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Lattices of Spaces

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Let R_1 and R_2 be two topological spaces with the same basic set and having the property that $O(R_1) \supset O(R_2)$, where $O(R_i)$ $(i = 1$, or $2)$ denotes the family of all open subsets in R_i. Then we say that R_1 is finer than R_2 and write $R_1 \underset{*}{>} R_2$ (by a binary relation $\underset{*}{>}$). The family consisting of all topological spaces with the same basic set forms a lattice under this ordering. Edwin Hewitt [1] has discussed the various problems on this lattice. In this paper, we shall consider the lattices consisting of all spaces with the same basic set on the basis of the neighbourhoods, the closure and the convergence.

§ 1. General properties of the lattices $\mathfrak{L}$ and $\mathfrak{L}_A$.

Definition. A set X is called a space if a closure operator assigns to each subset M of X a closure $\bar{M}$, and satisfies the condition that $\bar{\phi} = \phi$ (ϕ is the empty set), and $\overline{M \cup N} = \bar{M} \cup \bar{N}$ for each pair M and N. Further a set X is said to be a space with additive topology if it is a space and satisfies the condition that $\bar{M} \supset M$ for each subset M of X.

Let $\mathfrak{L}$, $\mathfrak{L}_A$ and $\mathfrak{L}_T$ denote the family consisting of all spaces on the same basic set[1] E, the family of all spaces with additive topology on E, and the family of all topological spaces on E respectively. We are going to construct in $\mathfrak{L}$ a lattice by defining a suitable ordering, that is, for two spaces R_1, R_2 of $\mathfrak{L}$, $R_1 > R_2$ (we say that R_1 is finer than R_2) if and only if the identical mapping from R_1 onto R_2 is continuous.[2]

This definition is equivalent to the following one: $R_1 > R_2$ if and only if for any point x in E, $N_x(R_1) \supset N_x(R_2)$, where $N_x(R_i)$ $(i = 1$, or $2)$ denotes the family of all neighbourhoods of x in R_i.

Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be any family of spaces in $\mathfrak{L}$. Then the join of the spaces R_λ is the space where family of all neighbourhoods of x is

1) If E is any set of points and if R is any space whose points are the points of E, then E is said to be the basic set of the space R.

2) See [2], p. 28.

defined as the family of all subsets consisting of some finite intersection of sets in the family $\sum_{\lambda\in\Lambda} N_x(R_\lambda)$, and the meet of the spaces R_λ is the space where family of all neighbourhoods of x is just the sets in the family $\prod_{\lambda\in\Lambda} N_x(R_\lambda)$. Hence we see that the family $\mathfrak{L}$ forms a complete lattice under the above ordering.

Further we see easily that a phalanx $x(\alpha|A)$ converges to x on the join $\bigvee_{\lambda\in\Lambda} R_\lambda$ if and only if it converges to x on all R_λ, and that an ultraphalanx $x(\alpha|A)$ converges to x on the meet $R_1 \bigwedge R_2$ if and only if $x(\alpha|A)$ converges to x on R_1 or on R_2. In fact, it is clear that if an ultraphalanx $x(\alpha|A)$ converges to x on R_1 or on R_2, then it also converges to x on the meet $R_1 \bigwedge R_2$. Conversely, suppose that an ultraphalanx $x(\alpha|A)$ fails to converge to x on R_1 and on R_2. Then there exist a neighbourhood of x in R_1, N_1, and a neighbourhood of x in R_2, N_2, and for some $\alpha_1 \subset A$, $\alpha_1 < \alpha$ implies $x(\alpha) \notin N_1 \bigcup N_2$. This show that $x(\alpha|A)$ fails to converge to x on the meet $R_1 \bigwedge R_2$, since $N_1 \bigcup N_2$ is a neighborhood of x in $R_1 \bigwedge R_2$. As it is easily seen, this fact implies that $\mathfrak{L}$ is a distributive lattice.

By the fact mentioned above, we have

Theorem 1. *The family $\mathfrak{L}$ forms a distributive, complete and atomic lattice.*

We can prove the following theorem:

Thorem 2. *If $|E|$[3] ≥ 2, every space in $\mathfrak{L}$ is a join of two compact spaces*[4] *in $\mathfrak{L}$.*

Proof. Let a, b be two distinct fixed points in E. We define now the space R_a (similarly R_b): R_a is the space such that $N_a(R_a)$ is the family of a single elment $\{E\}$, and for all x but a, $N_x(R_a)$ is the family of all subsets of E. Then the join $R_a \bigvee R_b$ is the greatest element R_I in $\mathfrak{L}$. For every spaces R in $\mathfrak{L}$, $R = R \bigwedge R_I = R \bigwedge (R_a \bigvee R_b) = (R \bigwedge R_a) \bigvee (R \bigwedge R_b)$. Now, the spaces $R \bigwedge R_a$ and $R \bigwedge R_b$ are compact, since R_a, R_b are compact spaces.

Corollary. *Any sub-semi-lattice relative to the join*[5] *(or sublattice) containing the family of all compact spaces in $\mathfrak{L}$ coincides with the whole lattice $\mathfrak{L}$, provided that $|E| \geq 2$.*

Corollary. *Every ideal in $\mathfrak{L}$ is a join of two ideals whose elements are all compact spaces, provided that $|E| \geq 2$.*

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3) $|A|$ denotes the potency of the set A.

4) See [2], p. 36.

5) Let L be any algebraic system with an idempotent, commutative, and associative binary operation $x \bigcirc y$. Then the system L is called a semi-lattice relative to the operation $\bigcirc$.

Proof. Clear, since the family consisting of all ideals in $\mathfrak{L}$ forms a distributive lattice[6] under set-inclusion.

Theorem 3. *Every space R in $\mathfrak{L}$ is a meet of two T_0-spaces[7] in $\mathfrak{L}$.*

Proof. Let the points of E be well ordered:

$$E = x_1,\ x_2,\ x_3,\ \dots,\ x_n,\ \dots,\ x_\lambda,\ \dots \qquad (\lambda < \lambda_0),$$

where λ_0 is some ordinal number with the corresponding cardinal number $|E|$.

We define now two spaces R_1 and R_2: R_1 is the space where family of all neighborhoods of x_λ is $E[M|M \supset M_\lambda]$ for each λ, M_λ being the set of elements $x_1, x_2, \dots, x_\lambda$. R_2 is the space where family of all neighbourhoods of x_λ is $E[N|N \supset N_\lambda]$ for each λ, N_λ denoting the set of elements $x_\lambda, x_{\lambda+1}, \dots, x_\mu, \dots (\mu < \lambda_0)$. Then $R_1 \wedge R_2$ is the least element in $\mathfrak{L}$. We see here that the theorem will be proved by the same method as with Theorem 2.

Corollary, *Any sub-semi-lattice relative to the meet[5] (or sublattice) containing the family of all T_0-spaces in $\mathfrak{L}$ coincides with the whole lattice $\mathfrak{L}$.*

REMARK. (1) In Theorem 3, T_0-space cannot be replaced by T_1-spaces or by T_2-spaces,[7] for the family of all T_1-spaces in $\mathfrak{L}$ forms a proper dual-ideal in $\mathfrak{L}$.

(2) By Theorem 2 and Theorem 3, we see easily that any join-irreducible element[8] in $\mathfrak{L}$ is a compact space and any meet-irreducible element[8] in $\mathfrak{L}$ is a T_0-space.

(3) The family $\mathfrak{L}_A$ is a sublattice in $\mathfrak{L}$ and the previous theorem in $\mathfrak{L}$ are all established in the lattice $\mathfrak{L}_A$.

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6) See, for instance, Birkhoff [3,] p. 141.

7) We define the separation axiom of the space R:

T_0-axiom—for two distinct points in R, there exists some neighbourhood (need not be an open set) of one of them, which fails to contain the other.

T_1-axiom—for two distinct points p, q in R, there exist a neighhorhood (need not be an open set) of p which fails to contain q, and a neighborhood of q which fails to contain p.

T_2 (Hausdorff)-axiom—for two distinct points p, q in R, there exist a neighborhood of p, N_1, and a neighborhood of q, N_2 such that $q \notin N_1$, $p \notin N_2$ and $N_1 \cap N_2 = \phi$. A neighborhood need not be an open set.

A spcce is said to be a T_0-space, a T_1-space, or a T_2-(Hausdorff) space if it satisfies T_0-, T_1-, or T_2-axiom respectively.

8) An element a of a lattice is called join (meet)-irreducible if
$x \vee y = a$ ($x \wedge y = a$) implies $x = a$ or $y = a$.

We remark finally that for any subset M in E, the closure $\bar{M}^R$ of M in the meet R of the family $\{R_\lambda\}_{\lambda\in\Lambda}$ is equal to the set $\sum_{\lambda\in\Lambda}\bar{M}^{R_\lambda}$, where $\bar{M}^{R_\lambda}$ denotes the closure of M in R_λ for each λ, and that the closure $\bar{M}^{R'}$ of M in the join R' of the completely ordered family $\{R\}_{\lambda\in\Lambda}$ (see, § 2) in $\mathfrak{L}$ is equal to the set $\prod_{\lambda\in\Lambda}\bar{M}^{R_\lambda}$.

§ 2. The lattice $\mathfrak{L}_T$.

Let R_1 and R_2 be two spaces in $\mathfrak{L}_T$ having the property that $O(R_1)\supset O(R_2)$, where $O(R_i)\,(i=1,$ or $2)$ denotes the family of all open subsets in R_i. Then we say that R_1 is finer than R_2 and write $R_1 \underset{*}{>} R_2$. The family $\mathfrak{L}_T$ forms a lattice under the ordering. But the lattice $\mathfrak{L}_T$ does not form a sublattice in $\mathfrak{L}$, provided that $|E|\geq 3$: let R_1 be the space where family of all neighborhoods of x is the family $N_x(R_1) = E[N\,|\,N\ni a,\ x]$ for any x, and let R_2 be the space such that for a fixed point a, $N_a(R_2)$ is the family of a single element $\{E\}$, and for any point x but a, $N_x(R_2)=E[N\,|\,N\ni x]$. Then we see easily that R_1 and R_2 are both topological spaces, but the meet $R_1\wedge R_2$ in $\mathfrak{L}$ is not a topological space, provided that $|E|\geq 3$. Furthermore $\mathfrak{L}_T$ is a complete and atomic lattice, but fails to satisfy the distributive law: let R_1 be the topological space where family of all open subsets is $O(R_1)=E[M\,|\,M\ni a,\ b]\cup\{\phi\}$, where two points a, b distinct fixed points of E. Let R_2 and R_3 be two topological spaces such that families of all open subsets are $O(R_2)=E[M\,|\,M\not\ni a]\cup\{E\}$ and $O(R_3)=E[M\,|\,M\not\ni b]\cup\{E\}$ respectively. Then we see easily that $R_1\wedge_*(R_2\vee_* R_3)$ is not equal to $(R_2\wedge_* R_2)\vee_*(R_1\wedge_* R_3)$, where the symbols $\vee_*$ and $\wedge_*$ are the join and the meet in $\mathfrak{L}_T$ respectively.

We remark finally that for two topological spaces R_1 and R_2 (in $\mathfrak{L}_T$), the join $R_1\vee_* R_2$ in $\mathfrak{L}_T$ coincides with the join $R_1\vee R_2$ in $\mathfrak{L}$.

Theorem 4. *Every T_1-topological space is a join of two compact* (= bicompact) *T_1-topological spaces.*

Proof. We may restrict ourselves to the case of $|E|\geq 2$. Let a, b be two distinct fixed points in E. Let R_a (similarly, R_b) denote the topological space such that the family of open subsets consist of all subsets (in E) which fail to contain the point a and of all subsets (in E) having a finite complemetary set. The space R_a is clearly a bicompact T_1-space. We see here that the join $R_a\vee_* R_b$ in $\mathfrak{L}_T$ is the greatest element in $\mathfrak{L}_T$, in other words, a discrete space S_I. Let R be any T_1-space in $\mathfrak{L}_T$. Then $R=R\wedge S_I=R\wedge(R_a\vee_* R_b)=R\wedge(R_a \vee R_b)=(R\wedge R_a)\vee(R\wedge R_b)$ by the distributivity of $\mathfrak{L}$, but we can prove easily that $R \wedge R_a$ and $R \wedge R_b$ are both topological spaces. In fact, let M be any infinite subset in $R^* = R \wedge R_a$. Then $\bar{M}^{R*} = \bar{M}^R \cup \bar{M}^{R_a}$ $= \bar{M}^R \cup (M \cup a) = \bar{M}^R \cup a$, $\bar{\bar{M}}^{R*} = \overline{\bar{M}^R \cup a}^{R*} = \overline{\bar{M}^R \cup a}^{R} \cup \overline{\bar{M}^R \cup a}^{R_a} =$ $(\bar{M}^R \cup a) \cup (\bar{M}^R \cup a) = \bar{M}^R \cup a$, therefore, $R^* = R \wedge R_a$ is a topological space. Hence $R = (R \wedge R_a) \vee (R \wedge R_b) = (R \wedge R_a) \vee_* (R \wedge R_b)$. Since the family of all T_1-spaces forms a dual-ideal in $\mathfrak{L}$, the spaces $R \wedge R_a$ and $R \wedge R_b$ are both bicompact T_1-topological spaces. This completes the proof.

In this theorem, T_1-spaces cannot be replaced by T_2-spaces, for non regular, T_2-spaces cannot be a join of two bicompact T_2-spaces by the following lemma.

Lemma. *Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be a family of spaces in $\mathfrak{L}_T$. Then the join $\vee_{\lambda \in \Lambda} R_\lambda$ in $\mathfrak{L}_T$ is homeomorphic to the diagonal line D in the Cartesian product $P_{\lambda \in \Lambda} R_\lambda$.*

Proof. The diagonal line D in the Cartesian product $P_{\lambda \in \Lambda} R_\lambda$ is represented as the set $E[(x, x. x, \ldots); x \in E]$. We define a function $\Phi(x)$ from $R = \vee_{\lambda \in \Lambda} R_\lambda$ onto D; $\Phi(x) = (x, x, x, \ldots) \in D$ for any x in R. Clearly $\Phi(x)$ is a homeomorphism from R onto the relative subspace D in the product space $P_{\lambda \in \Lambda} R_\lambda$, since a phalanx $x(\alpha | A)$ converges to x on the join $\vee_{\lambda \in \Lambda} R_\lambda$ if and only if it converges to x on all R_λ

Theorem 5. *Every T_1-topological space is homeomorphic to the diagonal line D in the Cartesian product of two bicompact T_1-topological spaces.*

Proof. It is evident from Theorem 4 and Lemma.

REMARK. Every T_1-topological space R is homeomorphic to the diagonal line D in the product space $R_1 \times R_2$ of two bicompact T_1-spaces R_1 and R_2. Hence $\bar{D}$ (the closure in $R_1 \times R_2$) is a compactification of R, that is, $T = \bar{D}$ is a bicompact T_1-space and contains R as a dense subset. But it is different from the Wallman's compactification of R. This compactification T has the property that $|T| = |R|$.

DEFINITION. A property of the topological spaces is said to be hereditary if it is enjoyed by evrey subspace (in its relative topology) of a space enjoying the property. The family $\{R_\lambda\}_{\lambda \in \Lambda}$ is said to be completely ordered if for any $\lambda_1, \lambda_2, \in \Lambda$, $R\lambda_1$ is finer than $R\lambda_2$ or conversely.

E. Hewitt[9] has posed the following problem: Can the hereditary property be preserved under the formation of join of complete ordered

9) See [1].

families?

In this connection, we have the following theorem.

Theorem 6. *Let P be a hereditary property. Let spaces $R_\lambda (\lambda \in \Lambda)$ in $\mathfrak{L}_T$ have the property P respectively, and let the Cartesian product $P_{\lambda \in \Lambda} R_\lambda$ also have the property P. Then the join $\bigvee_{\lambda \in \Lambda} R$ has the property P.*

Proof. Clear by Lemma.

This theorem implies the following theorem.

Theorem 7. *Let $\{R\}_{\lambda \in \Lambda}$ be any family of regular (Hausdorff) spaces in $\mathfrak{L}_T$. Then the join $\bigvee_{\lambda \in \Lambda} R$ is also a regular space. If all spaces R are completely regular (Hausdorff), then the join $\bigvee_{\lambda \in \Lambda} R$ also enjoys the property of the completely regularity.*

This theorem has been proved by M. J. Norris [4] by using the open sets.

Corollary. *Let $\{R\}_{\lambda \in \Lambda}$ be any family of uniform spaces[10] in $\mathfrak{L}_T$. Then the join $\bigvee_{\lambda \in \Lambda} R_\lambda$ is also a uniform space.*

Theorem 8. *Let $\{R_i\}_{i=1}^\infty$ be any completely ordered family of perfectly normal spaces in $\mathfrak{L}_T$. Then the join $R = \bigvee_{i=1}^\infty R_i$* is also a perfectly *normal space.*

Proof. We remark first that for any subset M in E, the closure $\bar{M}$ in R is equal to the set $\prod_{i=1}^\infty \bar{M}^{R_i}$, $\bar{M}^{R_i}$ being the closure in R_i for each i. Suppose that $A \subset R$ is a closed set. Since the space R_i is perfectly normal, there exists a continuous function $f_i(x)$ in R_i such that $0 \leq f_i(x) \leq 1$ for all points x in R_i and $f_i(x) = 0$ if and only if $x \in \bar{A}^{R_i}$.

For $x \in R$, let us put

$$f(x) = \sum_{i=1}^\infty \frac{1}{2^i} f_i(x).$$

Clearly $f(x)$ is a continuous function on R such that $0 \leq f(x) \leq 1$ for all points x in R and $f(x) = 0$ if and only if $x \in A$, therefore, R is a perfectly normal space.

DEFINITION. A Hausdorff space is said to be a *minimal Hausdorff space* if it has the family of open sets such that no proper subfamily can be the family of open sets for a Hausdorff topology on E. Similarly, minimal regular space and a minimal completely regular space are defined.

A family $\mathfrak{A}$ of real-valued functions on E will be said to distinguish between every pair of distinct points of E if for any pair of distinct

10) See [2], p. 73.

points of E, x, y, there exists a function $f \in \mathfrak{A}$ such that $f(x) \neq f(y)$. For any family $\mathfrak{A}$ of real-valued functions on E, we define the space $R_{\mathfrak{A}}$ as follows: if f is an element of $\mathfrak{A}$ and α and β are real numbers such that $\alpha < \beta$, then the set $E[x \mid x \in R,\ \alpha < f(x) < \beta]$ will be open in $R_{\mathfrak{A}}$, and the family of all open subsets of $R_{\mathfrak{A}}$ is formed by taking arbitrary unions and finite intersection of all such sets, with arbitrary α, β, and $f \in \mathfrak{A}$. If the family $\mathfrak{A}$ distinguishes between every pair of distinct points of E, then the space $R_{\mathfrak{A}}$ is a completely regular space.

M. Katětov has proved that a Hausdorff space is minimal Hausdorff if and only if it is semi-regular and H-closed.

Theorem 9. *A completely regular space is minimal (completely regular) if and only if it is bicompact.*

Proof. It is evident that a bicompact completely regular (=bicompact Hausdorff) space is minimal. Conversely, suppose that R is s non bicompact, completely regular space: and $\mathfrak{B} = \{V_\lambda\}_{\lambda \in \Lambda}$ be an open covering of R for which no finite subcovering exists. Take three distinct points, x, y, and a in R, where a is a fixed point. Since there exists a subfamily $\{V\lambda_1, V\lambda_2, \dots, V\lambda_n\}$ of $\mathfrak{B}$ such that $F = R - \sum_{i=1}^{n} V\lambda_i$ fails to contain the points x, y, and a, and since further the space R is completely regular, there is a continuous function f_{xy} on R such that $f_{xy}(p) = 0$ for $p \in (a) \cup (x) \cup F$ and $f_{xy}(y) = 1$. Similarly, for two distinct points y, a, there exists a continuous function f_y on R such that $f_y(p) = 0$ for $p \in (a) \cup F$ and $f_y(y) = 1$. We put $\mathfrak{A} = E[f_{xy} \mid x \in R - a,\ y \in R - a,\ x \neq y] \cup E[f_y \mid y \in R - a]$. Then we see easily that the space $R_{\mathfrak{A}}$ is completely regular and the family $O(R_{\mathfrak{A}})$ is a proper subfamily of $O(R)$. Therefore R is not minimal and the proof is complete.

Corollary. *A Stone space is minimal (Stone) if and only if it is bicompact.*

Proof. It is evident, since a minimal Stone space is necessarily completely regular.

A bicompact regular space is minimal regular, but I do not know whether a minimal regular space is bicompact or not.

For any space R in $\mathfrak{L}_T$, let $C(R)$ be the family of all real-valued continuous functions on R. If R and R' are two spaces in $\mathfrak{L}_T$ such that $R \underset{*}{>} R'$, then evidently $C(R) \supset C(R')$.

Conversely, we see

Theorem 10. *Let R and R' be a completely regular space and a space* *(which need not be completely regular) in* $\mathfrak{L}_T$ *respectively. If* $C(R') \supset C(R)$, *then* $R' \underset{*}{>} R$ *in* $\mathfrak{B}_T$.

We shall prove first the following Lemma.

Lemma. *If* R *and* R' *are two completely regular spaces in* $\mathfrak{L}_T$ *such that* $C(R) = C(R')$, *then* R *coincides with* R'.

Proof. we see easily that if S is a completely regular space, then the space $R_{C(S)}$ coincides with S. Hence $R = R_{C(R)} = R_{C(R')} = R'$, since $C(R) = C(R')$.

Proof of Theorem 10. If $C(R') \supset C(R)$, then $C(R) = C(R) \wedge C(R') = C(R \wedge_* R')$. Since the space R is completely regular, the space $R \wedge_* R'$ is also completely regular. By Lemma, $R \wedge_* R'$ coincides with R, that is, $R' \underset{*}{>} R$ in $\mathfrak{L}_T$.

Moreover we have,

Theorem 11. *A* T_0*-space* R *(in* $\mathfrak{L}_T$*) is completely regular if and only if it has the property that* $C(R') \supset C(R)$ *implies* $R' \underset{*}{>} R$ *for any space* R' *in* $\mathfrak{L}_T$.

Proof. Suppose that R has such a property. Then $C(R)$ distinguishes between every pair of distinct points of R. Otherwise, the space $R_{C(R)}$ fails to satisfy the T_0-separation axiom, and $C(R_{C(R)}) = C(R)$. But $R_{C(R)} \underset{*}{\not>} R$, since R is a T_0-space, which is a contradiction. Hence the space $R_{C(R)}$ is completely regular, and we see at the same time that $C(R_{C(R)}) = C(R)$ and $R \underset{*}{>} R_{C(R)}$. But, by the property of R, $R_{C(R)} \underset{*}{>} R$, since $C(R_{C(R)}) = C(R)$. Therefore R coincides with $R_{C(R)}$, and consequently, R is completely regular.

The converse is clear by Theorem 10. This completes the proof.

We remark finally that any expansion of a regular space is a Hausdorff space but in general not a regular space, and any expansion of a completely regular space is a Stone space but is not always a completely regular space, where the space R' is said to be an expansion of R if $R' \underset{*}{>} R$.

§ 3. Two dual-Ideals in $\mathfrak{L}_A$.

The topology of a space R with additive topology is also defined by the convergence, that is to say, a notion of the convergence determines, for each function $x(a \mid \mathfrak{A})$ from a directed system $\mathfrak{A}$ to R and for each point x of R, whether $x(a \mid \mathfrak{A})$ converges to x or not (the statement "$x(a \mid \mathfrak{A})$ converges to x" is written $x(a) \underset{\mathfrak{A}}{\rightarrow} x$, or merely

$x(a) \to x$), and satisfies the following conditions.[11]

(I) $x(a|\mathfrak{A})$ converges to x if and only if for each subsystem $\mathfrak{A}'$ which is cofinal in $\mathfrak{A}$, there exists some phalanx $x(\delta|D)$, which is contained in $x(\mathfrak{A}')$ and converges to x, D being the family of all subsets in E.

(II) If $x(a|\mathfrak{A})$ is ultimately equal[12] to x, then it converges to x.

DEFINITION. We will call a space X a *Fréchet's L-space* if a notion of the convergence determines, for each sequence $\{x_n\}$ and for each point x in X, whether x_n converges to x or not (the statement " x_n converges to x " is written $x_n \to x$) and satisfies the following conditions.

(I) If x_n converges to x, then $x_{n_i}(n_1 < n_2 < \cdots)$ also converges to x.

(II) If $x_n = x$ for all n, then x_n converges to x.

Each space with additive topology can be regarded as a Fréchet's L-space by this definition. We define next the L-equivalence: Two spaces R_1 and R_2 in $\mathfrak{L}_A$ are said to be L-equivalent if and only if whenever a sequence x_n converges to x on R_1, then it also converges to x on R_2, and conversely. Indeed, two spaces R_1 and R_2 in $\mathfrak{L}_A$ belong to the same type of Fréchet's L-spaces if they are L-equivalent.

Considering the L-equivalence, we see easily that the equivalence-class including the least element in $\mathfrak{L}_A$ forms an ideal $\mathfrak{A}_0$ and the class including the greatest element (= the discrete space) a dual-ideal $\mathfrak{A}_I$ in $\mathfrak{L}_A$. The spaces belonging to this class are very interesting. We will call such spaces I-spaces. That is, a spaces is an *I-space* if and only if any essential[13] sequence $\{x_n\}$ fails to converge to all points of E and each sequence which is ultimately equal to x converges to x only on it.

DEFINITION. A space is said to be a *C-space* if each infinite subset contains the closure of an infinite subset.

The family of all C-spaces on E forms a dual-ideal in $\mathfrak{L}_A$. For, let X, Y be two C-spaces in $\mathfrak{L}_A$. We may prove that Z is a C-space when we put $Z = X \wedge Y$. Let M be an infinite subset of the space Z. Then there exists an infinite set N and $M \supset \bar{N}^X$, since M is a subset of X and X is a C-space. Also, there is an infinite set N_1 and $N \supset \bar{N}_1^Y$, since Y is also a C-space. $\bar{N}_1^Z = \bar{N}_1^X \vee \bar{N}_1^Y \subset \bar{N}^X \vee \bar{N}_1^Y \subset M$. Hence the space Z is a C-space.

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11) See [2], pp. 17-24.

12) $x(a|\mathfrak{A})$ is ultimately equal to x if some $a' \in \mathfrak{A}$ $a > a'$ implies $x(a) = x$.

13) A function $x(a|\mathfrak{A})$ from a directed system $\mathfrak{A}$ into a space X is said to be essential if $x(a|\mathfrak{A})$ is not ultimately equal to all points of X.

14) Problem of E. Čech: See Fund. Math. 34 (1947).

E. Čech[14] has posed the following problem: Does there exist a non identical set-function f on the set X satisfing the following three properties: (i) $f(M) \supset M$ for each subset M of X, (ii) $f(M \cup N) = f(M) \cup f(N)$ for each pair M, N and (iii) for every subset M of X, there is a subset N such that $M = f(N)$? If there are such an f and an X, X is a C-space and a T_1-space.

Example of C-spaces. a) The ω_μ-additive spaces[15] ($\mu > 0$); the topological spaces which satisfy the following axioms:

(I) For every α-sequence of subsets $\{M_\xi\}$, $\alpha < \omega_\mu$,

$$\overline{\sum_{0 \leq \xi < \alpha} M_\xi} = \sum_{0 \leq \xi < \alpha} \bar{M}_\xi$$

(II) $\bar{M} = M$ for every finite subset M.

(III) $\bar{\bar{M}} = \bar{M}$ for every subset M.

b) The case where the potency of the space is equal to $\aleph_0$; suppose that X is the set of all rational points of the straight line E^1. For a subset M of X, $\tilde{M}$ denotes the closure of M with regard to the usual topology on E^1. Let p be a fixed point of X, and for each subset M of X, we define now the closure $\bar{M}$ as follows: $\bar{M} = M \cup p$ if and only if the set $\tilde{M}$ contains an open interval of E^1 with regard to the usual topology, and for the others, $\bar{M} = M$. Then X is also a C-space. The countable C-spaces are I-spaces, as we see later, if we assume the Hausdorff's separation axiom, therefore (b) is also an example of I-spaces.

We shall give hereafter some characteristics of I-spces, C-spaces, and countably compact spaces, and relations between them.

In this paragraph, "the set" or "the subset" denotes a set with an infinite potency.

A subset of X is called a *hereditary closed set* if and only if its subsets are all the closed sets in X. We see here that the hereditary closed sets are very scattered sets.

Theorem 12. *In a space with additive topology satisfing the T_1-axiom, the following properties are equivalent.*

(i) *Space X is a C-space.*

(ii) *Each subset contains a closed set in X.*

(iii) *Each subset is a sum of hereditary closed sets in X.*

Proof. (iii)$\to$(i). It is evident from the definition.

(i)$\to$(ii). Let M be any subset of X. Then there exists a subset M_1

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15) See Sikorski [5].

and $M \supset \bar{M}_1$, since X is a C-space. Similarly, for each n, there is a subset $\bar{M}_{n+1}$ and $M_n \supset \bar{M}_{n+1}$. If for some n, the set M_n is the closed set, then this subset M_n is a desired one. But if for each n, M_n fails to be the closed set, then $M_n \supsetneqq M_{n+1}$ $(n = 1, 2, 3, \ldots)$. Consequently, any set $M_n - M_{n+1}$ contains at least one point p_n. Put $P = E[p_n | n = 1, 2, 3, \cdots]$, then it is clear that $\bar{P} - P \subset \prod_{n=1}^{\infty} M_n$. Since the set $\prod_{n=1}^{\infty} M_n$ is a closed set, $P^\mu - P \subset \prod_{n=1}^{\infty} M_n$ and $P^\mu \subset M$ for each ordinal number μ, where we write; $P^1 = P$, $P^2 = \bar{P}$, $P^\lambda = (\overline{P^{\lambda-1}})$ for any non-limit ordinal number λ, and $P^\lambda = \sum_{\mu<\lambda} P^\mu$ for any limit ordinal number λ. P^ξ can be a closed set for some ordinal number ξ. This set P^ξ is a required one.

(ii)$\rightarrow$(iii). Let M be any subset. Then we can find the closed sets M_n such that $M \supset M_1$ and $M_n \supsetneqq M_{n+1}$ $(n = 1, 2, 3, \ldots)$ by (ii). Hence $M_n - M_{n+1}$ has at least a point p_n. If we put $P = E[p_n | n = 1, 2, 3, \ldots]$, there exists a closed subset Q such that $Q \subset P$ by (ii). We see here that set Q is contained in M and is a hereditary closed set. This completes the proof.

From this theorem, we see easily that for any subset M of a C-space, $|\mathfrak{N}_M| \geq 2^{\aleph_0} \cdot |M|$, where $\mathfrak{N}_M = E[N | M \supset \bar{N}, |N| \geq \aleph_0]$.

Theorem 13. *In a perfectly normal (T-)space, the following properties are equivalent.*

(i) *X is an I-space.*

(ii) *If $|\mathfrak{A}| = \aleph_0$, any essential function $x(a | \mathfrak{A})$ into X fails to converge to all points of X.*

(iii) *X is a C-space.*

(iv) *Each subset of X is a non-compact (= non bicompact) space.*

Proof. Verifications of the equivalence (i)$\leftrightarrow$(ii) and of (iv)$\rightarrow$(i) are quite elementary and are therefore passed over.

(i)$\rightarrow$(iii). Let M be any subset. Then there exists a subset N such that $M \supset N$ and N'[16] has at most one point by the hypothesis of the perfectly mormality. We may assume here that the potency of the subset N is countably infinite. If $N' = \{x\}$ and the elements of N are $x_1, x_2, \ldots, x_n, \ldots$, that is, $N = E[x_n | n = 1, 2, 3, \ldots]$, then $x_n \nrightarrow x$ (the denial of the convergence) by the hypothesis of (ii). Hence there is a neighbourhood of x, $N(x)$, and $N_1 = N\text{-}N(x)$ is an infinite set and $M \supset \bar{N}_1$.

(iii)$\rightarrow$(iv). Suppose that X is a C-space. By Theorem 12, any subset M of X contains a hereditary closed set N. Put $N_p = N - p$

16) N' denotes the set of all cluster points of the set N.

for each point p in N, then the family $\{N_p\}_{p \in N}$ is of the closed sets satisfiing the finite intersection property, and $\prod_{p \in N} N_p = \phi$. This show that each subset of X is non-compact.

Corollary. *In a Hausdorff countable space (= space with additive topology), the properties of* (i), (ii), (iii), (iv) *of Theorem* 13 *are equivalent.*

Theorem 14. *In a perfectly normal (T-)space or a Hausdorff countable T-space X, any subset contains a hereditary closed set or an essential converging sequence. If each subset of X contains a hereditary closed set, then X is an I-space (=a C-space), but if each subset contains an essential converging sequence, then X is countably compact space.*

Proof. For each subset M, there is a subset N such that $M \supset N$ and N' consist of at most a point by the hypothesis of the perfectly normality (or by Hausdorff's separation axiom in the case of the countable space.) If $\bar{N}$ is a C-space, then its subsets N contains a hereditary closed sets in the relative substance $\bar{N}$ of X, since $\bar{N} \supset N$, by Theorem 12. But, since $\bar{N}$ is a closed set in X, N contains a hereditary closed set in X. Secondary, if $\bar{N}$ is a non C-space, then $\bar{N}$ contains an essential converging sequence $\{x_n\}$ by Theorem 13. But $\bar{N} - N \subset N'$ and N' has at most one point, therefore, we have an essential convering sequence which is contained in subset N. The proof is complete.

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