sqrt{2}} \right) > \frac{\pi}{2} + \varepsilon_0 \quad \left(\frac{\pi}{4} \geq \theta \geq 0 \right),$$

and the argument of $w = re^{\theta+\pi}$ is $\theta+\pi$. Therefore the distance from $w=0$ to the image J is larger than a positive number δ_1 . The same fact holds true when $-\frac{\pi}{4} \leq \theta \leq 0$.

Since $\omega^{**}(p)$ attains 1 only on $D, E, F, H, C, I,$ and J , and since the distance from $w=0$ to the images of E, D, F, H, C, I, J has a positive distance larger than δ_3 , and further since the length of β^* is less than π , we see that $\omega^{**}(z) \leq \delta_4 < 1$, where δ_4 is a positive

number whenever \$S\$ is on \$\alpha\$.

2°) \$F_i^n\$ has a slit \$S_i^n\$. We denote by \$\alpha'\$ the part of the circle such that \$|z| = 1 - \frac{p_{n+1} - p_n}{2p_n p_{n+1}}\$, \$\frac{1}{2}(\arg S_i + \arg S_{i+1}) + \frac{\pi}{4} < \arg z < -\frac{1}{2}(\arg S_i + \arg S_{i+1}) + \frac{7\pi}{4}\$, and denote by \$\omega^{***}(z)\$ the harmonic measure of \$S_i\$ with respect to the domain (unit-circle-\$S_i\$). Then clearly \$\omega^{***}(z) \le \delta_5 < 1 : z \in \alpha'\$ and \$\omega(p) \le \omega^{***}(z)\$.

Let \$\tilde{\omega}_n(p)\$ be a harmonic function \$0 \le \tilde{\omega}_n(p) \le 1\$ such that \$\tilde{\omega}_n(p) = 0\$ on the boundary of \$F_{i_1}^1, F_{i_2}^2, \dots, F^n\$ and \$= 1\$ on the circle on \$F_0\$ with radius \$= 1 - \frac{1}{p_{n+1}}\$ and on the part of slits \$S_{n+1}^i\$ contained in the part \$|z| \ge 1 - \frac{1}{p_{n+1}}\$. On the other hand \$F_0\$ has no common point with \$F_i^{n+1}, F_j^{n+2}, \dots\$. Thus we see from 1°) and 2°)

$$\tilde{\omega}_n(p) \le \max(\delta_4, \delta_5),$$

where the projection of \$p\$ is on the circle \$|z| = 1 - \frac{p_{n+1} - p_n}{2p_n p_{n+1}}\$.

Let \$\{V_i(p)\}\$ be non negative continuous super-harmonic functions on \$F\$ such that \$V_i(p) \le 1\$ and \$\lim_{\substack{|p|=1 \\ p \in F_0}} V_i(p) = 1\$, and denote \$V(p)\$ its lower envelope. Then

$$V(p) \le \max(\delta_4, \delta_5) \quad \text{on } |z| = r_n \quad n = 1, 2, \dots,$$

thus

$$V(p) \le \max(\delta_4, \delta_5)V(p), \quad \text{and} \quad V(p) \equiv 0.$$

We denote by \$\hat{F}\$ the symmetric surface with respect to the unit circle and identify the boundary of \$F_i^n (n \ge 1)\$ with that of \$\hat{F}\$, then we have a planer Riemann surface \$\hat{F}\$ over the \$z\$-plane.

Proposition. \$\hat{F}\$ is contained in the class \$O_{AB}\$.⁶⁾

If there were a non-constant bounded analytic function \$A(p) = U(p) + iV(p)\$ on \$\hat{F}\$, where \$\tilde{p}\$ is the symmetric point of \$p\$ with respect to the unit circle, then we have

$$U(p) - U(\tilde{p}) = 0, \quad V(p) - V(\tilde{p}) = 0,$$

which implies the constancy of \$A(p)\$. It is clear that \$\hat{F}\$ has not the Gross's property.

Theorem (W. Gross). *Let \$z = z(p) : p \in R\$ be a meromorphic function and let \$R\$ be a Riemann surface belonging to \$O_\sigma\$. If we denote by \$p = p(z)\$ its inverse function, if \$p = p(z)\$ is regular at \$z_0\$, then we can continue \$p(z)\$ analytically on half lines \$z = z_0 + re^{i\theta}\$ (\$0 \le r < \infty\$) except a set of \$\theta\$ of angular measure zero.*

Thus our example is not contained in O_g .

When the genus of an abstract Riemann surface is finite, it is known

$$O_g = O_{HB} = O_{HD} < O_{AB} < O_{AD} = O_{ABD}.$$

Since there is a Riemann surface of finite genus of O_{AD} on which a non-constant bounded analytic function exists, O_{AD} has not necessarily Iversen's property. In the previous⁷⁾ paper we proved that O_g is the only class in which any Riemann surface always has Gross's property. Now even when we confine ourselves to Riemann surfaces of finite genus, we know that O_g is the maximal class in which Gross's theorem holds.

Denote by P_I, P_G the class of Riemann surfaces having Iversen's or Gross's property respectively. Then

1) Case of infinite genus

$$P_I \supset O_{HB} \supset O_{HP} \supset O_g, \quad P_I \not\supset O_{AB}, \quad P_I \not\supset O_{HD}, \quad P_G \not\supset O_{HP}.$$

2) Case of finite genus

$$P_I \overset{?}{\supset} O_{AB}, \quad P_I \supset O_g, \quad P_I \not\supset O_{AD}, \quad P_G \supset O_g, \quad P_G \not\supset O_{AB}.$$

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