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On a Non-Parametric Test

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1. Introduction. Let X be a random variable having the distribution function ($d.f.$) $F(x)$. We want to test the hypothesis H_0 that $F(x)$ is identical with a specified continuous $d.f. F_0(x)$. F. N. David [1] has recently proposed the following test (though this is slightly modified in comparison with the original one):

Let $x_1, x_2, \dots, x_N$ be N independent observations of X. As $F_0(x)$ is continuous, there are real numbers $\{a_i\}$, $i=1, \dots, n-1$, such that $F_0(a_i)-F_0(a_{i-1})=1/n$, $i=1, \dots, n$, where $a_0=-\infty$, $a_n=+\infty$. Let C be the set of intervals on the real line on each of which $F_0(x)$ is constant and C' be its complementary set. The intersection of $(a_{i-1}, a_i]$ with C' will be called "part". Let v be the number of parts which contain no x's and w be the number of x's which fall in C. If either w is positive or v is too large we reject H_0.

David conjectured that under the null hypothesis H_0 v is asymptotically normally distributed when $n, N\to\infty$, $N/n\to$ const. This can be proved by the method of B. Sherman [2]. Furthermore this test is consistent and unbiased against a rather general class of alternative hypotheses. As Lehmann [3] says, very little work has been done on the existence of unbiased tests for non-parametric problems. It is remarkable that David's test has this property.

2. Distribution of v under H_0. Put $u=n-v$, i.e., u is the number of parts which contain at least one x. First we shall determine the distribution of u under H_0.

Denote by P_k the probability that N x's "fill" k given parts (i.e., every x_i falls in some of them and each of them contains at least one x). The probability that N x's fall into k given parts is

$$\left(\frac{k}{n}\right)^N=\sum_{i=1}^{k}\binom{k}{i}P_i .$$

Therefore, for every positive integer ν,

$$\begin{aligned}\sum_{k=1}^{\nu}(-1)^{\nu-k}\binom{\nu}{k}\left(\frac{k}{n}\right)^N &= \sum_{k=1}^{\nu}(-1)^{\nu-k}\binom{\nu}{k}\sum_{i=1}^{k}\binom{k}{i}P_i \\ &= \sum_{i=1}^{\nu}\binom{\nu}{i}P_i\sum_{k=i}^{\nu}(-1)^{\nu-k}\binom{\nu-i}{k-i} \\ &= P_\nu ,\end{aligned}$$

because

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}=1 \quad \text{if} \quad n=0 \ ,$$
$$=0 \quad \text{if} \quad n>0 \ .$$

Thus

$$P(u=\nu)=\binom{n}{\nu}P_\nu$$
$$=n^{-N}\binom{n}{\nu}\sum_{k=1}^{\nu}(-1)^{\nu-k}\binom{\nu}{k}k^N \ , \quad \nu=1,\dots,n \ .$$

Replacing ν by $n-\nu$,

$$P(v=\nu)=n^{-N}\binom{n}{\nu}\sum_{k=1}^{n-\nu}(-1)^{n-\nu-k}\binom{n-\nu}{k}k^N \ , \quad \nu=0,1,\dots,n-1 \ .$$

3. Moments of v under H_0. Define $x^{(s)}$ for every non-negative integer s as

$$x^{(s)}=x(x-1)(x-2)\dots(x-s+1) \quad \text{if} \quad s>0 \ ,$$
$$x^{(0)}=1 \ .$$

Then the s th factorial moment is

$$E(v^{(s)})=\sum_{\nu=0}^{n-1}\nu^{(s)}P(v=\nu)$$
$$=\sum_{\nu=s}^{n-1}\frac{\nu\,!}{(\nu-s)\,!}n^{-N}\binom{n}{\nu}\sum_{k=1}^{n-\nu}(-1)^{n-\nu-k}\binom{n-\nu}{k}k^N$$
$$=\frac{n\,!}{n^N}\sum_{k=1}^{n-s}\frac{k^N}{k\,!(n-s-k)\,!}\sum_{\nu=s}^{n-k}(-1)^{n-\nu-k}\binom{n-k-s}{\nu-s}$$
$$=\frac{n\,!\,(n-s)^N}{n^N(n-s)\,!} \ .$$

Putting $s=1, 2,$

$$E(v)=n(n-1)^N n^{-N} \ ,$$
$$Ev(v-1)=n(n-1)(n-2)^N n^{-N} \ ,$$

whence, if $N=nr$ (r is a constant),

$$E(v/n)=e^{-r}(1-r/2n)+O(n^{-2}) \ ,$$
$$D^2(v/n)=e^{-2r}(e^r-1-r)n^{-1}+O(n^{-2}) \ .$$

4. Asymptotic normality of v under H_0.

Theorem 1. *v/n is asymptotically normally distributed with mean e^{-r} and variance $e^{-2r}(e^r-1-r)n^{-1}$, where $r=N/n=$* const.

As the proof is almost parallel to that of Theorem 2 of B. Sherman [2], we shall only sketch it.

It is sufficient to prove that moments of $(n/c)^{\frac{1}{2}}(v/n-e^{-r})$ tend to the moments of the standard normal distribution, where $c=e^{-2r}(e^r-1-r)$. It can be shown that if the limiting moments of even order exist the limiting moments of odd order are zero. Thus we may restrict ourselves to even order moments.

Denoting by $B_r^{(n)}$ the Bernoulli's number of order n and degree r,

$$v^k=\sum_{q=0}^{k}\binom{k}{q}B_q^{(q-k)}v^{(k-q)} .$$

Then

$$E\left[\left(\frac{n}{c}\right)^{\frac{1}{2}}\left(\frac{v}{n}-e^{-r}\right)\right]^{2m}=\left(\frac{n}{c}\right)^{m}\sum_{k=0}^{2m}\binom{2m}{k}\left(-e^{-r}\right)^{2m-k}E\left(\frac{v}{n}\right)^{k}$$

$$=\left(\frac{n}{c}\right)^{m}\sum_{k=0}^{2m}\binom{2m}{k}\left(-e^{-r}\right)^{2m-k}n^{-k}\sum_{q=0}^{k}\binom{k}{q}B_q^{(q-k)}E(v^{(k-q)})$$

$$=\frac{n^m(2m)!}{(e^r-1-r)^m}\sum_{q=0}^{2m}\frac{1}{q!}\sum_{k=q}^{2m}\frac{(-1)^k e^{kr}}{(2m-k)!(k-q)!}B_q^{(q-k)}\frac{n!\,(n-k+q)^N}{n^{N+k}(n-k+q)!}$$

$$\equiv\frac{n^m(2m)!}{(e^r-1-r)^m}\left[a_0+\frac{a_1}{n}+\frac{a_2}{n^2}+\cdots+\frac{a_m}{n^m}+\cdots\right] .$$

We have to show that $a_i=0$, $i=0,1,2,\dots,m-1$. Then $\lim_{n\to\infty}\left[\left(\frac{n}{c}\right)^{\frac{1}{2}}\left(\frac{v}{n}-e^{-r}\right)\right]^{2m}=a_m(2m)!\,(e^r-1-r)^{-m}$. If we denote by a_{iq} the coefficient of n^{-i} in the expansion in powers of n^{-1} of

$$\sum_{k=q}^{2m}\frac{(-1)^k e^{kr}}{(2m-k)!(k-q)!}B_q^{(q-k)}\frac{n!\,(n-k+q)^N}{n^{N+k}(n-k+q)!} ,$$

we have

$$a_i=\sum_{q=0}^{i}a_{iq}/q! , \qquad i=0,1,\dots,m .$$

Now

$$\frac{n!(n-k+q)^N}{n^{N+k}(n-k+q)!}=\frac{1}{n^k}\left(1-\frac{k-q}{n}\right)^N n(n-1)\cdots(n-k+q+1)$$

$$=x^q[1-(k-q)x]^{r/x}(1-x)(1-2x)\cdots(1-(k-q-1)x)$$

$$\equiv x^qF(x) ,$$

where $x=1/n$ and

$$F(x)=[1-(k-q)x]^{r/x}\prod_{j=1}^{k-q-1}(1-jx) .$$

Expanding this, we have

$$F(x)=a_{kq0}+a_{kq1}x+a_{kq2}x^2+\cdots ,$$

where

$$a_{kq0}=e^{-(k-q)r} ,$$

$$a_{kqp}=\frac{1}{p!}\left.\frac{d^pF(x)}{dx^p}\right|_{x=0} , \qquad p=1,\dots,i-q .$$

a_{kqp} can be written as

$$a_{kqp} = \frac{1}{p} \sum_{s=0}^{p-1} a_{kq(p-s-1)} b_{kqs} ,$$

where b_{kqp} is a polynomial in k of degree $s+2$ and particularly $b_{kq0} = -(1+r)k^2/2 + Ak + B$, A, B depending on q only. Hence

$$a_{kq(i-q)} = e^{-(k-q)r} B_{kqi} ,$$

where

$$\begin{aligned} B_{kqi} &= [(i-q)!]^{-1}(b_{kq0})^{i-q} + \text{terms of lower degree in } k \\ &= [(i-q)!]^{-1}(-(1+r)/2)^{i-q} k^{2(i-q)} + \sum_{j=0}^{2(i-q)-1} A_j k^j . \end{aligned}$$

Now

$$\begin{aligned} a_{iq} &= \sum_{k=q}^{2m} \frac{(-1)^k e^{kr}}{(2m-k)!(k-q)!} B_q^{(q-k)} a_{kq(i-q)} \\ &= e^{qr} \sum_{k=q}^{2m} \frac{(-1)^k}{(2m-k)!(k-q)!} B_q^{(q-k)} B_{kqi} . \end{aligned}$$

As $B_q^{(q-k)}$ is the polynomial in k of degree q with 2^{-q} as the coefficient of the term of the highest degree and as

$$\sum_{k=q}^{2m} \frac{(-1)^k}{(2m-k)!(k-q)!} k^l = \begin{cases} 0 & \text{if} \quad l < 2m-q , \\ 1 & \text{if} \quad l = 2m-q , \end{cases}$$

we obtain

$$\begin{aligned} a_{iq} &= 0 \qquad \text{if} \quad i < m , \\ a_{mq} &= e^{qr} 2^{-q} [(m-q)!]^{-1} (-(1+r)/2)^{m-q} , \end{aligned}$$

and

$$\begin{aligned} a_i &= 0 \qquad \text{if} \quad i < m , \\ a_m &= \sum_{q=0}^{m} a_{mq}/q! = 2^{-m}(e^r - 1 - r)^m / m! . \end{aligned}$$

Thus

$$\begin{aligned} \lim_{n\to\infty} \left[\left(\frac{n}{c}\right)^{\frac{1}{2}} \left(\frac{v}{n} - e^{-r}\right) \right]^{2m} &= a_m (2m)! (e^r - 1 - r)^{-m} \\ &= 2^{-m} (2m)!/m! , \end{aligned}$$

which is the $2m$ th moment of the standard normal distribution.

5. Computation of the power. The power of the test with respect to the alternative hypothesis H_1 is, for a given integer $l (\leq n)$,

$$P = P(w > 0 \mid H_1) + P(w = 0, u \leq l \mid H_1) .$$

Denoting by $p_0, p_1, \ldots, p_n$ the probability of C and n parts under H_1, it is readily seen that

$$P(w>0\,|\,H_1)=1-(1-p_0)^N\ .$$

Denote by $P_{i_1\ldots i_k}$ the probability that $w=0$ and N x's fill (see the section 3) i_1 th, ..., i_k th parts. The probability that $w=0$ and Nx's fall in the union of the i_1 th, ..., i_j th parts is

$$(p_{i_1}+\cdots+p_{i_j})^N=P_{i_1\ldots i_j}+\sum_{(t_1,\ldots,t_{j-1})}^{j}P_{i(t_1)\ldots i(t_{j-1})}+\cdots+\sum_{(t_1)}^{j}P_{i(t_1)}\ ,$$

where $\sum\limits_{(t_1\ldots t_l)}^{j}$ denotes the summation over all combinations $(t_1,\ldots,t_l)$ drawn from $(1,2,\ldots,j)$. In another from

$$(p_{i_1}+\cdots+p_{i_j})^N=\sum_{l=1}^{j}\sum_{(t_1\ldots t_l)}^{j}P_{i(t_1)\ldots i(t_l)}\ .$$

Then for every positive integer k,

$$\begin{aligned}
&\sum_{j=1}^{k}(-1)^{k-j}\binom{n-j}{k-j}\sum_{(i_1\ldots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N\\
&=\sum_{j=1}^{k}(-1)^{k-j}\binom{n-j}{k-j}\sum_{(i_1\ldots i_j)}^{n}\sum_{l=1}^{j}\sum_{(t_1\ldots t_l)}^{j}P_{i(t_1)\ldots i(t_l)}\\
&=\sum_{\nu=1}^{k}\sum_{(m_1\ldots m_\nu)}^{n}P_{m_1\ldots m_\nu}\sum_{j=\nu}^{k}(-1)^{k-j}\binom{n-j}{k-j}\binom{n-\nu}{j-\nu}\\
&=\sum\sum P_{m_1\ldots m_\nu}(-1)^{k-\nu}\binom{n-\nu}{n-k}\sum_{j=\nu}^{k}(-1)^{j-\nu}\binom{k-\nu}{j-\nu}\\
&=\sum_{(m_1\ldots m_k)}^{n}P_{m_1\ldots m_k}=P(w=0\,,\ u=k\,|\,H_1)\ .
\end{aligned}$$

Thus

$$P(w=0\,,\ u=k\,|\,H_1)=\sum_{j=1}^{k}(-1)^{k-j}\binom{n-j}{k-j}\sum_{(i_1\ldots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N\ ,\qquad k=1,\ldots,n\ .$$

Therefore

$$\begin{aligned}
P(w=0\,,\ u\le l\,|\,H_1)&=\sum_{k=1}^{l}P(w=0\,,\ u=k\,|\,H_1)\\
&=\sum_{k=1}^{l}\sum_{j=1}^{k}(-1)^{k-j}\binom{n-j}{k-j}\sum_{(i_1\ldots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N\\
&=\sum_{j=1}^{l}\sum_{k=j}^{l}(-1)^{k-j}\binom{n-j}{k-j}\sum^{n}(p_{i_1}+\cdots+p_{i_j})^N\\
&=\sum_{j=1}^{l}(-1)^{l-j}\binom{n-j-1}{l-j}\sum(p_{i_1}+\cdots+p_{i_j})^N\ .
\end{aligned}$$

Thus

$$P=1-(1-p_0)^N+\sum_{j=1}^{l}(-1)^{l-j}\binom{n-j-1}{l-j}\sum_{(i_1\ldots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N\ .$$

6. Moments of v under H_1 with $p_0=0$. If $p_0=0$, so $P(w=0)=1$. Then, from the result of the preceding section

$$P(u=k\,|\,H_1)=P(w=0,\,u=k\,|\,H_1)$$
$$=\sum_{j=1}^{k}(-1)^{k-j}\binom{n-j}{k-j}\sum_{(i_1\dots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N\ .$$

Hence

$$E(v^{(s)}\,|\,H_1)=\sum_{k=s}^{n-1}k^{(s)}P(v=k\,|\,H_1)=\sum_{k=s}^{n-1}k^{(s)}P(u=n-k\,|\,H_1)$$
$$=\sum_{k=s}^{n-1}\frac{k\,!}{(k-s)\,!}\sum_{j=1}^{n-k}(-1)^{n-k-j}\binom{n-j}{k}\sum_{(i_1\dots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N$$
$$=\sum_{j=1}^{n-s}\frac{(n-s)\,!}{(n-s-j)\,!}\sum_{(i_1\dots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N\sum_{k=s}^{n-j}(-)^{n-k-j}\binom{n-s-j}{k-s}$$
$$=s\,!\sum_{(i_1\dots i_{n-s})}^{n}(p_{i_1}+\cdots+p_{i_{n-s}})^N$$
$$=s\,!\sum_{(i_1\dots i_s)}^{n}(1-p_{i_1}-\cdots-p_{i_s})^N\ .$$

Putting $s=1,2$, we get

$$E(v\,|\,H_1)=\sum_{i=1}^{n}(1-p_i)^N\ ,$$
$$E(v(v-1)\,|\,H_1)=\sum_{i\neq j}^{n}(1-p_i-p_j)^N\ .$$

7. Consistency.

Theorem 2. *The test based on v is consistent against the class of alternative hypothesis H_1 as far as we are concerned with absolute continuous d.f.'s whose density functions are differentiable.*

Proof. If $p_0>0$, $(1-p_0)^N$ tends to zero and so P tends to 1 when $N\to\infty$, that is, the test is consistent against this H_1.

If $p_0=0$, $F_1(x)$ is absolutely continuous with respect to $F_0(x)$ and its relative density is differentiable. Putting $Y=F_0(X)$, Y is distributed uniformly over $[0,1]$ under H_0 and according to *d. f.* $G(y)=F_1(F_0^{-1}(y))$ under H_1. As $p_0=0$, $G(y)$ is defined uniquely with differentiable derivative $g(y)$ such that

$$p^i=\int_{(i-1)/n}^{i/n}g(y)dy\ ,\quad i=1,\dots,n\ .$$

By Taylor expansion

$$p_i=\frac{1}{n}g\Big(\frac{i}{n}\Big)-\frac{1}{2n^2}g'\Big(\frac{i}{n}\Big)+O(n^{-3})\ ,$$
$$(1-p_i)^N=e^{-rg(i/n)}\Big[1+\frac{r}{2n}\Big\{g'\Big(\frac{i}{n}\Big)-g^2\Big(\frac{i}{n}\Big)\Big\}+O(n^{-2})\Big]\ ,$$

whence

$$E(v/n)=\sum_{i=1}^{n}(1-p_i)^N/n$$
$$=\int_0^1 e^{-rg(y)}dy-\frac{r}{2n}\int_0^1 g^2(y)e^{-rg(y)}dy+O(n^{-2}) \ .$$

In the same manner

$$D^2(v/n)=\frac{1}{n}\left[\int_0^1\left(e^{-rg}-e^{-2rg}\right)dy-r\left(\int_0^1 ge^{-rg}dy\right)^2\right]+O(n^{-2}) \ .$$

As

$$\int_0^1 e^{-rg(y)}dy\geq e^{-r}$$

with the equality if and only if $g(y)\equiv 1$, the test is, as in Sherman's case, consistent against H_1.

8. Unbiasedness.

Theorem 3. *The test based on v is unbiased against the class of all alternative hypotheses.*

Proof. We need only prove that for any integer $l(\leq n)$ the following relation holds:

$$P\geq P(u\leq l|H_0) \ ,$$

where P is the power. This is trivial for $l=0$ or $l=n$.

As P attains its minimum with respect to p_0 at $p_0=0$, we have only to prove that, for $l=1,\dots,n-1$,

$$P_0\equiv\sum_{j=1}^{l}(-1)^{l-j}\binom{n-j-1}{l-j}\sum_{(i_1\dots i_j)}^{n}(p_{i_1}+\cdots+p_{i_j})^N$$

attains its minimum with respect to $(p_1,\dots,p_n)$ at $p_1=\cdots=p_n=1/n$.

As the case $l=1$ is simple, we consider the cases $2\leq l\leq n-1$.

If all p_i are not equal, there are $i, j\,(=1,\dots,n)$ such that $p_i<p_j$. We can assume without any loss of generality that $p_{n-1}<p_n$. Put $p_n-p_{n-1}=2\varepsilon$ and

$$p_i'=p_i\ ,\quad i=1,\dots,n-2\ ;$$
$$p_{n-1}'=p_{n-1}+x\ ,\quad p_n'=p_n-x\ ;$$
$$P'=\sum_{j=1}^{l}A_j\sum_{(i_1\dots i_j)}^{n}(p_{i_1}'+\cdots+p_{i_j}')^N\ ,$$

where

$$A_j=(-1)^{l-j}\binom{n-j-1}{l-j}\ .$$

P' is a function of x and we have only to show $dP'/dx\leq 0$ for $0\leq x<\varepsilon$ in order to complete the proof of the theorem. Now

$$P' = A_1 \sum_{i=1}^{n} p_i'^N + \sum_{j=2}^{l} A_j \sum_{(i_1 \ldots i_j)}^{n} (p'_{i_1} + \cdots + p'_{i_j})^N$$
$$= A_1 \left\{ \sum_{i=1}^{n-2} p_i^N + (p_{n-1}+x)^N + (p_n - x)^N \right\}$$
$$+ \sum_{j=2}^{l} A_j \left\{ \sum_{(i_1 \ldots i_j)}^{n-2} (p_{i_1} + \cdots + p_{i_j})^N + \sum_{(i_1 \ldots i_{j-1})}^{n-2} (p_{i_1} + \cdots + p_{i_{j-1}} + p_{n-1} + x)^N \right.$$
$$\left. + \sum_{(i_1 \ldots i_{j-1})}^{n-2} (p_{i_1} + \cdots + P_{i_{j-1}} + p_n - x)^N + \sum_{(i_1 \ldots i_{j-2})}^{n-2} (p_{i_1} + \cdots + p_{i_{j-2}} + p_{n-1} + p_n)^N \right\}.$$

Therefore

$$\frac{1}{N} \frac{dP'}{dx} = A_1 \left\{ (p_{n-1}+x)^{N-1} - (p_n - x)^{N-1} \right\}$$
$$+ \sum_{j=2}^{l} A_j \sum_{(i_1 \ldots i_{j-1})}^{n-2} \left\{ (p_{i_1} + \cdots + p_{i_{j-1}} + p_{n-1} + x)^{N-1} - (p_{i_1} + \cdots + p_{i_{j-1}} + p_n - x)^{N-1} \right\}.$$

In order to prove $dP'/dx \leq 0$ for $0 \leq x < \varepsilon$, we have only to show that

$$A_1 p^{N-1} + \sum_{j=2}^{l} A_j \sum_{(i_1 \ldots i_{j-1})}^{n-2} (p + p_{i_1} + \cdots + p_{i_{j-1}})^{N-1}, \quad p \geq 0,$$

is a monotone increasing function of p. This is now a polynomial of p and the coefficents B_s of p^{N-1-s} are

$$B_0 = A_1 + \sum_{j=2}^{l} A_j \binom{n-2}{j-1},$$
$$B_s = \binom{N-1}{s} \sum_{j=2}^{l} A_j \sum_{(i_1 \ldots i_{j-1})}^{n-2} (p_{i_1} + \cdots + p_{i_{j-1}})^s, \quad s = 1, \cdots, N-1.$$

It suffices to show that $B_s \geq 0$, $s = 0, 1, \cdots, N-1$.

Now

$$B_0 = \sum_{j=1}^{l} A_j \binom{n-2}{j-1} = \sum_{j=1}^{l} (-1)^{l-j} \binom{n-j-1}{l-j} \binom{n-2}{j-1}$$
$$= \binom{n-2}{l-1} \sum_{j=1}^{l} (-1)^{l-j} \binom{l-1}{l-j} = 0,$$

as $l > 1$. On the other hand

$$B_s = \binom{N-1}{s} \sum_{j=1}^{l-1} A_{j+1} \sum_{(i_1 \ldots i_j)}^{n-2} (p_{i_1} + \cdots + p_{i_j})^2$$
$$= \binom{N-1}{s} \sum_{j=1}^{l-1} (-1)^{l-1-j} \binom{n-2-j}{l-1-j} \sum_{(i_1 \ldots i_j)}^{n-2} (p_{i_1} + \cdots + p_{i_i})^s.$$

By the result of section 5

$$P(w=0, u=k \mid H_1) = \sum_{j=1}^{k} (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \ldots i_j)}^{n} (p_{i_1} + \cdots + p_{i_j})^N.$$

Comparing these two equations, we obtain

$$B_s \geq 0,$$

which completes the proof.

9. Remark. There is another type of the non-parametric problem. Given two random samples independently from two populations, it is asked whether they have the same d.f.. The run test of Wald and Wolfowitz [4] occupies the same position in this problem as David's test in our problem. Wolfowitz [5] recently proved that their test statistic is asymptotically normally distributed even under the alternative hypothesis. Therefore we may perhaps expect so in our case, though we have not yet succeeded in proving it.

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References

[1] F. N. David: Two combinatorial tests of whether a sample has come from a given population, Biometrika, 37 (1950), 97-110.
The author has not yet read it and he was only informed with it by W. Hoeffding's review, Mathematical Review, 12 (1951), p. 38.

[2] B. Sherman: A random variable related to the spacing of sample values, Ann. of Math. Stat., 21 (1950), 339-361.

[3] E. L. Lehmann: Consistency and unbiasedness of certain nonparametric tests, Ann. of Math. Stat., 22 (1951), 165-179.

[4] A. Wald and J. Wolfowitz: On a test whether two samples are from the same population, Ann. of Math. Stat., 11 (1940), 147-162.

[5] J. Wolfowitz: Non-parametric statistical inference Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, Univ. of California Press, Berkeley, 1949.