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A Generalization of a Theorem of I. Kaplansky

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I. Kaplansky[1] showed that a bicompact space X is determined by the lattice of all continuous real functions on X. On the other hand it is proved by I. Gelfand, A. N. Kolmogoroff, M. H. Stone, E. Hewitt and others[2] that the rings of continuous real functions on spaces X, which are not always bicompact, determine spaces X.

In this note we discuss the characterization of some sublattices of lattices of all regular open sets on locally bicompact spaces and applying this characterization we show that locally bicompact spaces and complete metric spaces are determined respectively by sublattices of all continuous real functions on them. Furthermore we prove that e-complete spaces[3] and completely regular spaces are determined respectively by the lattices and the topological lattices of all continuous real functions on them.

§ 1. Lattices of regular open sets of locally bicompact spaces.

Definition 1. In a distributive lattice L with the smallest elements 0, we define a *binary relation* $a \supset b$ for two elements a and b of L as follows:

$$a \supset b \quad \text{if and only if} \quad a \wedge c \rightarrow b \wedge c = 0 \,.$$

Furthermore we say that *a is equivalent to b* if $a \supset b$ and $b \supset a$. Let $p(L)$ be the set of all equivalence classes and for two equivalence classes $[a]$ and $[b]$ we define $[a] \geqq [b]$ if $a \supset b$. Then $p(L)$ is also a distributive lattice.

Obviously $L = p(L)$ if and only if L satisfies Wallman's disjunction property[4] and $p(L) = p(p(L))$.

Definition 2. We say that a distributive lattice L with the smallest element 0 satisfying Wallman's disjunction property an *R-lattice*, if

1) I. Kaplansky [1].

2) I. Gelfand and A. N. Kolmogoroff [2], M. H. Stone [3], E. Hewitt [4], J. Nagata [5], D. Gale [6] and M. E. Shanks [7].

3) T. Shirota [11]. We say that completely regular space is e-complete if the structure with basis for uniformity made up of all countable normal coverings is complete.

4) H. Wallman [9].

there exists a binary relation $\gg$ in L which satisfies the following condition:

i) if $h \geqq f$ and $f \gg g$, then $h \gg g$,

ii) if $f_1 \gg g_1$ and $f_2 \gg g_2$, then $f_1 \wedge f_2 \gg g_1 \wedge g_2$,

iii) if $f \gg g$, there exists an h such that $f \gg h \gg g$,

iv) for all $f > 0$, there exist g_1 and $g_2 > 0$ such that $g_1 \gg f \gg g_2$, and

v) if $h \gg f \gg g$, there exists an $h(f, g)$ such that $h(f, g) \vee f = h$ and $h(f, g) \wedge g = 0$. Moreover if $h_1 \gg h$ and $g \gg g_1$, then $h_1(g, g_1) \gg h(f, g)$.

Then obviously if $f \gg g$ and $g \geqq h$, then $f \gg h$ and if $f \gg g$, then $f \geqq g$.

Now we have the following theorems.

Theorem 1. *A distributive lattice L with the smallest element 0 is an R-lattice if and only if it is isomorphic to a sublattice of the lattice of all regular open sets on a locally bicompact space X which is an open basis and whose elements have bicompact closures.*

Proof. (I) Let L be an R-lattice. Then we say that a subset I of L satisfying the following conditions is an open maximal ideal of L:

1) $0 \notin I$,

2) $f \in I$ and $g \in I \rightarrow f \wedge g \in I$,

3) for any $f \in I$ there exists $g \in I$ such that $f \gg g$,

and

4) it is maximal with respect to the properties 1), 2) and 3).

Moreover let X be the set of all open maximal ideals and let $U(f)$ be its subset $\{I \mid I \ni f\}$. Then we show that considering $\{U(f) \mid f \in L\}$ as an open basis of X, X is a locally bicompact space.

First we remark that for any $f > 0$, $U(f)$ is not void, since by the conditions i)-iv) of R-lattice and by the definition of the open maximal ideal for any $f > 0$ there exists an I such that $I \ni f$. Furthermore X is obviously a T_1-space. Next we show that if $f \gg g$, $U(f) \supset U(g)^a$.[5] For if $I \in U(g)^a$, then $g \wedge f_1 \neq 0$ for any $f_1 \in I$, hence by i), ii), iii), and 4) $I' = \{h \mid h \geqq g_1 \wedge f_1$ for some $f_1 \in I$ and for some $g_1 \gg g\}$ is equal to I, but $f \in I'$, hence $f \in I$, i.e., $U(f) \ni I$. Accordingly X is a regular space. Now we show that for any h $U(h)^a$ is bicompact. For this assume that for some h_1 $U(h_1)^a$ is not bicompact. Then there exists a closed filter $\{F_\alpha\}$ of $U(h_1)^a$ such that $\Pi F_\alpha = \phi$. Let $h_0 \gg h_1$. Then for any $I \in U(h_0)$ there exists an F_α such that $F_\alpha \not\ni I$. Hence there exists an $f \in I$ such that $U(f) \frown F_\alpha = \phi$. Moreover let h'_i and g_j $(i = 1, 2;$ $j = 1, 2, 3)$ be elements of I such that $h_0 \gg h'_1 \gg h'_2 \gg h_1$, $h'_2 \gg g_1$ and

5) A^a = the closure of A. A^c = the complement of A.

$f \gg g_1 \gg g_2 \gg g_3$. Then $U(g_1)^a \frown F_\alpha = \phi$ and $h_1'(g_2, g_3) \gg h_2'(g_1, g_2)$. Hence $U(h_2'(g_1, g_2))^a \cup U(g_1)^a \supset U(h_2')^a \supset F_\alpha$, that is, $U(h_1'(g_2, g_3)) \supset F_\alpha$. Furthermore let h' and g' be the elements of I such that $h_0 \gg h' \gg h_1'$ and $g_3 \gg g'$. Then $h'(g_3, g') \gg h_1'(g_2, g_3)$ and $h'(g_3, g') \frown g' = 0$. Now we set $I' = \{h \mid U(h') \supset F_\alpha$ for some $F_\alpha \in \{F_\alpha\}$ and for some $h' \ll h\}$. Then I' satisfies the conditions 1), 2) and 3). If I_0 is the open maximal ideal containing I', then for all $I \in U(h_0)$ there exists g such that $g \in I$, $g \wedge h = 0$ for some $h \in I_0$. In particular for I_0 there exist g and h such that $g, h \in I_0$ and $g \wedge h = 0$, which is a contradiction.

(II) We prove that $U(f)$ is a regular open set of X. For this we have only to show that $U(f)^{aca} \supset U(f)^c$, i.e., that if $I \notin U(f)$, $U(g) \not\subset U(f)^a$ for all $g \in I$. Let $f \notin I$, then $f \not> g$ for all $g \in I$. By the disjunction property there exists $h > 0$ such that $f \wedge h = 0$ and $g > h$. Evidently $U(f) \frown U(h) = \phi$ and $U(g) \supset U(h)$. Accordingly $U(f)^a \not\supset U(g)$. Furthermore we show that the correspondence $f \leftrightarrow U(f)$ is one to one and isomorphic. For this we have to show that $U(f) \wedge U(g) = U(f \wedge g)$ and $U(f) \vee U(g) = U(f \vee g)$ in the lattice of all regular open subsets of X. But by the definition of $U(f)$ the first relation and $U(f) \vee U(g) \leqq U(f \vee g)$ is evident. Accordingly we have only to prove that $U(f \vee g) \subset (U(f) \bigcup U(g))^{-c-c}$, i.e., that $U(f \vee g) \subset U(f)^a \bigcup U(g)^a$, but it is evident. Furthermore by the disjunction axiom this correspondence is one-to-one.

(III) Conversely let L be a sublattice with the smallest element ϕ of the lattice of all regular open sets on a locally bicompact space X which is an open basis and whose elements have bicompact closures and let $U_1 \gg U_2$ if $U_1 \supset U_2{}^a$. Obviously the binary relation $\gg$ satisfies the conditions of Definition 2. The proof of Theorem 1 is thus complete.

Theorem 2. *Let L be an R-lattice. Then there exists uniquely a locally bicompact space X which satisfies the property in Theorem 1 and where $U(f) \supset U(g)^a$ if and only if $f \gg g$.*

Proof. We first prove that if $f \not\gg g$, then $U(f) \not\supset U(g)^a$. Let $f \not\gg g$. Then by iv) there exists an h_0 such that $h_0 \gg f$ and $h_0 \gg g$. Now if there exist f_1 and f_2 such that $f_2 \ll f_1 \ll f$ and $h_0(f_1, f_2) \wedge g = 0$, then we have $h_0(f_1, f_2) \vee f_1 = h_0, h_0(f_1, f_2) \wedge f_2 = 0$, and $f_1 \wedge g = (f_1 \vee h_0(f_1, f_2)) \wedge g = h_0 \wedge g = g$. Since $f \gg f_1$ and $h_0 \gg g$, we have by iii) $f \geqq f \wedge h_0 \gg f_1 \wedge g = g$, which implies $f \gg g$, contrary to our assumption. Thus, if $f_2 \ll f_1 \ll f$, then $h_0(f_1, f_2) \wedge g \neq 0$. Now let $I' = \{h \mid h \geqq h_0(f_1, f_2) \wedge g_1$ for some $f_2 \ll f_1 \ll f$, for some $h_0 \gg f \vee g$ and for some $g_1 \gg g\}$. Then I' satisfies the conditions 1), 2) and 3) because by ii) and v) if $f_1 \gg g_1$ and $f_2 \gg g_2$, then $f_1 \vee f_2 \gg g_1 \vee g_2$. Let I be an open maximal ideal containing I'. Then $I \in U(g)^a - U(f)$. Since $h_0(f_1, f_2) \in I$ and $h_0(f_1, f_2) \wedge f_2 = 0$, by 3) $I \not\ni f$.

Moreover if $e \in I$, then $e \wedge g \neq 0$. For, if $e \wedge g = 0$ for some $e \in I$, there exist by 3) e_1, e_2, h_0 and h_0' such that $e_2 \in I$ and $e_2 \ll e_1 \ll e \ll h_0 \ll h_0'$ and $h_0 \gg f \vee g$. Then $h_0(e, e_1) \wedge g = (h_0(e, e_1) \vee e) \wedge g = h_0 \wedge g = g$ and $h_0'(e_1, e_2) \gg h_0(e, e_1)$, hence $h_0'(e_1, e_2) \gg g$. Accordingly $h_0'(e_1, e_2) \in I$, but $e_2 \in I$ and $h_0'(e_1, e_2) \wedge e_2 = 0$, which is a contradiction.

Moreover the uniqueness of X is evident by the property: $U(f) \supset U(g)^a \leftrightarrow f \gg g$ (See the proof of Theorem 7). The proof of Theorem 2 is complete.

Definition 3. We denote the space X obtained in Theorem 2 by $X(L)$ and call it the *representative space* of L.

It is remarkable that a lattice L with the smallest element 0 is an R-lattice with the largest element 1 if and only if $X(L)$ is bicompact and that in a Boolean algebra L if $a \gg b$ whenever $a > b$ then $X(L)$ is the representative Boolean space of L. We show in the next section other applications of Theorem 2.

§ 2. Applications of Theorem 2.

Definition 4. Let L be a sublattice with the smallest element of the lattice $C(X)$ of all continuous real functions on a space X. Then for two elements f and g of L we define $f \gg g$ if for any subset $\{h_\alpha\}$ of L with upper bounds in L such that $h_\alpha \subset g$, there exists an upper bound h of $\{h_\alpha\}$ such that $h \subset f$. Moreover we define $[f] \gg [g]$ in $p(L)$ if $f \gg g$ in L.

Lemma 1. *Let $C_+(gX)$ be a lattice of all (bounded) uniformly continuous non-negative real functions on a uniform structure gX over a completely regular space X. Then the correspondence: $[f] \leftrightarrow U[f] = P(f)^{acac}$ is a lattice isomorphic mapping of $p(C_+(gX))$ into the lattice of all regular open sets of X and $[f] \gg [g]$ if and only if $U[f]^c$ and $U[g]$ are completely separated by a function of $C_+(gX)$, where $P(f) = \{x \mid f(x) > 0\}$.*

Proof. We prove only the last statement of the lemma. Let $P(f)^{aca}$ and $P(g)^a$ be completely separated by g_1, i.e., $g_1(x)=1$ for any $x \in P(g)^a$ and $g_1(x) = 0$ for any $x \in P(f)^{ac}$, and let $\{h_\alpha\}$ be a subset of L such that $h_\alpha \subset g$ and $h_\alpha \leqq h_1$ for some h_1. Then $(\sum P(h_\alpha))^a \subset P(g)^a$. Now let $h = g_1 \cdot h_1$, then $P(h)^a \subset P(g_1)^a$ and $P(g_1)^a \subset P(f)^a$, hence $h \subset f$ and $h(x) = g_1(x) \cdot h_1(x) = h_1(x)$ for any $x \in P(g)^a$, i.e., $h_\alpha \leqq h$. Thus we see that $f \gg g$. Conversely let $f \gg g$ and for any $x \in P(g)$ let $h_x \in C_+(gX)$ be a function such that $h_x(x) = 1$ and $h_x(y) = 0$ for any y not belonging to some neighborhood of x which is contained in $P(g)$. Then $h_x \subset g$ and we may assume that $h_x \leqq 1$. By the definition there exists an h

such that $h \geqq h_x$ and $h \subset f$. Hence $h(P(g)^a) \geqq 1$ and $P(f)^a \supset P(h)$, i.e., $P(f)^{ac} \frown P(h) = \phi$, accordingly $h(P(f)^{ac}) \equiv 0$.

Theorem 3. *Let gX be a uniform structure over a completely regular space X and let $C_+(gX)$ be a lattice of all (bounded) uniformly continuous non-negative real functions on gX. Then $p(C_+(gX))$ is an R-lattice and its representativ espace $X(p(C_+(gX)))$ is homeomorphic to the completion $\overline{tgX}$, where tgX is the maximal totally bounded structure*[6] *of totally bounded structures less than the gX.*

Proof. By Lemma 1 $p(C_+(gX))$ is an R-lattice with the largest element. Hence by Theorem 2 there exists uniquely a bicompact space such that it satisfies the properties of Theorem 2. Now we show that $\overline{tgX}$ has these properties. Obviously for any equivalence class $[f]$ of $p(C_+(gX))$, there exists an f_0 such that f_0 is bounded and $f_0 \in [f]$. Accordingly we see easily that as R-lattices $p(C_+(gX)) = p(C_{+b}(gX)) = p(C_+(\overline{tgX}))$ where $C_{+b}(gX)$ is the lattice of all bounded functions in $C_+(gX)$. Further by Lemma 1 $\overline{tgX}$ satisfies the properties of Theorem 2 with respect to $L = p(C_+(\overline{tgX}))$.

Theorem 4. *Let X be a locally bicompact completely regular space and let $C_{+,\kappa}(X)$ be the lattice of all continuous non-negative real functions with a bicompact support on X. Then $X(p(C_{+,\kappa}(X))) = X$.*

Proof. By an analogous way as in the proof of Theorem 3 we see that $p(C_{+,\kappa}(X))$ is an R-lattice and is isomorphic to a sublattice of all regular open sets of X such that it is an open basis and its elements have bicompact closures and such that $f \gg g$ if and only if $P(f)^{acac} \supset P(g)^a$ i.e. $U(f) \supset U(g)^a$. Hence by Theorem 2 we have Theorem 4.

Theorem 5. *Let X be a locally bicompact completely regular space. Then X is determined by the lattice $c_k(X)$ of all continuous real functions with bicompact support on X.*

Proof. For any $f \in C(X)$ let $C_{f,k}(X) = \{g \mid g \in C_k(x) \ \& \ g \geqq f\}$. Then the correspondence between $C_{f,k}(X)$ and $C_{+,k}(X)$: $g \leftrightarrow g - f$ is an isomorphic mapping preserving the relations $\supset$ and $\gg$.

Moreover by Theorem 4 $X = X(p(C_{+,k}(X)))$. Hence $X = X(p(C_{f,k}(X)))$, in other words X can be expressed in terms of lattice structure of $C_k(X)$. Accordingly we have Theorem 5.

Corollary 1. *Let X be a locally bicompact completely regular space.*

6) T. Shirota [10].

Then X is determined by the lattice $C_\infty(X)$ of all continuous functions which are zero at infinity on X.

Proof. Let $C_{+,\infty}(X)$ be the sublattice of all non-negative functions in $C_\infty(X)$. We show that $C_{+,\infty}(X)$ characterizes $C_{+,k}(X)$. Obviously $f \in C_{+,k}(X)$ if and only if there exists a countable subset $\{g_n\}$ of $C_{+,\infty}(X)$ such that if $h < f$, then $g_n \geqq h$ for some g_n. Moreover $C_{+,\infty}(X)$ is isomorphic to $C_{f,\infty}(X)$, where $C_{f,\infty}(X) = \{g \mid g \in C_\infty(X) \ \& \ f \leqq g\}$, for any $f \in C_\infty(X)$. Accordingly $C_\infty(X)$ characterizes $C_{+,\infty}(X)$ and Theorem 5 implies our corollary.

Corollary 2. *Let X be a locally bicompact and fully normal space. Then X is determined by the lattice $C(X)$ of all continuous real functions on X.*

Proof. We have only to show that $C_+(X)$ characterizes $C_{+,k}(X)$. For this we remark that $f \notin C_{+,k}(X)$ if and only if there exists a countable subset $\{f_n\}$ of $C_+(X)$ such that i) $f \geqq f_n > 0$ and such that ii) for any countable subset $\{g_n\}$ satisfying the condition: if $g_n < f_n$ for any n there exists an upper bound of $\{g_n\}$.

Lemma 2. *Let gX and $g'X'$ be complete metric spaces. Then gX and $g'X'$ are unimorphic if and only if $\overline{tgX}$ and $\overline{t'g'X'}$ are homeomorphic.*

Proof. By the analogous method to the one used by Čech[7] we easily see that if A is a closed G_δ-set of $\overline{tgX}-X$, the cardinal number $|A|$ of A is not less than $2^{\aleph_0}$. Accordingly if $\overline{tgX}$ and $\overline{t'g'X'}$ are homeomorphic, X and X' are homeomorphic. Moreover if $\overline{tgX}$ and $\overline{tg'X}$ are homeomorphic where any point of X corresponds to itself, gX and $g'X$ are unimorphic.

By Theorem 3 and by the above lemma we have

Theorem 6. *Let X be a complete metric space. Then X is determined by the lattice of all uniformly continuous real functions on X. Moreover X is determined by the lattice of all bounded uniformly continuous real functions on it.*

Proof. Let $C_u(X)$ and $C_{b,u}(X)$ be the lattice of all uniformly continuous real functions and of all bounded uniformly continuous real functions on X respectively. Then $C_u(X)$ and $C_{b,u}(X)$ determine the sublattice $C_{+,u}(X)$ and $C_{+,b,u}(X)$ respectively, and moreover by Theorem 3 each of them characterizes $\overline{tgX}$. Furthermore by the above lemma $\overline{tgX}$ determines gX.

7) E. Čech, On bicompact space, Ann. of Math. 38 (1938).

Remark. A. N. Milgram[8] showed that a bicompact space X is determined by the multiplicative semi-group $C(X)$. Here we remark that his theorem can also be obtained by our method.

Considering $C(X)$ as a multiplicative semi-group, we define binary relations $\supset$ and $\gg$ and an equivalence relation $\sim$ as follows:

$$f \supset g \text{ if } f\cdot h = 0 \text{ implies } g\cdot h = 0,$$
$$f \gg g \text{ if there exists an } h \text{ such that } g\cdot h = g \text{ and } f \supset h$$

and

$$f \sim g \text{ if } f \supset g \text{ and } g \supset f.$$

Furthermore we define an ordering relation on the system of equivalence classes by letting $[f] \geqq [g]$ if $f \supset g$, and we use $p'(C(X))$ for the ordered system. Then obviously $p(C_+(X))$ and $p'(C(X))$ are lattice-isomorphic and $[f] \gg [g]$ if and only if $U[|f|]^c$ and $U[|g|]$ are completely separated. Accordingly Theorems and Corollaries in § 2 are valid for multiplicative semi-group $C(X)$ and for its sub-groups. Moreover by the analogous methods to the one in § 3 and § 4 Theorem 7 and Theorem 8 are valid for multiplicative semi-group $C(X)$ and topological semi-group $C(X, T)$ respectively.

§ 3. *e*-complete spaces.[9]

E. Hewitt obtained the result that Q-space X is determined by the ring $C(X)$ of all continuous real functions on X. Now we prove the following

Theorem 7. *An e-complete space X is determined by the lattice $C(X)$ of all continuous real functions on X.*

For this the following definition and lemma are used.

Definition 4. Let X be a completely regular space. Then a sequence $\{f_n \mid n = 1, 2, 3, \ldots\}$ of $C_+(X)$ is called *a normal sequence* if it satisfies the following conditions:

i) if $\{e_n\}$ is a sequence such that $e_n \subset f_n$ for any n, there exists an upper bound of $\{e_n\}$ in the lattice $C_+(X)$.

ii) there exists a sequence $\{g_n\}$ such that $f_n \gg g_n$ for any n and if f is not the zero function, then $f \wedge g_n \neq 0$ for some g_n.

Moreover if $\{f_n\}$ is a normal sequence of $C_+(X)$, then we say that $\{[f_n]\}$ in $p(C_+(X))$ is a normal collection.

Lemma 3. *If $\{[f_n]\}$ is a normal collection, $\{U[f_n]\}$ is a normal covering. Conversely for any normal countable covering $\mathfrak{U}$ there exists a*

8) A. N. Milgram [8].

9) E. Hewitt [4] and T. Shirota [11].

normal sequence $\{f_n\}$ *such that* $\{U[f_n]\}$ *is a refinement of* $\mathfrak{U}$. *Hence the set of all normal collections corresponds to a basis for the e-structure eX of X.*[9]

Proof. We first show that $\{U[f_n]\}$ is a covering of X. Obviously by i) $(\sum_n P(g_n))^a = X$. Moreover $\{P(g_n)^a\}$ is locally finite. For if there exists a point $x \in X$ such that for any neighborhood $U(x)$, $U(x)$ intersects an infinite number of $\{P(g_n)^a\}$, then, setting e_n: $e_n(x) = 0$ for $x \in P(f_n)^{aca}$ and $e_n(x) = n$ for $x \in P(g_n)^a$, which is possible since $f_n \gg g_n$, we see that for any $U(x)$ there exists a $y \in U(x)$ such that $\bigvee e_n(y) \geqq n$, i.e., $\bigvee e_n(x) = \infty$, but $e_n \subset f$ which contradicts the assumption ii). Thus we see that $\{P(g_n)^a\}$ is a locally finite closed covering. Moreover $U[f_n] = P(f_n)^{acac} \supset P(g_n)^a$. Accordingly $\{U[f_n]\}$ is a normal covering.

Conversely let $\mathfrak{U}$ be a normal countable covering, $\mathfrak{V}$ a locally finite refinement $\{V_n\}$ of $\mathfrak{U}$, $\mathfrak{V}_1$ a normal covering $\{V_n'\}$ such that $V_n'^a \subset V_n$, $\mathfrak{V}_2$ a star refinement of $\mathfrak{V}_1$ and finally let $F_n = X - S(X - V_n', \mathfrak{V}_2)$. Then $V_n' \supset F_n$, $V_n'^c \frown S(F_n, \mathfrak{V}_2) = \phi$ and $\sum F_n = X$. Let f_n be a non-negative continuous function such that $f_n(x) = 0$ for $x \ni V_n'^c$ and $f(x) = 1$ for $x \in F_n$. Furthermore let g_n be $(f_n - \frac{1}{2}) \bigvee 0$. Then $f_n \gg g_n$. For, let e_n be $f_n - g_n$, then $e_n(x) = \frac{1}{2}$ for $x \in P(g_n)^a$ and $e_n(x) = 0$ for $x \in P(f_n)^{aca}$. Moreover $P(g_n) \supset F_n$, hence $\{g_n\}$ satisfies the condition i) and since $P(f_n) \subset V_n'$, we have $P(f_n)^a \subset V_n$. Accordingly $\{P(f_n)^a\}$ is a locally finite refinement of $\mathfrak{U}$. Thus $\{f_n\}$ satisfies the conditions i) and ii) and $\{U[f_n]\}$ is a refinement of $\mathfrak{U}$.

The proof of Theorem 6. (I) We construct a space $X_e(p(C_+(X))$ as follows:

The point of X_e is the c-open maximal ideal I of $p(C_+(X))$ satisfying following conditions:

1) it is an open maximal ideal of $p(C_+(X))$

and

2) if $\{[f_n]\}$ is a normal collection, there exists an $[f_n]$ such that it is contained in I.

Moreover let $\{U'[f]\} = \{\{I \mid I \ni [f]\} \mid [f] \in p(C_+(X))\}$ be an open basis of X_e.

(II) Let X be an e-complete space, and let I be a c-open maximal ideal of $p(C_+X))$. Then we show that there exists uniquely a point $p \in X$ such that the total intersection $\prod_{[f] \in I} U[f] = \{p\}$. First by the open-maximality we see easily that both $\{U[f] \mid [f] \in I\}$ and $\{P(f)^a \mid [f] \in I\}$ satisfy the finite intersection property and by 2) and by Lemma 3 for any normal countable covering $\mathfrak{V}$ there exists an $[f] \in I$ such that

$U[f] \subset U \in \mathfrak{B}$, moreover there exists a $[g] \in I$ such that $P(g)^a \subset U[f]$. Hence $\{U[f]\}$ and $\{P(f)^a\}$ is a Cauchy family of eX. Since X is e-complete there exists uniquely a point $p \in X$ such that $\Pi\, U[f] = \Pi\, P(f)^a = \{p\}$. Conversely for any $p \in X$ let $I(p) = \{[f] | U[f] \ni p\}$. Then we show that $I(p)$ is a c-open maximal ideal. To see the maximality, if there exists an open ideal I such that $I \supsetneqq I(p)$ then we can find a $[f] \in I$ such that $U[f] \not\ni p$. Let g be such function that $[g] \in I$ and $[g] \ll [f]$. Then $P(f) \supset P(g)^a$, hence $P(g)^a \not\ni p$. Evidently there exists an e such that $e(p) = 1$ and $e(x) = 0$ for $x \in U(p)^c$ where $U(p)$ is a neighborhood of p and $U(p) \frown P(g)^a = \phi$. Accordingly $[e] \in I(p) \subset I$, but $[e] \wedge [g] = [0]$, which is a contradiction. Thus we see that the correspondence $p \leftrightarrow I(p)$ is a one-to-one mapping of X onto X_e.

(III) We show that the above mapping is a homeomorphism of X onto X_e. Obviously by definitions $p \in U[f]$ if and only if $I(p) \ni [f]$, i.e., if and only if $I(p) \in U'[f]$. Accordingly X and X_e are homeomorphic.

(IV) Thus we see that e-complete space in determined by $C_+(X)$, which in turn is characterized by $C(X)$, hence we obtain our Theorem.

§ 4. Completely regular spaces.

Let gX be a uniform structure over a completely regular space X and let $C(gX, T)$ be a topological lattice with topology T of all (bounded) uniformly continuous real functions on X.

Definition 5. We say that a topological lattice $C(X, T)$ is *point-admissible,* if $f(x) \in U$ for a fixed point $x \in X$ and if for some open set U of the space of reals there exists an open set $N(f)$ of f such that $g(x) \in U$ for any $g \in N(f)$.

Definition 6. A subset $\{f_\alpha | A\}$ of $C_+(gX, T)$ is called a *covering set* of $C_+(gX, T)$ if the sublattice generated by the set $\{g | g \subset f$ for some $f \in \{f_\alpha | A\}\}$ is dense in $C_+(gX, T)$. A subcollection $\{[f_\alpha] | A\}$ of $p(C_+(gX, T))$ is called a *covering collection* if $\{f_\alpha | A\}$ is a covering set.

Lemma 4. *Let gX be a uniform structure and let $\{[f_\alpha] | A\}$ be a covering collection, where the topology T of $C(gX, T)$ is weaker than the compact open topology KOT of $C(gX)$ and it is point-admissible. Then $\{U[f_\alpha] | A\}$ is an open covering of X. Conversely for any open covering $\mathfrak{U}$ there exists a covering collection $\{U[f_\alpha] | A\}$ such that it is a refinement of $\mathfrak{U}$.*

Proof. Let $\{[f_\alpha] | A\}$ be a covering collection and assume that there exists a point $x \in X$ such that $x \notin \sum_{\alpha \in A} U[f_\mu]$. Then if f is a function in $C_+(gX, T)$ such that $f(x) > 0$, there exists an open set $N(f)$ such that $g(x) > 0$ for any $g \in N(f)$. But for any $[f_\alpha] \in \{[f_\alpha] | A\}$, $f_\alpha(x) = 0$, hence $\bigvee_{i=1}^{n} g_{x_i}(x) = 0$ for any finite number of α_i and for $g_{x_i} < f_{\alpha_i}$, accordingly, if h is generated by $\{g | g < f_\alpha$ for some $f_\alpha\}$, then $h(x) = 0$, hence $h \notin N(f)$, which is a contradiction.

Conversely let $\mathfrak{U}$ be an open covering of X. Then for any $x \in X$, let f_x be a continuous non-negative function such that $f_x(x) = 1$ and $f_x(y) = 0$ for any $y \in U(x)^c$ where $U(x)$ is a neighborhood of x and $U(x)^a < U$ for some $U \in \mathfrak{U}$. Then obviously the set $\{f_x | X\}$ is a covering set in the compact open topology and hence in $C(gX, T)$. Furthermore $U[f_x] < U(x)^a < U \in \mathfrak{U}$, hence $\{U[f_x] | X\}$ is a refinement of $\mathfrak{U}$.

Theorem 8. *Let X be a completely regular space and let $C(X, T)$ be point-admissible and weaker than $C(X, KOT)$. Then $C(X, T)$ determines X.*

Proof. First we define the t-open maximal ideal I of $p(C_+(X, T))$ as follows:

1) it is an open maximal ideal of $p(C_+(X, T))$

and

2′) if $\{[f_\alpha] | A\}$ is a covering collection, there exists an $[f_\alpha]$ such that $[f_\alpha] \in I$.

Then we see that for any t-open maximal ideal I there exists uniquely a point p of X such that $p \in U[f]$ ($[f] \in I$). For, obviously $\Pi\, U[f_\alpha] = \Pi\, P(f_\alpha)^a$ and $P(f_\alpha)^a$ satisfies the finite intersection property. Then, if $\Pi\, P(f_\alpha)^a = \phi$, $\{P(f)^{ac} | [f] \in I\}$ is an open covering, and hence by Lemma 4 there exists a covering collection $\{[f_\alpha] | A\}$ such that $\{U[f_\alpha]\}$ is a refinement of $\{P(f)^{ac}\}$, but by 2′) there exists a $[f_\alpha]$ such that $[f_\alpha] \in I$. Then $U[f_\alpha] < P(f)^{ac}$ for some $[f] \in I$, hence $\phi = U[f_\alpha] \cap P(f)^a = U[f_\alpha] \cap U[f]$, which is a contradiction. Furthermore the intersection $\Pi\, U[f_\alpha]$ is obviously a point of X.

The remainder of the proof is very nearly analogous to that of Theorem 6, therefore it will be here omitted.

Corollary. *Let gX be a totally bounded uniform structure or a metric space and let $C(gX, T)$ be a point-admissible topological lattice of all (bounded) uniformly continuous real functions on X whose topology is weaker than the compact-open topology of $C(gX)$. Then $C(gX, T)$ determines X.*

Proof. By the same method as in Theorem 7, $C(gX, T)$ determines the space X and by Theorem 3 and Lemma 2, it determines the completion $\overline{gX}$, hence also gX.

Remark. Let L be one of the lattices $C_{+,k}(X)$, $C_{+,u}(X)$, $C_+(X)$ and $C_+(X, t)$ in Theorem 5, 6 7 and 8 respectively. Then we call a sublattice L' with the smallest element 0 to be characteristic if it satisfies the following condition:

(1) When a subset A of X is completely separated from a subset B of X by a function f in L, i.e., if $f(x)=1$ for $x \in A$ and $f(x)=0$ for $x \in B$, then for any $g \in L$ there exists a function h in L' such that

$$h(x) \geqq g(x) \text{ for } x \in A \text{ and } h(x)=0 \text{ for } x \in B.$$

Evidently by the same way as in the proof of Theorem 5, 6, 7 and 8 we see that in the above theorems we can replace $C_k(X)$, $C_u(X)$, $C(X)$ and $C(X, T)$ by characteristic sublattices of $C_{+,k}(X)$, $C_{+,u}(X)$, $C_+(X)$ and $C_+(X, T)$ respectively.

Moreover in $C_{+,k}(X)$, $C_{+,b,u}(X)$ and $C_{b,+}(X, T)$ the condition with respect to the characteristic sublattice is simplfied as follows:

(2) When a subset A of X is completely separated from a subset B of X, then for any positive number r there exists a function f in L' such that $f(x) \geqq r$ for $x \in A$ and $f(x)=0$ for $x \in B$.

Furthermore in $C_{+,k}(X)$ we can reduce the condition (2) to the following:

(3) For any neighborhood $U(x)$ of any point x and for any positive number r there exists a function f in L' such that

$$f(x) \geqq r \text{ and } f(y)=0 \text{ for } y \in U(x)^c.$$

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