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On a Theorem of Kaplansky

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I. Kaplansky has proved an interesting theorem: any division ring is commutative if, for every element x, some power $x^{n(x)}$ is in the centre.[1] As special cases this theorem contains the well-known theorem of Wedderburn of finite division rings as well as its generalization due to Jacobson.[2] On the other hand I. N. Herstein has proved that any ring in which $x^n - x$ is in its centre for every element x and for a fixed integer $n > 1$, is commutative. Moreover he has conjectured that the rings in which $x^{n(x)} - x$ is in the centre for every element x and for an integer $n(x)$ (depending on x and larger than 1) may be commutative.[3]

In this note we shall prove a generalization of Kaplansky's theorem and, as its applications, we shall generalize a result of Hua[4] and show that Herstein's conjecture is valid for semi-simple rings in the sense of Jacobson.

Theorem. *Let D be a division ring with centre Z and let $c_i (i = 0, 1, \cdots, r)$ be $r+1$ fixed non- zero elements in the prime subfield of D. If, for every element x in D, there are integers $n_0(x) > n_1(x) > \cdots > n_r(x) > 0$ such that* i) $\sum_{i=0}^{r} c_i x^{n_i(x)}$ *is in Z and* ii) $n_1(x)$ *is smaller than an integer M (not depending on x), then D is commutative.*

Here, if we put $r = 0$, we have Kaplansky's theorem. Hence we prove only the case $r > 0$.

To prove our theorem, it is sufficient according to Kaplansky to prove the following[5]

Lemma. *Let K be a field, $L (\neq K)$ an extension of K and let $c_i (i = 0, 1, \cdots, r)$ be $r+1$ fixed non-zero elements in the prime subfield of L. If, for every element x in L, there are integers $n_0(x) > n_1(x) > \cdots$*

1) Cf. Kaplansky [5].
2) Cf. Jacobson [4] Th. 8.
3) Cf. Herstein [1].
4) Cf. Hua [2] Th. 7.
5) Cf. Kaplansky [5].

$n_r(x) > 0$ *such that* (i) $\sum_{i=0}^{r} c_i x^{n_i(x)}$ *is in* K *and* (ii) $n_1(x)$ *is smaller than a fixed integer* M, *then* L *has prime characteristic and is either purely inseparable over* K *or algebraic over its prime subfield.*

Proof. For every element x, $m_0(x) > m_1(x) > \cdots > m_r(x) > 0$ denote the system of $r+1$ integers satisfying (i) such that $m_1(x)$ is the minimum of $n_1(x)$. Hence $m_1(x)$ is smaller than M by (ii).

(a) First we prove that L has prime characteristic.

Assume that L has characteristic zero. Then the prime subfield P of L is the field of rational numbers. Therefore, we may assume that $c_i (i = 0, \cdots, r)$ are $r+1$ fixed non-zero integers. Now let a be an element in L but not in K. Then a can be sent into an element $b\ (\neq a)$ by a suitable isomorphism θ which leaves K elementwise fixed. Here b need not be in L. By θ, a^{-1} and $i(a+1)$, i an arbitrary integer, are sent into b^{-1} and $i(b+1)$ respectively. Since $\sum_{\kappa=0}^{r} c_\kappa (a^{-1})^{m_\kappa(a^{-1})}$ and $\sum_{\kappa=0}^{r} c_\kappa (i(a+1))^{m_\kappa(i(a+1))}$ are in K, we have readily

$$(1) \qquad \sum_{\kappa=0}^{r} c_\kappa (a^{-1})^{m_\kappa(a^{-1})} - \sum_{\kappa=0}^{r} c_\kappa (b^{-1})^{m_\kappa(a^{-1})} = 0$$

and

$$(2) \qquad \sum_{\kappa=0}^{r} c_\kappa \left\{ (i(b+1))^{m_\kappa(i(a+1))} - (i(a+1))^{m_\kappa(i(a+1))} \right\} = 0 .$$

Multiplying (1) by $(ab)^{m_0(a^{-1})}$, we have

$$(3) \qquad \Big(\sum_{\kappa=0}^{r} c_\kappa a^{m_0(a^{-1}) - m_\kappa(a^{-1})}\Big) b^{m_0(a^{-1})} - \sum_{\kappa=0}^{r} c_\kappa a^{m_0(a^{-1})} b^{m_0(a^{-1}) - m_\kappa(a^{-1})} = 0 .$$

Dividing (2) by $i^{m_r(i(a+1))}$, we have

$$(4) \qquad \sum_{\kappa=0}^{r} c_\kappa i^{m_\kappa(i(a+1)) - m_r(i(a+1))} \left\{ (b+1)^{m_\kappa(i(a+1))} - (a+1)^{m_\kappa(i(a+1))} \right\} = 0 .$$

Since $b - a \neq 0$, dividing (4) by $(b+1)-(a+1)$, we have

$$(5) \qquad c_0 i^{m_0(i(a+1)) - m_r(i(a+1))} b^{m_0(i(a+1)) - 1} + \cdots \text{terms with powers of } b \cdots + \sum_{\kappa=0}^{r} c_\kappa i^{m_\kappa(i(a+1)) - m_r(i(a+1))} \Big(\sum_{\lambda=0}^{m_\kappa(i(a+1)) - 1} (a+1)^\lambda \Big) = 0 .$$

Eliminating b from (3) and (5), we have a relation:

$$F(a\,;\,i)\equiv\begin{vmatrix} \sum c_\kappa a^{m_0(a^{-1})-m_\kappa(a^{-1})} \cdots\cdots c_0 a^{m_0(a^{-1})} & & & \\ \ddots & \ddots & & 0 \\ 0 \quad \ddots & & \ddots & \\ \sum c_\kappa a^{m_0(a^{-1})-m_\kappa(a^{-1})} & & c_0 a^{m_0(a^{-1})} & \\ \cdots\cdots\cdots\cdots\cdots\cdots * & & & \\ \ddots & & \ddots & 0 \\ 0 \quad \ddots & & & \ddots \\ & \cdots\cdots\cdots\cdots\cdots\cdots & & * \end{vmatrix}=0,$$

where $*$ is the last term of (5). Replacing a by X, we have an equation $F(X\,;\,i)=0$ satisfied by a. It is obvious that the coefficients of $F(X\,;\,i)$ are integers. The constant term $F(o\,;\,i)$ of $F(X\,;\,i)$ is a product of c_0 and $c(i)=\sum_{\kappa=0}^{r} c_\kappa m_\kappa(i(a+1))\, i^{m_\kappa(i(a+1))-m_r(i(a+1))}$. Since $M> m_\kappa(i(a+1))$ $(\kappa\geqq 1)$ for every i, it is obvious that, if we take as i an integer j such that $|j|>rM\times\operatorname*{Max}_{\kappa\geqq 1} c_\kappa$, then $c(j)\neq 0$. Hence $F(X\,;\,j)=0$ is a non-trivial equation and consequently a is algebraic over P. Let $g(X)\equiv\sum_{i=0}^{N}\alpha_i X^i$ be a primitive irreducible polynomial in $P[X]$ satisfied by a. Then the constant term α_0 of $g(X)$ divides the constant term of $F(X\,;\,i)$ for every i. Hence α_0 divides the constant term of $F(X\,;\,\alpha_0)$ which is a product of powers of c_0 and $c(\alpha_0)=\sum_{\kappa=0}^{r} c_\kappa m_\kappa(\alpha_0(a+1))\,\alpha_0^{m_\kappa(\alpha_0(a+1))-m_r(\alpha_0(a+1))}$. Therefore, α_0 divides a product of suitable powers of c_0 and $c_r m_r(\alpha_0(a+1))$. Now let p be a prime which does not divide c_0, c_r and α_N, and is larger than M. Then pa satisfies the equation $\alpha_N X^N+p\alpha_{N-1}X^{N-1}+\cdots+p^N\alpha_0=0$ which is primitive and irreducible. Therefore, considering pa in place of a, we see that the constant term $p^N\alpha_0$ divides a product of suitable powers of c_0 and $c_r m_r(p^N\alpha_0(pa+1))$. But p does not divide c_0 and c_r, so it divides $m_r(p^N\alpha_0(pa+1))<M$. This is a contradiction. Hence L has prime characteristic.

(b) Secondly we prove the latter half of the lemma. Now let L have characteristic $p\neq 0$. If L is purely inseparable over K, then there is nothing to prove. Therefore, let a be a separable element in L but not in K. Since L is algebraic over K, if K is algebraic over its prime subfield P, then L is algebraic over P. So we assume that in K there is at least one transcendental element over P. Let z be such an element. Since a is separable over K, it is sent into an element $b\,(\neq a)$ by a suitable isomorphism θ whicn leaves K elementwise fixed. By θ, a^{-1} and $z(a+1)$ are set into b^{-1} and $z(b+1)$ respectively. In the same way as in (a), we have

$$(6)\quad \left(\sum_{\kappa=0}^{r} c_\kappa a^{m_0(a^{-1})-m_\kappa(a^{-1})}\right) b^{m_0(a^{-1})} - \sum_{\kappa=0}^{r} c_\kappa a^{m_0(a^{-1})} b^{m_0(a^{-1})-m_\kappa(a^{-1})} = 0$$

and

$$(7)\quad c_0 z^{m_0(z(a+1))-m_r(z(a+1))} b^{m_0(z(a+1))-1} + \cdots \text{terms with powers of } b$$
$$\cdots + \sum_{\kappa=0}^{r} c_\kappa z^{m_\kappa(z(a+1))-m_r(z(a+1))} \Big(\sum_{\lambda=0}^{m_\kappa(z(a+1))-1} (a+1)^\lambda \Big) = 0 .$$

Eliminating b from (6) and (7), we have an equation $F(X)=0$ satisfied by a. It is easy to see that the coefficients of $F(X)$ are in $P[z]$ and the constant term of $F(X)$ is a product of suitable powers of c_0 and $c(z)=\sum_{\kappa=0}^{r} c_\kappa m_\kappa(z(a+1))\, z^{m_\kappa(z(a+1))-m_r(z(a+1))}$. Here we may assume that $m_0(x), m_1(x), \cdots, m_r(x)$ are not all congruent to zero mod. p for every separable element x over K. For, if $m_i(x)=p^\mu m_i'(x)$ for $i=0,\cdots,r$, then $\sum_{i=0}^{r} c_i x^{m_i(x)} = (\sum_{i=0}^{r} c_i x^{m_i'(x)})^{p^\mu}$ is in K. Since x is separable over K, $\sum_{i=0}^{r} c_i x^{m_i'(x)}$ is separable over K, so $\sum_{i=0}^{r} c_i x^{m_i'(x)}$ is in K. This contradicts the minimality of $m_1(x)$. Therefore $c(z)$ is not zero, since z is transcendental over P. Therefore $F(X)=0$ is a non-trivial equation and consequently a is algebraic over $P(z)$. Furthermore the domain of integrity $P[z]$ of $P(z)$ is a unique factorization domain. Let $g(X)\equiv\sum_{i=0}^{N} \alpha_i X^i$ be a primitive irreducible polynomial in $P[z][X]$ satisfied by a. Now assume that α_0 is not in P and π is a prime divisor of α_0. Since $\pi(\pi^M a+1)$ is sent into $\pi(\pi^M b+1)$ by θ,

$$\sum_{\kappa=0}^{r} c_\kappa \left\{ (\pi(\pi^M b+1))^{m_\kappa(\pi(\pi^M a+1))} - (\pi(\pi^M a+1))^{m_\kappa(\pi(\pi^M a+1))} \right\} = 0 .$$

Dividing this by $\pi^{m_r(\pi(\pi^M a+1))}\{(\pi^M b+1)-(\pi^M a+1)\}$, we have the relation:

$$c_0 \pi^{m_0(\pi(\pi^M a+1))-m_r(\pi(\pi^M a+1))} (\pi^M b)^{m_0(\pi(\pi^M a+1))-1} + \cdots \text{terms with powers of } b$$
$$\cdots + \Big\{ \cdots \text{terms with powers of } a \cdots$$
$$+ \sum_{\kappa=0}^{r} c_\kappa m_\kappa(\pi(\pi^M a+1))\, \pi^{m_\kappa(\pi(\pi^M a+1))-m_r(\pi(\pi^M a+1))} \Big\} = 0 .$$

In this relation, terms with powers of b or a are divisible by π^M. Hence if $m_i(\pi(\pi^M a+1))\equiv 0\ (p)$ for $i>s$ and $m_s(\pi(\pi^M a+1))\not\equiv 0\ (p)$ for an $s\neq 0$, we divide the above relation by $\pi^{m_s(\pi(\pi^M a+1))-m_r(\pi(\pi^M a+1))}$. Now we eliminate b from the relation thus obtained and (6). Then we have an equation $G(X)=0$ satisfied by a where the constant term $G(0)$ of $G(X)$ is either a product of powers of c_0 and $\sum_{\kappa=0}^{s} c_\kappa m_\kappa(\pi(\pi^M a+1))\, \pi^{m_\kappa(\pi(\pi^M a+1))-m_s(\pi(\pi^M a+1))}$ for some $s\neq 0$ or a product of powers of c_0

and $c_0 m_0(\pi(\pi^M a+1))\,\pi^{m_0(\pi(\pi^M a+1))-m_r(\pi(\pi^M a+1))}$. It is easy to see that the coefficients of $G(X)$ are in $P[z]$. Therefore, α_0 is a divisor of the constant term $G(0)$ of $G(X)$ and consequently π is a divisor of $G(0)$. If $G(0)$ is a product of powers of c_0 and $\sum_{\kappa=0}^{s} c_\kappa m_\kappa(\pi(\pi^M a+1))\,\pi^{m_\kappa(\pi(\pi^M a+1))-m_s(\pi(\pi^M a+1))}$ for an $s\neq 0$, then a product of powers of $c_0\neq 0$ and $c_s m_s(\pi(\pi^M a+1))\neq 0$ is divisible by π. This is a contradiction. Therefore, $G(0)$ is a product of a power of π and an element in P. Thus we see that the constant term of a primitive irreducible polynomial in $P[z][X]$ satisfied by a separable element is either in P or a product of a power of an irreducible polynomial in $P[z]$ and an element in P.

Now if we take $a+H(z)$, $H(z)$ an arbitrary polynomial in $P[z]$, in place of a, then the constant term of a primitive irreducible polynomial in $P[z][X]$ satisfied by $a+H(z)$ must be either in P or a product of a power of an irreducible polynomial in $P[z]$ and an element in P. Now we take z^i as $H(z)$, where i is an integer larger than the degrees of $\alpha_\kappa(\kappa=0,\cdots,N)$. Then $a+z^i$ satisfies $g(X-z^i)$ which is a primitive irreducible polynomial in $P[z][X]$. Obviously the constant term $g(-z^i)$ of $g(X-z^i)$ is not in P. Hence $g(-z^i)=\beta h(z)^l$, where $h(z)$ is an irreducible polynomial in $P[z]$ and β is an element in P. Take $z^i+h(z)^t$ as $H(z)$, where t is an integer larger than l. Then the constant term $g(-(z^i+h(z)^t))$ of $g(X-(z^i+h(z)^t))$ which is a primitive irreducible polynomial in $P[z][X]$ satisfied by $a+z^i+h(z)^t$, is not in P and is divisible by $h(z)$. Therefore, $g(-(z^i+h(z)^t))$ must be a product of a power of $h(z)$ and an element in P. But this is impossible. Thus we have a contradiction. Therefore K is algebraic over P and L is algebraic over P.

Corollary. *Let D be a division ring with centre Z and let $f(X)$ be a fixed polynomial of degree n whose coefficients are in the prime subfield of D. If $x^{n(x)}+f(x)$ is in Z for every x in D and for an integer $n(x)$ (depending on x and larger than n), then D is commutative.*

Remark. It is probably true that we can drop the condition (ii) and take the assumption that $c_i(i=0,\cdots,r)$ are in Z, in place of the assumptian that c_i are in the prime subfield. But this is still an open question.

As the first application of our theorem, we shall generalize a result of Hua[6] as follows:

Theorem. *Any non-commutative division ring D is generated by*

6) Cf. Hua [2] Th. 7.

elements of the form $\sum_{i=0}^{r} c_i x^{n_i(x)}$, *where* $c_i (i=0, 1, \cdots, r)$ *are the fixed non-zero elements in the prime subfield of* D *and* $n_0(x) > n_1(x) > \cdots > n_r(x) > 0$ *are intergers such that* $n_i(x) = n_i(a^{-1}xa)$ *for all* $a \neq 0$ *in* D *and* $n_1(x)$ *is smaller than a fixed integer* M.

Proof. Let D' be the division ring generated by the elements $\sum_{i=0}^{r} c_i x^{n_i(x)}$, then D' is invariant under inner automorphisms of D. If $D' \neq D$, then D' is contained in the centre of D, by a result of Hua.[7] Then D is commutative. This is a contradiction. Therefore $D' = D$.

As the second application of our theorem, we show that Herstein's conjecture is valid for semi-simple rings in the sense of Jacobson:

Theorem. *Let* A *be a semi-simple ring with centre* Z *and let* c *be an integer. If there is an integer* $n(x)$ *larger than 1 for every element* x *and* $x^{n(x)} + cx \in Z$, *then* A *is commutative.*

Proof. Since A is semi-simple, A is a subdirect sum of primitive rings.[8] Since our assumption is valid for residue class rings of A, it is sufficient to prove our assertion in the case where A is a primitive ring. Any primitive ring is isomorphic to a dense ring R of linear transformations in a vector space V over a division ring D.[9] Let V be more than one-dimensional and let α and β be two linear independent vectors. Since R is dense, there is an element a in R such that $\alpha a = \beta$ and $\beta a = 0$. Then, for any integer $n > 1$, $\alpha(a^n + ca) = c\beta$ and $\beta(a^n + ca) = 0$. If $c\beta = 0$, then $c\gamma = 0$ for all vectors in V. Hence $cb = 0$ for all b in R, so $cx = 0$ for all x in A. But this case was proved by Kaplansky.[10] If $c\beta \neq 0$, then $a^n + ca$ is not in the centre of R. For, $a^n + ca$ does not commute with the linear transformation in R such that $\alpha \to \beta$ and $\beta \to \alpha$. Hence V is one-dimensional, so R is a division ring. Then, by our theorem, R is commutative.

Putting $c = -1$, we see that Herstein's conjecture is valid for semi-simple rings.

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7) Cf. Hua [2] Th. 1.
8) Cf. Jacobson [3].
9) Cf. Jacobson [3].
10) Cf. Kaplansky [5].