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Extensions and Classification of Maps [1)]

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I. INTRODUCTION

Given two topological spaces X and R, let us consider an arbitrarily given map [2)] $f: X_0 \to R$ defined on a given subset X_0 of X. The problem of determining whether $f: X_0 \to R$ can be extended continuously throughout X, in other words, whether there exists a map $f^*: X \to R$ such that $f^*|X_0=f$, is known as the *extension problem of maps.* The map f^* is called an *extension* of f. Next, let us consider the totality of the maps of X into R. These maps are divided into disjoint *homotopy classes,* those in each class being homotopic to each other [40] [3)]. The problem of enumerating these classes by means of some convenient invariants is known as the *classification problem of maps.*

The two problems of maps described above are of extremely great importance in modern topology. Neither of them has been solved in general form while a great number of particular results are already known. A majority of the literatures of these results are provided in the bibliography given at the end of the paper.

The object of the present work is to give a general investigation to both problems by considering the singular polytope [4)] of the given space X. The singular polytope $P(X)$ and the related notions are studied in Chapter II. The general theory of continuous extension is given in Chapter III and that of homotopy classification in Chapter IV. One might easily see that the results concerning homotopy and classification are in a form more satisfactory than those of the extension problem. By suitable specializations, our theorems will contain a major portion of the known results. Throughout the paper, we frequently assume the n-simplicity of the space R in the sense of Eilenberg [17].

II. THE SINGULAR POLYTOPE OF A SPACE

1. The singular complex $S(X)$

For the convenience of our investigation given in the sequel, we shall briefly recall Eilenberg's definition, [23, p. 420], of the singular complex $S(X)$ of a topological space X with some modified terminology.

Let

$$s=\langle v_0, \dots, v_m \rangle$$

be an ordered geometric m-simplex, i. e. with ordered vertices. Denote by $s^{(i)}$ the face of s opposite the i-th vertex v_i, i. e. [5]

$$s^{(i)}=\langle v_0, \dots, \hat{v}_i, \dots, v_m \rangle.$$

For any two given ordered geometric m-simplices s_1 and s_2, there is a unique barycentric map

$$B_{s_1 s_2}: \quad s_1 \to s_2$$

which preserves the order of the vertices.

Let X be a topological space. By a *continuous* m-simplex in X, we understand a (continuous) map

$$T: \quad s \to X$$

of an ordered geometric m-simplex s with values in X.

Two continuous m-simplices

$$T_1: \quad s_1 \to X, \qquad T_2: \quad s_2 \to X$$

are said to be *equivalent* (notation: $T_1 \equiv T_2$), if $T_2 B_{s_1 s_2} = T_1$. The continuous m-simplices in X are thus divided into disjoint equivalence called the *singular* m-simplices in X. We shall denote by $[T]$ the singular simplex which contain the continuous simplex $T: s \to X$ and call T a *representative* of $[T]$.

We remark that, for a given singular m-simplex ξ and a given ordered geometric m-simplex s, there is a *unique* continuous m-simplex $T: s \to X$ such that $\xi=[T]$.

Let $C_m(X)$ be the free abelian group generated by the singular m-simplices in X. The elements of $C_m(X)$ are called the *integral singular m-chains* in X.

Given a continuous m-simplex

$$T: \quad s \to X, \qquad s=\langle v_0, \dots, v_m \rangle,$$

consider the continuous $(m-1)$-simplices

$$T^{(i)}: \quad s^{(i)} \to X, \qquad (i=0, \dots, m),$$

defined by the partial maps $T^{(i)}=T\,|\,s^{(i)}$. We define the boundary of the singular m-simplex $[T]$ to be

$$\partial[T]=\sum_{i=0}^{m}(-1)^i[T^{(i)}],$$

which is clearly independent of the choice of the representative $T: s\to X$ for the singular m-simplex $[T]$. Therefore we get a homomorphism

$$\partial: \quad C_m(X)\to C_{m-1}(X)$$

and we easily verify that $\partial\partial=0$. This boundary operation ∂ can be used to define incidence numbers and leads to a closure finite abstract complex $S(X)$, called the *singular complex* of the space X.

2. The singular polytope [4] $P(X)$

For every integer $m\geqq 0$ and every singular m-simplex $\xi\in S(X)$, let us associate with an open geometric m-cell σ_ξ, called the *open singular m-cell* corresponding to the singular m-simplex ξ, which is the interior of some ordered geometric m-simplex s_ξ, i.e.

$$\sigma_\xi=\mathrm{Int}\ s_\xi, \qquad s_\xi=<v_0,\dots,v_m>.$$

We assume that no two of these open singular cells have a point in common. Let each open singular cell σ_ξ have the euclidean topology and the affine relation of the geometric simplex s_ξ.

Now, we are going to define the *closed singular m-cells*. Let $\xi\in S(X)$ be an arbitrary singular m-simplex in X and s be the ordered geometric m-simplex associated with ξ as above. Then there is a unique representative

$$T: \quad s_\xi\to X, \qquad s_\xi=<v_0,\dots,v_m>,$$

of the singular simplex ξ, i.e. $[T]=\xi$. A singular p-simplex $\eta\in S(X)$, $(p\leqq m)$, is termed as a *face* of the singular m-simplex (notation: $\eta<\xi$), if there exists a p-face $<v_{i_0},\dots,v_{i_p}>$ with $i_0<\dots<i_p$ such that the continuous p-simplex

$$T\,|<v_{i_0},\dots,v_{i_p}>$$

represents η. Define the *closed singular m*-cell $Cl\sigma_\xi$ as a set by taking

$$Cl\sigma_\xi=\bigcup\nolimits_{\eta<\xi}\sigma_\eta.$$

There is a natural function $\mu_\xi: s_\xi\to Cl\sigma_\xi$ defined as what follows. For each p-face, $(0\leqq p\leqq m)$, $s'=<v_{i_0},\dots,v_{i_p}>$ of s_ξ, we define μ_ξ on the interior Int s' of s' to be the unique barycentric map of Int s'

onto σ_η which preserves the order of vertices, where $\eta=[T|s']$ and $i_0<\dots<i_p$. The topology of $Cl\sigma_\xi$ is defined by calling a set $M\subset Cl\sigma_\xi$ to be *open* if its inverse image $\mu_\xi^{-1}(M)\subset s_\xi$ is open.

Let us denote by $P(X)$ the union of all open singular cells corresponding to the singular simplices in X. We define a topology of $P(X)$ as follows: *A* set M of $P(X)$ is said to be *open* if $M\cap Cl\sigma_\xi$ is an open set of $Cl\sigma_\xi$ for every closed singular cell $Cl\sigma_\xi$. The topological space $P(X)$ thus obtained will be called the *singular polytope* of X. It is a polyhedral realisation of the singular complex $S(X)$; however, it is neither simplicial nor locally finite.

We remark that, for each singular simplex $\xi\in S(X)$, the natural function

$$\mu_\xi:\quad s_\xi\to Cl\sigma_\xi\subset P(X),$$

described above, is a continuous map of s_ξ onto $Cl\sigma_\xi$ and $\mu_\xi|\sigma_\xi$ is the identity map on σ_ξ. Following J. H. C. Whitehead, [73, p. 221], it will be called the *characteristic map* for the singular cell σ_ξ.

Obviously, $P(X)$ is a CW-complex in the sense of J. H. C. Whitehead, [73, p. 223]. Hence we have the following statements.

(2.1) *The singular polytope $P(X)$ of a topological space X is a non-metrizable normal Hausdorff space.*

(2.2) *A transformation $f: P(X)\to R$ of $P(X)$ into an arbitrary topological space R is continuous, if and only if the partial transformation $f|Cl\sigma_\xi$ is continuous for each closed singular cell $Cl\sigma_\xi$.*

3. The projection $\omega: P(X)\to X$

There is a natural projection ω of $P(X)$ into X described as what follows. For an arbitrary point $p\in P(X)$, let σ_ξ be the (unique) open singular cell which contains p. Since σ_ξ is the interior of the associated ordered geometric simplex s_ξ, p is a point of s_ξ. There is a unique continuous simplex

$$T_\xi:\quad s_\xi\to X$$

which represents ξ, i. e. $\xi=[T_\xi]$. We define the projection $\omega: P(X)\to X$ by taking

$$\omega(p)=T_\xi(p),\qquad (p\in\sigma_\xi\subset P(X)).$$

(3.1) *The projection $\omega: P(X)\to X$ is a continuous map of $P(X)$ onto X.*

Proof. It is easy to see that ω is onto. For an arbitrary point $x\in X$ and an ordered geometric simplex s, consider the singular simplex ξ represented by the trivial continuous simplex $T: s \to X$ defined by $T(s)=x$. Then, we have $\omega(\sigma_\xi)=x$.

According to (2. 2), to prove the continuity of ω is to prove that of the partial map $\omega_0=\omega | Cl\sigma_\xi$ for each closed singular cell $Cl\sigma_\xi$. Let $U \subset X$ be an arbitrary open set, it remains to show that the inverse image $V=\omega_0^{-1}(U)$ is an open set of $Cl\sigma_\xi$.

Consider the ordered geometric simplex s_ξ associated with the singular simplex ξ. The interior of s_ξ is σ_ξ. There is a unique continuous simplex $T_\xi: s_\xi \to X$ such that $\xi=[T_\xi]$. Remembering the definition of the characteristic map $\mu_\xi: s_\xi \to Cl\sigma_\xi$ in § 2, one can easily see that

$$T_\xi=\omega_0\mu_\xi. \tag{3. 2}$$

Therefore, $\mu_\xi^{-1}(V)=T_\xi^{-1}(U)$. Since T_ξ is continuous and U is open, $\mu_\xi^{-1}(V)$ is an open set of s_ξ. According to the topology of $Cl\sigma_\xi$, V is an open set of $Cl\sigma_\xi$. This completes the proof.

4. Pathwise connectedness of X and $P(X)$

We say that two points x_0, x_1 of X can be connected by a path if there exists a continuous map $f: I \to X$ of the closed unit segment I of real numbers into X such that $f(0)=x_0$ and $f(1)=x_1$. Such a map f is called a *path*.

By the *path-component* $\Gamma(x_0)$ of X containing $x_0 \in X$, we understand the set of all points $x \in X$ which can be connected to x_0 by a path. Obviously $x_0 \in \Gamma(x_0)$, and $\Gamma(x_1)=\Gamma(x_0)$ if $x_1 \in \Gamma(x_0)$. Hence the set $\Gamma=\Gamma(x_0)$ does not depend on the choice of the basic point x_0 from Γ and might be called a *path-component* of X.

A topological space X is said to be *pathwise connected* if every pair of points of X can be connected by a path, or in other words, if X has only a single path-component.

(4. 1) *A topological space X is pathwise connected if and only if its singular polytope $P(X)$ is so.*

Proof. *Sufficiency.* Let x_0, x_1 be any pair of points of X. Since ω maps $P(X)$ onto X, there exists y_0, y_1 of $P(X)$ with $\omega(y_0)=x_0$ and $\omega(y_1)=x_1$. Since $P(X)$ is, by hypothesis, pathwise connected, there is a path $g: I \to P(X)$ such that $g(0)=y_0$ and $g(1)=y_1$. Then, the path $f=\omega g$ connects x_0 and x_1; and X is pathwise connected.

Necessity. Assume that X be pathwise connected. In order to prove the pathwise connectedness of $P(X)$, clearly it needs only to show that any pair of vertices p_0, p_1 of $P(X)$ are connected by a closed singular 1-cell $Cl\sigma_\xi$. Call $x_0=\omega(p_0)$ and $x_1=\omega(p_1)$. Since X is pathwise connected, there exists a path $T: I \to X$ such that $T(0)=x_0$ and $T(1)=x_1$. Since $I=\langle 0,1\rangle$ is an ordered geometric 1-simplex, T is a continuous 1-simplex and represents a singular 1-simplex $\xi=[T]$. Clearly $Cl\sigma_\xi$ connects p_0 and p_1. This completes the proof.

5. Subspaces and subpolytopes

Let X_0 be a subspace [6] of X. The singular simplexes of $S(X)$ represented by the continuous simplexes whose images are contained in X_0 form a closed subcomplex $S(X_0)$ ef $S(X)$. Therefore, the corresponding open singular cells of $P(X)$ form a subpolytope $P(X_0)$ of $P(X)$. Here we do not assume the closedness of X_0 as a subset of X; however, the subpolytope $P(X_0)$ obtained is *closed* both as a subcomplex and as a subset of the singular polytope $P(X)$ of X. This might be one of the advantages in using the singular polytope. Further, the following statement is obvious:

(5.1) *The projection* $\omega: P(X) \to X$ *maps* $P(X_0)$ *onto* X_0.

6. The induced map $f^\#: P(X) \to P(Y)$

In the present section, let X, Y be topological spaces and let $f: X \to Y$ be a continuous map. f induces naturally a map $f^\#: P(X) \to P(Y)$ described as follows: For an arbitrary point $p \in P(X)$, let σ_ξ be the (unique) open singular cell of $P(X)$ which contains p and let s_ξ be the associated ordered geometric simplex. Then $p \in \sigma_\xi = \operatorname{Int} s_\xi$. There is a unique continuous simplex $T_\xi: s_\xi \to X$ which represents ξ, i.e. $\xi=[T_\xi]$. The continuous simplex $fT_\xi: s_\xi \to Y$ represents a singular simplex $\eta=[fT_\xi] \in S(Y)$. Let σ_η be the corresponding open singular cell of $P(Y)$ and s_ξ be the associated ordered geometric simplex. Obviously, s_ξ and s_η are of the same dimension. Let $B_\xi: s_\xi \to s_\eta$ denote the unique barycentric map of s_ξ onto s_η which preserves the order of the vertices. Then the map $f^\#$ is defined by taking

$$f^\#(p)=B_\xi(p) \in \sigma_\eta, \qquad (p \in \sigma_\xi \subset P(X)).$$

(6.1) $f^\#$ *is continuous.*

Proof. According to (2.2), it needs only to prove that the partial map $f_0^\#=f^\# | Cl\sigma_\xi$ is continuous for an arbitrary closed singular cell $Cl\sigma_\xi$ of $P(X)$. Remembering the characteristic map $\mu_\xi: s_\xi \to Cl\sigma_\xi$, one can easily see that

(6.2) $$f_0^\# \mu_\xi = \mu_\eta B_\xi.$$

Let U be an arbitrary open set of $P(Y)$ and let $V=f_0^{\#-1}(U)\subset Cl\sigma_\xi$. It remains to prove that V is an open set of $Cl\sigma_\xi$. It follows from (6.2) that $\mu_\xi^{-1}(V)=B_\xi^{-1}\mu_\eta^{-1}(U)$. Since μ_η and B_ξ are both continuous, $\mu_\xi^{-1}(V)$ is an open set of s_ξ. Hence, by the topology of $Cl\sigma_\xi$, V is an open set of $Cl\sigma_\xi$. This completes the proof.

Now let us consider the following diagram of maps:

$$\begin{tikzcd} P(X) \arrow[r, "f^{\#}"] \arrow[d, "\omega_X"] & P(Y) \arrow[d, "\omega_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The following commutativity relation is an immediate consequence of the definition of $f^{\#}$:

(6.3) $$f\omega_X = \omega_Y f^{\#}.$$

Let $X_0\subset X$ and $Y_0\subset Y$ be subspaces, then the following statement is obvious:

(6.4) *If* $f(X_0)\subset Y_0$, *then* $f^{\#}(P(X_0))\subset P(Y_0)$.

7. The barycentric subdivisions of $P(X)$

For each singular simplex $\xi\in S(X)$, let us denote by s'_ξ the *barycentric first derived*, [54, p. 3], of the ordered geometric simplex s_ξ associated with ξ. Since the characteristic map $\mu_\xi: s_\xi\rightarrow Cl\sigma_\xi$ reduces to the identity map if it is restricted within the interior σ_ξ of s_ξ, μ_ξ induces a simplicial subdivision of σ_ξ into $\mu_\xi(\text{Int } s'_\xi)$, named the *barycentric first derived* σ'_ξ of σ_ξ, which is a finite set of open geometric simplices. If we replace each open singular cell σ_ξ by its barycentric first derived σ'_ξ, we obtain a subdivision of $P(X)$, called the *first barycentric subdivision* $P'(X)$ of $P(X)$.

More generally, let us denote by $s_\xi^{(n)}$ the *barycentric n-th derived*, [54, p. 3], of s_ξ. Then the characteristic map μ_ξ induces a simplicial subdivision of σ_ξ into $\mu_\xi(\text{Int } s_\xi^{(n)})$, called the *barycentric n-th derived* $\sigma_\xi^{(n)}$ of σ_ξ. If we replace each open singular cell σ_ξ by its barycentric n-th derived $\sigma_\xi^{(n)}$, we obtain the *n-th barycentric subdivision* $P^{(n)}(X)$ of the singular polytope $P(X)$ of X. It is clear that the characteristic map μ_ξ of s_ξ onto $Cl\sigma_\xi$ maps each open simplex of $s_\xi^{(n)}$ barycentrically onto some open simplex of $P^{(n)}(X)$.

By a *simplicial polytope* P, we understand the union of a collection of closed geometric simplices $\{s_\alpha\}$, where α runs over a certain abstract set A, such that (i) every face of an arbitrary simplex s_α of the collection belongs to the collection and (ii) the intersection $s_\alpha \cap s_\beta$ of any two simplices of the collection is either vacuous or a face on both of them, with the topology defined as follows: A set $M \subset P$ is said to be open if and only if, for each closed geometric simplex s_α of the collection, $M \cap s_\alpha$ is an open set of s_α in its euclidean topology. Simplicial polytopes are called topological polyhedra by J. H. C. Whitehead, [72, p. 316].

(7.1) *For each $n \geqq 2$, the n-th barycentric subdivision $P^{(n)}(X)$ of the singular polytope $P(X)$ of X is a simplicial polytope.*

Proof. If, for every closed singular cell $Cl\sigma_\xi$, the finite subpolytope $Cl\sigma_\xi \cap P^{(n)}(X)$ of $P^{(n)}(X)$ is simplicial, then clearly so is $P^{(n)}(X)$. It is classical that $Cl\sigma_\xi \cap P^{(n)}(X)$ is simplicial if $n \geqq 2$. Hence (7.1) is proved.

8. The injection $j: X \to P(X)$

Throughout the present section, we assume that X be a simplicial polytope and X_0 be a closed subpolytope of X. Let the vertices of X be partially ordered in such a way that those of every closed simplex of X are ordered. If such a partial ordering has been given, then each closed simplex of X becomes an ordered geometric simplex. Then we may define a natural map $j: X \to P(X)$, called the *injection* of the simplicial polytope X into its singular polytope $P(X)$ associated with the given partial ordering, which will be described as follows.

Let s be an arbitrary closed simplex of X. Then the identity map on s defines a continuous simplex $T: s \to X$ and represents a singular simplex $\xi = [T]$ of X. Let s_ξ be the ordered geometric simplex associated with ξ and σ_ξ its interior. Then, we define the map $j: X \to P(X)$ by taking

$$j(x) = \mu_\xi B_s(x), \qquad (x \in \operatorname{Int} s \subset X),$$

where B_s denotes the barycentric map of s onto s_ξ preserving the order of vertices and μ_ξ denotes the characteristic map of s_ξ onto $Cl\sigma_\xi$.

(8.1) *The injection $j: X \to P(X)$ is a homeomorphism of X onto a closed subpolytope $j(X)$ of $P(X)$ with the property that ωj is the identity map on X.*

Proof. To show the continuity of the injection j, it needs only to prove that of the partial map $j|s$ for any closed simplex s of X. It is easily verified that $j|s = \mu_\xi B_s$. Since both B_s and μ_ξ are continuous, so is $j|s$. This proves that j is a continuous map. Since the image of each closed simplex s of X is a closed singular cell $Cl\sigma_\xi$ of $P(X)$, the image $j(X)$ is a closed subcomplex and hence a closed subpolytope of $P(X)$. Since the continuous simplex $T: s \to X$ in the definition of j is defined by the identity map on s, it is obvious that ωj is the identity map on X. This shows that $j^{-1}=\omega|j(X)$ and hence j is a homeomorphism of X onto $j(X)$. Q. E. D.

The following statement is obvious:

(8.2) *j maps X_0 into $P(X_0)$.*

III. The General Theory of Continuous Extension

Throughout the present chapter, we assume that R be a pathwise connected topological space and (X, X_0) a given pair, i. e. a topological space X and a subspace $X_0 \subset X$ which need not be closed. We shall use the following abridged notations:

$$P=P(X), \qquad P_0=P(X_0).$$

As usual, we denote by P^n the n-dimensional skeleton of P, i. e. the set of all open singular cells with dimensions not exceeding n. Further, let

$$\overline{P}^n=P_0 \cup P^n.$$

Remembering the projection $\omega: P \to X$, we denote the partial map on P_0 onto X_0 by $\omega_0=\omega|P_0$.

Hereafter, a map will understand to be a continuous map and an extension means a continuous extension.

9. n-Extensibility

A map $f: X_0 \to R$ is said to be *n-extensible with respect to X*, if the map $f\omega_0: P_0 \to R$ has an extension $\phi: \overline{P}^n \to R$. Since R is pathwise connected, the following assertion is obvious.

(9.1) *Every map $f: X_0 \to R$ is 1-extensible with respect to X.*

For a given map $f: X_0 \to R$, the least upper bound of the set of integers n such that f is n-extensible with respect to X is called the *extension index* of f with respect to X.

(9.2) *Homotopic maps have the same extension index.*

Proof. Suppose that $f, g: X_0 \to R$ be any two homotopic maps and that f be n-extensible with respect to X. It needs only to show that g is also n-extensible with respect to X. Since $f \simeq g$, we have

$f\omega_0 \simeq g\omega_0$. Since $f\omega_0$ has an extension over $\bar{P}^n$, it follows from the homotopy extension property of the CW-complexes, [73. p. 228], that $g\omega_0$ has also an extension over $\bar{P}^n$. This completes the proof.

(9.3) *Let $\kappa: (M, M_0) \to (X, X_0)$ be a map of a pair (M, M_0) into (X, X_0) and let $\kappa_0 = \kappa | M_0$. If a map $f: X_0 \to R$ is n-extensible with respect to X, then the map $f\kappa_0: M_0 \to R$ is n-extensible with respect to M.*

Proof. According to §6, the map κ induces a map $\kappa^\#: P(M) \to P(X)$. Let $\kappa_0^\# = \kappa^\# | P(M_0)$. It is obvious that $\kappa^\#$ maps $\bar{P}^n(M)$ into $\bar{P}^n(X)$. Since f is n-extensible with respect to X, the map $f\omega_{X_0}$ has an extension ϕ over $\bar{P}^n(X)$. Then, the map $f\omega_{X_0}\kappa_0^\#$ will have $\phi\kappa^\#$ over $\bar{P}^n(M)$ as an extension. According to (6.3), we have $f\kappa_0\omega_{M_0} = f\omega_{X_0}\kappa^\#$. Hence $f\kappa_0$ is n-extensible with respect to M. This completes the proof.

(9.4) *A necessary and sufficient condition for a map $f: X_0 \to R$ to be n-extensible with respect to X is that, for an arbitrary map κ: $(M, M_0) \to (X, X_0)$ of a simplicial polytope M and a closed subpolytope $M_0 \subset M$, the map $f\kappa_0$, where $\kappa_0 = \kappa | M_0$, can be extended over $\bar{M}^n = M_0 \cup M^n$.*

Proof. *Necessity.* Assume that f be n-extensible with respect to X. Then, there is an extension $\phi: \bar{P}^n(X) \to R$ of the map $f\omega_{X_0} = f\omega_X | P(X_0)$. The map κ induces a map $\kappa^\#: P(M) \to P(X)$ which clearly maps $\bar{P}^n(M)$ into $\bar{P}^n(X)$. Since M is a simplicial polytope and $M_0 \subset M$ is a closed subpolytope, there is an injection $j: M \to P(M)$ which maps $\bar{M}^n$ into $\bar{P}^n(M)$. By (6.3) and (8.1), we have

$$\omega_X \kappa^\# j(y) = \kappa\omega_M j(y) = \kappa(y), \qquad (y \in M).$$

Hence, the map $f\kappa_0 = f\kappa | M_0$ has an extension $\phi^* = \phi\kappa^\# j | \bar{M}^n$ over $\bar{M}^n$. This proves the necessity.

Sufficiency. Assume that the condition holds. Let Q and $Q_0 \subset Q$ denote the second barycentric subdivision of $P = P(X)$ and $P_0 = P(X_0)$. According to (7.1), Q is a simplicial polytope and Q_0 is a closed subpolytope of Q. As a topological space and a subspace, we have $P = Q$ and $P_0 = Q_0$. Now, since the projection $\omega: P \to X$ maps the pair (Q, Q_0) into (X, X_0), it follows from our condition that the map $f\omega_0 = f\omega | Q_0$ has an extension $\phi^*: \bar{Q}^n \to R$, where $\bar{Q}^n = Q_0 \cup Q^n$. Since $\bar{P}^n \subset \bar{Q}^n$, we may take $\phi = \phi^* | \bar{P}^n$. This proves that f is n-extensible with respect to X and thus completes the proof.

(9.5) *If X is a simplicial polytope and X_0 a closed subpolytope of X, then a necessary and sufficient condition for a map $f: X_0 \to R$ to be n-extensible with respect to X is that f has an extension $f^*: \bar{X}^n \to R$*

over $\bar{X}^n = X_0 \cup X^n$.

Proof. *Necessity.* Assume $f: X_0 \to R$ to be n-extensible with respect to X. Then, by definition, the map $f\omega_0: P_0 \to R$ has an extension $\phi: \bar{P}^n \to R$. Since X is a simplicial polytope and X_0 a closed subpolytope of X, it follows from (8. 1) that there exists an injection $j: X \to P$ which clearly maps $\bar{X}^n$ into $\bar{P}^n$. According to (8 1), we have $f\omega j | X_0 = f$. Therefore, f has an extension $\phi j | X^n = f^*$. This proves the necessity of the condition.

Sufficiency. Assume the existence of an extension $f^*: \bar{X}^n \to R$ of the given map f. Since both X and P are CW-complexes, [73, p. 223], it follows from a cellular approximation theorem of J. H. C. Whitehead, [73, p. 229], that there exists a homotopy $h_t: P \to X$ $(0 \leq t \leq 1)$ such that $h_0 = \omega$ and $h_1(P^n) \subset X^n$ for each n. Clearly we may choose h_t in such a way that $h_t(P_0) \subset X_0$ for each $0 \leq t \leq 1$. Now, $fh_1 | P_0$ has an extension $f^*h_1 | \bar{P}^n$; therefore, it follows from a homotopy extension theorem of J. H. C. Whitehead, [73, p. 228], that the map $f\omega_0 = fh_0 | P_0$ has also an extension over $\bar{P}^n$. This completes the proof.

10. Extensibility

A map $f: X_0 \to R$ is said to be *extensible over* X, if there exists an extension $f^*: X \to R$ of f, i. e. $f = f^* | X_0$. The following assertion is immediate:

(10. 1) *The extensibility of a map* $f: X_0 \to R$ *over* X *implies the* n-*extensibility of* f *with respect to* X *for every positive* n.

In the remainder of the present section, we are going to study the converse of (10. 1).

We recall the notion of the homotopy extension property, [40, p. 992], as follows: X_0 is said to have the homotopy extension property in X relative to R, if any partial homotopy $f_t: X_0 \to R$ $(0 \leq t \leq 1)$ of an arbitrary map $f_0: X \to R$ has an extension $f_t^*: X \to R$ $(0 \leq t \leq 1)$ such that $f_0^* = f_0$. In particular, for the following three important special cases, X_0 has the homotopy extension property in X relative to R:- (i) if X is a CW-complex and X_0 a closed subcomplex of X, [73, p. 228]; (ii) if both X and X_0 are absolute neighborhood retracts and X_0 is closed in X, [39]; (iii) if R is an absolute neighborhood retract and X_0 a closed subspace of a metric space X, [49, p. 86, Borsuk's Theorem].

Following J. H. C. Whitehead, [73, p. 214], we say that a pair (M, M_0) *dominates* a pair (X, X_0), if there are two maps

$$\xi : (X, X_0) \to (M, M_0), \qquad \eta : (M, M_0) \to (X, X_0),$$

and a homotopy $\lambda_t : (X, X_0) \to (X, X_0)$, $(0 \leq t \leq 1)$, such that $\lambda_0 = \eta\xi$ and λ_1 is the identity map on X.

Let C denote the class of all pairs (X, X_0) each of which is dominated by a *simplicial pair* (M, M_0), i.e. M being a simplicial polytope and M_0 a closed subpolytope of M. For each pair (X, X_0) of C, we shall use $\Delta(X, X_0)$ to denote the minimum value of $\dim(M \setminus M_0)$ for all simplicial pairs (M, M_0) dominating (X, X_0). Let C_0 denote the subclass of C consisting of all pairs (X, X_0) of C such that $\Delta(X, X_0)$ be finite. In particular, C_0 contains the following important subclasses, where (X, X_0) satisfies one of the following conditions: (i) (X, X_0) is a simplicial pair with a finite $\dim(X \setminus X_0)$; (ii) X and X_0 are compact absolute neighborhood retracts; and (iii) X and X_0 are absolute neighborhood retracts of finite dimensions and X_0 is closed in X.

(10. 2) *Let (X, X_0) be a pair of C_0 such that X_0 is closed in X and has the homotopy extension property in X relative to R, and let $n = \Delta(X, X_0)$. Then, the n-extensibility of a map $f : X_0 \to R$ with respect to X implies the extensibility of f over X.*

Proof. Let (M, M_0) be a simplicial pair with $\dim(M \setminus M_0) = n$ which dominates (X, X_0). Let ξ, η, and λ_t be the maps and the homotopy given in the above definition of a dominating pairs. Since $\dim(M \setminus M_0) = n$, we have $\overline{M}^n = M$. If $f : X_0 \to R$ is n-extensible with respect to X, then it follows from (9. 4) that the map $f\eta | M_0$ has an extension $g : M \to R$. Hence, the map $f\eta\xi | X_0$ has an extension $g\xi : X \to R$. Since $\eta\xi | X_0$ is homotopic with the identity by the homotopy $\lambda_t | X_0$, it follows from the homotopy extension property that f has an extension $f^* : X \to R$. This completes the proof.

11. Algebraic condition for 2-extensibility

Let $f : X_0 \to R$ be a given map and assume that both X and X_0 be pathwise connected. Choose a fixed point $x_0 \in X_0$ and call $r_0 = f(x_0) \in R$. Throughout the present section, let us use the following notations

$$F = \pi_1(X, x_0), \qquad F_0 = \pi_1(X_0\, x_0), \qquad G = \pi_1(R, r_0)$$

for the fundamental groups of X, X_0, R with basic points x_0, x_0, r_0 respectively. The given map f induces a homomorphism

$$f^* : \pi_1(X_0, x_0) \to \pi_1(R, r_0)$$

as follows: Let $e \in F_0$ be an arbitrary element represented by a closed path $\sigma: I \to X_0$ with $\sigma(0)=x_0=\sigma(1)$. Then $f^*(e)$ is the element of G represented by the closed path $f\sigma: I \to R$.

In particular, the identity map $\iota: X_0 \to X$ on X_0 induces a homomorphism

$$\iota^*: \pi_1(X_0, x_0) \to \pi_1(X, x_0).$$

A homomorphism $k: F_0 \to G$ is said to be *extensible* with respect to X, if there exists a homomorphism $h: F \to G$ such that $k=h\iota^*$.

(11.1) *The map $f: X_0 \to R$ is 2-extensible with respect to X, if and only if its induced homomorphism $f^*: \pi_1(X_0, x_0) \to \pi_1(R, r_0)$ is extensible with respect to X.*

Proof. *Necessity.* Assume that f be 2-extensible with respect to X. Then, the map $f\omega_0: P_0 \to R$ has an extension $\phi: \bar{P}^2 \to R$.

We are going to construct a homomorphism $h: F \to G$ depending only on ϕ. Let $e \in F$ be an arbitrary element represented by a path $T: I \to X$ with $T(0)=x_0=T(1)$. Since the closed unit segment $I=\langle 0, 1\rangle$ is an ordered geometric 1-simplex, the path T is a continuous 1-simplex in X and hence represents a singular 1-simplex $\xi=[T]$. Let s_ξ denote the ordered geometric 1-simplex associated with ξ whose interior is the open singular cell σ_ξ. Let $B_\xi: I \to s_\xi$ denote the barycentric map of I onto s_ξ preserving the order of the vertices. Using the characteristic map $\mu_\xi: s_\xi \to Cl\sigma_\xi$, we define a path $\tau: I \to R$ by taking $\tau=\phi\mu_\xi B_\xi$. Clearly $\tau(0)=r_0=\tau(1)$, and hence τ represents an element $h(e) \in G$.

The element $h(e)$ does not depend on the choice of the representative path $T: I \to X$ for e. In fact, let $T': I \to X$ be another representative path of e with $T'(0)=x_0=T'(1)$. Take an ordered geometric 2-simplex $s=\langle v_0, v_1, v_2\rangle$. Let B_i, $(i=1, 2)$, denote the barycentric map of $I=\langle 0, 1\rangle$ onto $s_i=\langle v_0, v_i\rangle$ preserving the order of vertices. Call $s_3=\langle v_1, v_2\rangle$. Define a map $g: \partial s \to X$ on the boundary sphere ∂s of s by taking

$$g(y)=\begin{cases} TB_1^{-1}(y) & (y \in s_1), \\ T'B_2^{-1}(y) & (y \in s_2), \\ x_0 & (y \in s_3). \end{cases}$$

Since the paths T and T' represent the same element $e \in F$ and since $g(s_3)=x_0$, the map g has an extension $g^*: s \to X$. Since s is an ordered geometric 2-simplex, g^* is a continuous 2-simplex in X and hence represents a singular 2-simplex $\eta=[g^*]$. Let s_η be the ordered geometric 2-simplex associated with η whose interior is the open singular 2-cell σ_η. Let $B_\eta: s \to s_\eta$ denote the barycentric map of s onto s_η preserving the order of vertices. Consider the map

$$\psi = \phi \mu_\eta B_\eta : s \to R,$$

where $\mu_\eta: s_\eta \to Cl\sigma_\eta$ is the characteristic map for the singular simplex η. It is clear that $\tau = \psi B_1$ and $\tau' = \psi B_2$. Since $g(s_3) = x_0$, we obtain $\mu_\eta B_\eta(s_3) \subset P_0$. Therefore, we have

$$\phi \mu_\eta B_\eta(s_3) = f\omega_0 \mu_\eta B_\eta(s_3) = fg(s_3) = r_0.$$

Since ψ is defined throughout s, τ and τ' represent the same element $h(e) \in G$. This proves that $h(e)$ does not depend on the choice of the representative path $T: I \to X$ for $e \in F$.

Next, let us show that the correspondence $e \to h(e)$ defines a homomorphism $h: F \to G$. Suppose $e_i \in F$, $(i=1, 2, 3)$, to be arbitrary elements such that $e_2 = e_1 e_3$. Choose representative path $T_i: I \to X$, $(i=1, 2, 3)$, for e_i with $T_i(0) = x_0 = T_i(1)$. Take an ordered geometric 2-simplex $s = \langle v_0, v_1, v_2 \rangle$ and call

$$s_1 = \langle v_0, v_1 \rangle, \qquad s_2 = \langle v_0, v_2 \rangle, \qquad s_3 = \langle v_1, v_2 \rangle.$$

Let $B_i: I \to s_i$, $(i=1, 2, 3)$, denote the barycentric map of I onto s_i preserving the order of vertices. Define a map $g: \partial s \to X$ on the boundary sphere ∂s of s by taking $g|s_i = T_i B_i^{-1}$ on each s_i, $(i=1, 2, 3)$. It follows from $e_2 = e_1 e_3$ that g has an extension $g^*: s \to X$. g^* is a continuous 2-simplex in X and hence represents a singular 2-simplex $\eta = [g^*]$. As before, we obtain a map

$$\psi = \phi \mu_\eta B_\eta : s \to R.$$

It is easy to see that $\psi B_i: I \to R$ represents the element $h(e_i)$ of G for each $i=1, 2, 3$. Since ψ is defined throughout s, we have $h(e_2) = h(e_1)h(e_3)$. This proves that h is a homomorphism.

Last, let us show that $f^* = h\iota^*$. Let $e \in F_0$ be an arbitrary element represented by a path $T: I \to X_0$ with $T(0) = x_0 = T(1)$. Then, the element $\iota^*(e) \in F$ is represented by the same path T. Denote by $\xi = [T] \in S(X)$. Then, by construction, the element $h\iota^*(e) \in G$ is represented by the path

$$\tau = \phi \mu_\eta B_\eta : I \to R.$$

Since $T(I) \subset X_0$, we have $\mu_\eta B_\eta(I) \subset P_0$. Hence, we obtain

$$\tau = \phi \mu_\xi B_\xi = f\omega_0 \mu_\xi B_\xi = fT.$$

This shows that $f^*(e)=h\iota^*(e)$ and completes the proof of the necessity.

Sufficiency. Assume f^* to be extensible with respect to X. Then, there exists a homomorphism $h: F \to C$ such that $f^*=h\iota^*$.

By means of the homotopy extension property of P_0 in P and the pathwise connectedness of X_0 and X, one might easily prove the existence of a homotopy

$$\delta_t: (P, P_0) \to (X, X_0), \qquad (0 \leq t \leq 1),$$

such that $\delta_0=\omega$ and δ_1 maps every vertex of P at x_0.

Let σ_ξ be an arbitrary open singular 1-cell contained in P_0 and s_ξ the ordered geometric 1-simplex associated with ξ. Denote by $B_\xi: I \to s_\xi$ the barycentric map of I onto s_ξ which preserves the order of vertices. The map

$$\theta_\xi=\delta_1\mu_\xi B_\xi: I \to X_0$$

is a path in X_0 with $\theta_\xi(0)=x_0=\theta_\xi(1)$. It represents an element $e_0 \in F_0$ and an element $e=\iota^*(e_0) \in F$. Hence, the path $f\theta_\xi=f\delta_1\mu_\xi B_\xi: I \to R$ is a representative of the element $f^*(e_0)=h(e)$.

Now, we are going to construct an extension $\psi^*: \bar{P}^2 \to R$ of the partial map $f\delta_1|P_0$ by the methods described as what follows.

First, let σ_ξ be an arbitrary open singular 1-cell contained in $P \setminus P_0$ and s_ξ the ordered geometric 1-simplex associated with ξ. The path $\delta_1\mu_\xi B_\xi: I \to X$ represents an element $e_\xi \in F$. Choose a path $\tau_\xi: I \to R$ with $\tau_\xi(0)=r_0=\tau_\xi(1)$ which represents the element $h(e_3) \in G$. Since σ_ξ is the interior of s_ξ, we may define a map $\psi: \bar{P}^1 \to R$ by

$$\psi(y)=\begin{cases} f\delta_1(y) & (y \in \bar{P}^0=P_0 \cup P^0), \\ \tau_\xi B_\xi^{-1}(y) & (y \in \sigma_\xi \subset P^1 \setminus P_0). \end{cases}$$

The continuity of ψ is verified by the fact that δ_1 maps every vertex of P at $x_0 \in X_0$.

Next, let σ_η be an arbitrary open singular 2-cell contained in $P \setminus P_0$ and $s_\eta=\langle v_0, v_1, v_2 \rangle$ the ordered geometric 2-simplex associated with η whose interior is σ_η. Denote by

$$s_1=\langle v_0, v_1 \rangle, \qquad s_2=\langle v_0, v_2 \rangle, \qquad s_3=\langle v_1, v_2 \rangle$$

the three ordered sides of s_η. Let $B_i: I \to s_i$, $(i=1, 2, 3)$, denote the barycentric maps of I onto s_i preserving the order of vertices. Let $e_i \in F$, $(i=1, 2, 3)$, be the elements represented by the paths $\delta_1\mu_\eta B_i$: $I \to X$. Since $\delta_1\mu_\eta$ is defined throughout s_η, we have $e_2=e_1e_3$. It follows from the construction of ψ that the paths $\psi\mu_\eta B_i: I \to R$ represent the elements $h(e_i) \in G$, $(i=1, 2, 3)$. Since h is a homomorphism, $e_2=e_1e_3$ implies $h(e_2)=h(e_1)h(e_3)$. Hence, the map $\psi\mu_\eta|\partial s_\eta$ can be extended into the interior σ_η of s_η. Choose an extension $\psi_\eta: s_\eta \to R$ of $\psi\mu_\eta|\partial s_\eta$ for each $\sigma_\eta \subset P^2\setminus P_0$. Then the required map $\psi^*: \overline{P}^2 \to R$ is given by

$$\psi^*(y)=\begin{cases} \psi(y) & (y \in \overline{P}^1=P_0 \cup P^1), \\ \psi_\eta(y) & (y \in \sigma_\eta \subset P^2\setminus P_0). \end{cases}$$

The continuity of ψ^* is easily verified by the fact that $\psi^*\mu_\eta=\psi_\eta$ on s_η.

Since $f\delta_1|P_0$ is homotopic with $f\omega_0$, it follows from the homotopy extension property of P_0 in P relative to R that $f\omega_0$ has an extension $\phi: \overline{P}^2 \to R$. Hence, f is 2-extensible with respect to X and our proof is complete.

12. Obstruction cocycles $\mathbf{c}^{n+1}(\phi)$

In the present section, we are concerned with the task to establish the theorems of Eilenberg, [19], for the singular polytope P and its closed subpolytope P_0. With a purpose to simplify the arguments, we assume an additional condition that R be n-simple in the sense of Eilenberg [17], where $n \geq 1$ is a given integer. Let us denote by $\pi_n=\pi_n(R)$ the n-th homotopy group of R.

Let $\phi: \overline{P}^n \to R$ be a given map. ϕ defines an $(n+1)$-dimensional singular cochain $c^{n+1}(\phi)$ of X with coefficients in π_n as follows: Let $\xi \in S(X)$ be an arbitrary singular $(n+1)$-simplex. Let s_ξ be the ordered geometric $(n+1)$-simplex associated with ξ whose interior is the open singular cell σ_ξ. The characteristic map $\mu_\xi: s_\xi \to Cl\sigma_\xi$ maps the boundary sphere ∂s_ξ of s_ξ into $\overline{P}^n$. Since the sphere ∂s_ξ has been oriented by the order of the vertices, the map $\phi\mu_\xi|\partial s_\xi$ determines an element c_ξ of the homotopy group $\pi_n=\pi_n(R)$ since R is n-simple. The correspondence $\xi \to c_\xi$ defines $(n+1)$-dimensional singular cochain $c^{n+1}(\phi)$ of X with coefficients in π_n.

(12.1) *$c^{n+1}(\phi)$ is a singular cocycle of X modulo X_0, called the obstruction cocycle of ϕ.*

Proof. Let $\eta \in S(X)$ be an arbitrary singular $(n+2)$-simplex. To show that $c^{n+1}(\phi)$ is a singular cocycle, it suffices to prove that $\delta c^{n+1}(\phi)\cdot\eta=0$. Let s_η be the ordered geometric $(n+2)$-simplex associated with η and $\mu_\eta: s_\eta \to Cl\sigma_\eta$ the characteristic map for η. Denote by s_η^n the n-dimensional skeleton of s_η and consider the map $\theta: s_\eta^n \to R$ defined by $\theta=\phi\mu_\eta|s_\eta^n$. According to Eilenberg, [19, p. 237], θ determines a cocycle $c^{n+1}(\theta)$ of s_η. Let s be an arbitrary $(n+1)$-face of s_η, then μ_η maps the interior of s onto some open singular $(n+1)$-cell σ_ξ of P. It is easy to see that

$$c^{n+1}(\phi)\cdot\xi=c^{n+1}(\theta)\cdot s.$$

Hence it follows that

$$\delta c^{n+1}(\phi)\cdot\eta=\delta c^{n+1}(\theta)\cdot s_\eta=0.$$

This proves that $c^{n+1}(\phi)$ is a singular cocycle.

Next, let ξ be an arbitrary singular $(n+)$-simplex contained in $S(X_0)$. Since μ_ξ maps s_ξ onto $Cl\sigma_\xi\subset P_0$, $\phi\mu_\xi$ is defined over s_ξ. Therefore, $c^{n+1}(\phi)\cdot\xi=c_\xi=0$. This proves that $c^{n+1}(\phi)$ is a singular cocycle of X modulo X_0. Q. E. D.

Now, let $\phi, \psi: \bar{P}^n\rightarrow R$ be two maps such that $\phi|\bar{P}^{n-1}=\psi|\bar{P}^{n-1}$. ϕ and ψ determine an n-dimensional singular cochain $d^n(\phi,\psi)$ described as follows: Let $\xi\in S(X)$ be an arbitrary singular n-simplex. Obviously, the characteristic map $\mu_\xi: s_\xi\rightarrow Cl\sigma_\xi$ maps the boundary sphere ∂s_ξ of s_ξ into $\bar{P}^{n-1}$. Hence the maps $\phi\mu_\xi$ and $\psi\mu_\xi$ of s_ξ into R agree on ∂s_ξ. We define a map θ_ξ on the boundary sphere

$$\partial(s_\xi\times I)=(s_\xi\times 0)\cup(\partial s_\xi\times I)\cup(s_\xi\times 1)$$

of $s_\xi\times I$ with values in R by taking

$$\theta_\xi(y,t)=\begin{cases}\phi\mu_\xi(y) & (y\in s_\xi,\ t=0),\\ \phi\mu_\xi(y)=\psi\mu_\xi(y) & (y\in\partial s_\xi,\ t\in I),\\ \psi\mu_\xi(y) & (y\in s_\xi,\ t=1).\end{cases}$$

The sphere $\partial(s_\xi\times I)$ may be oriented in such a way that $s_\xi\times 0$ lies negatively and $s_\xi\times 1$ positively on $\partial(s_\xi\times I)$. Then the map θ_ξ determines an element $d_\xi\in\pi_n$ since R is n-simple. The association $\xi\rightarrow d_\xi$ defines an n-dimensional singular cochain $d^n(\phi,\psi)$.

(12. 2) *$d^n(\phi,\psi)$ is a singular cochain of X modulo X_0, satisfying*

$$\delta d^n(\phi,\psi)=c^{n+1}(\psi)-c^{n+1}(\phi).$$

Proof. That $d^n(\phi,\psi)\cdot\xi=0$ for every singular n-simplex ξ contained in $S(X_0)$ is easily verified by means of the fact that the maps ϕ and ψ agree on P_0.

To prove the equality, let $\eta\in S(X)$ be an arbitrary singular $(n+1)$-simplex. It suffices to prove that

$$\text{(i)}\qquad \delta d^n(\phi,\psi)\cdot\eta=c^{n+1}(\psi)\cdot\eta-c^{n+1}(\phi)\cdot\eta.$$

Let s_η be the ordered geometric $(n+1)$-simplex associated with η, and denote by s_η^n the n-dimensional skeleton of s_η. Consider the maps

$$f=\phi\mu_\eta|s_\eta^n, \qquad g=\psi\mu_\eta|s_\eta^n.$$

They determine two cocycles $c^{n+1}(f)$ and $c^{n+1}(g)$ of s_η with coefficients in π_n. Since f and g agree on the $(n-1)$-dimensional skeleton s_η^{n-1}, they determine an n-dimensional cochain $d^n(f,g)$ of s_η with coefficients in π_n such that

$$\delta d^n(f,g)=c^{n+1}(g)-c^{n+1}(f),$$

according to Eilenberg, [19, p. 237], with an obvious minor modification. It is easy to verify that

$$c^{n+1}(\phi)\cdot\eta=c^{n+1}(f)\cdot s_\eta, \qquad c^{n+1}(\psi)\cdot\eta=c^{n+1}(g)\cdot s_\eta,$$
$$\delta d^n(\phi,\psi)\cdot\eta=\delta d^n(f,g)\cdot s_\eta.$$

This proves our equality (i) and completes the proof.

(12.3) *For an arbitrary singular n-cochain d^n of X modulo X_0 with coefficients in π_n and an arbitrary map $\phi: \bar{P}^n \to R$, there is a map $\psi: \bar{P}^n \to R$ such that $\phi|\bar{P}^{n-1}=\psi|\bar{P}^{n-1}$ and $d^n(\phi,\psi)=d^n$.*

Proof. Since d^n is a singular n-cochain of X modulo X_0, $d^n\cdot\xi=0$ for every $\xi\in S(X_0)$. Now, let $\eta\in S(X)$ be an arbitrary n-simplex not in $S(X_0)$ and s_η the ordered geometric n-simplex associated with η whose interior is the open singular cell σ_η. Call $d_\eta=d^n\cdot\eta\in\pi_n$. Consider the map $f_\eta=\phi\mu_\eta: s_\eta\to R$. There is a map $g_\eta: s_\eta\to R$ such that $f_\eta|\partial s_\eta=g_\eta|\partial s_\eta$, and the maps f_η, g_η determine the element d_η. Define a map $\psi: \bar{P}^n\to R$ by taking

$$\psi(y)=\begin{cases}\phi(y) & (y\in\bar{P}^{n-1}),\\ g_\eta(y) & (y\in\sigma_\eta\subset P^n\setminus P_0).\end{cases}$$

The continuity and the relation $d^n(\phi,\psi)=d^n$ are immediate consequences of the construction. This completes the proof.

As a direct consequence of (12.2) and (12.3), we state the following existence theorem:

(12.4) *For a given map $\phi: \bar{P}^n\to R$ and any cocycle $c^{n+1}\sim c^{n+1}(\phi)$ modulo X_0, there exists a map $\psi: \bar{P}^n\to R$ such that $\phi|\bar{P}^{n-1}=\psi|\bar{P}^{n-1}$ and $c^{n+1}(\psi)=c^{n+1}$.*

With the same proof of Eilenberg, [19, p. 239], the following First Extension Theorem of Eilenberg can be proved.

(12.5) *For a given map $\phi: \bar{P}^n\to R$, we have:-*

(i) $c^{n+1}(\phi)=0$ *if and only if there is a map* $\phi^*: \bar{P}^{n+1}\to R$ *such that* $\phi^*|\bar{P}^n=\phi$.

(ii) $c^{n+1}(\phi)\sim 0$ *modulo* X_0 *if and only if there is a map* $\phi^*: \bar{P}^{n+1}\to R$ *such that* $\phi^*|\bar{P}^{n-1}=\phi|\bar{P}^{n-1}$.

13. Obstruction sets

In the present section, let $n\geqq 1$ be a given integer. Assume that R be n-simple and denote by $\pi_n=\pi_n(R)$ the n-th homotopy group of R. Denote by $H^{n+1}(X, X_0, \pi_n)$ the $(n+1)$-dimensional singular cohomology group of X modulo X_0 with π_n as the coefficient group.

Let $f: X_0\to R$ be a given map. We are going to define the $(n+1)$ *-dimensional obstruction set* $\Omega^{n+1}(f)\subset H^{n+1}(X, X_0, \pi_n)$ of the map f with respect to X. If f is not n-extensible with respect to X, we define $\Omega^{n+1}(f)$ to be the vacuous set. Now suppose f to be n-extensible with respect to X. Then there exists an extension $\phi: \bar{P}^n\to R$ of the partial map $f\omega_0: P_0\to R$. The obstruction cocycle $c^{n+1}(\phi)$ represents an element $\gamma^{n+1}(\phi)$ of the cohomology group $H^{n+1}(X, X_0, \pi_n)$, called *an* $(n+1)$*-dimensional obstruction element* of f. $\Omega^{n+1}(f)$ is defined to be the set of all $(n+1)$-dimensional obstruction elements of f.

(13. 1) *Homotopic maps have the same* $(n+1)$*-dimensional obstruction set.*

Proof. Assume that $f, g: X_0\to R$ be two homotopic maps. It follows from (9. 2) that our assertion is true if one and hence both of the maps are not n-extensible with respect to X. On the other hand, let $\gamma^{n+1}(\phi)$ be an arbitrary element of $\Omega^{n+1}(f)$, where $\phi: \bar{P}^n\to R$ is an extension of $f\omega_0$. Since $f\simeq g$, we have $f\omega_0\simeq g\omega_0$. Hence it follows from the homotopy extension property of P_0 in P relative to R that $g\omega_0$ has an extension $\psi: \bar{P}^n\to R$ which is homotopic with ϕ. Then $\gamma^{n+1}(\phi)=\gamma^{n+1}(\psi)\in\Omega^{n+1}(g)$. This proves that $\Omega^{n+1}(f)\subset\Omega^{n+1}(g)$. Similarly, one may prove that $\Omega^{n+1}(f)\supset\Omega^{n+1}(g)$. Q. E. D.

(13. 2) *A map* $f: X_0\to R$ *is* n*-extensible with respect to* X *if and only if* $\Omega^{n+1}(f)$ *is non-empty.*

(13. 3) FUNDAMENTAL EXTENSION LEMMA. *A map* $f: X_0\to R$ *is* $(n+1)$*-extensible with respect to* X *if and only if* $\Omega^{n+1}(f)$ *contains the zero element of* $H^{n+1}(X, X_0, \pi_n)$.

Proof. *Necessity.* Suppose $f: X_0\to R$ to be $(n+1)$-extensible with respect to X. Then the map $f\omega_0$ has an extension $\phi^*: \bar{P}^{n+1}\to R$. Let $\phi=\phi^*|\bar{P}^n$. According to (i) of (12. 5), we have $c^{n+1}(\phi)=0$. Hence $\Omega^{n+1}(f)$ contains the zero element $\gamma^{n+1}(\phi)$ of $H^{n+1}(X, X^0, \pi_n)$.

Sufficiecy. Suppose that $\Omega^{n+1}(f)$ contains the zero element of $H^{n+1}(X, X_0, \pi_n)$. There exists an extension $\phi: \bar{P}^n \to R$ of $f\omega_0$ such that $c^{n+1}(\phi) \sim 0$. According to (ii) of (12.5), there exists a map $\phi^*: \bar{P}^{n+1} \to R$ such that $\phi^* | \bar{P}^{n-1} = \phi | \bar{P}^{n-1}$. Since $\phi^* | P_0 = f\omega_0$, f is $(n+1)$-extensible with respect to X. This completes the proof.

(13.4) *Let $\kappa: (M, M_0) \to (X, X_0)$ be a map of a pair (M, M_0) into (X, X_0) and $\kappa_0 = \kappa | M_0$. Then, for an arbitrary map $f: X_0 \to R$, $\Omega^{n+1}(f\kappa_0)$ contains the image $\kappa^*(\Omega^{n+1}(f))$ under the induced homomorphism*

$$\kappa^*: H^{n+1}(X, X_0, \pi_n) \to H^{n+1}(M, M_0, \pi_n).$$

Proof. The assertion is obviously true if f is not n-extensible with respect to X. Assume that f be n-extensible with respect to X and $\gamma^{n+1}(\phi)$ be an arbitrary element of $\Omega^{n+1}(f)$ where $\phi: \bar{P}^n(X) \to R$ is an extension of the map $f\omega_{X_0}$. According to (6.1) and (6.4), the map κ induces a map

$$\kappa^\#: (P(M),\ P(M_0)) \to (P(X),\ P(X_0)),$$

which maps $\bar{P}^n(M)$ into $\bar{P}^n(X)$. Consider the map $\psi = \phi\kappa^\# | \bar{P}^n(M)$. By means of (6.3), we have

$$\psi | P(M_0) = f\omega_{X_0}\kappa^\# | P(M_0) = f\kappa_0\omega_{M_0}.$$

Hence, ψ is an extension of $f\kappa_0\omega_{M_0}$ and determines an $(n+1)$-dimensional obstruction element $\gamma^{n+1}(\psi) \in \Omega^{n+1}(f\kappa_0)$. Since we have obviously that $\gamma^{n+1}(\psi) = \kappa^*(\gamma^{n+1}(\phi))$, if follows that $\Omega^{n+1}(f\kappa_0) \supset \kappa^*(\Omega^{n+1}(f))$. This completes the proof.

Throughout the remainder of the section, we shall assume further that X be a locally finite simplicial polytope, i. e. a locally finite polyhedron with a fixed simplicial triangulation K, and X_0 be a closed subpolytope of K, i. e. $X_0 \wedge K$ is a closed subcomplex K_0 of K. We shall denote by $H^{n+1}(K, K_0, \pi_n)$ the $(n+1)$-dimensional cohomology group of the simplicial complex K modulo K_0 by using the (infinite) cochains with coefficients in $\pi_n = \pi_n(R)$, while $H^{n+1}(X, X_0, \pi_n)$ is still used to denote the singular cohomology group.

For a given map $f: X_0 \to R$, we are going to define the *obstruction set* $\Omega^{n+1}(f, K)$ of f in $H^{n+1}(K, K_0, \pi_n)$. If f is not n-extensible with respect to X, we define $\Omega^{n+1}(f, K)$ to be the vacuous set of $H^{n+1}(K, K_0, \pi_n)$. Now suppose f to be n-extensible with respect to X. It follows from (9.5) that there exists an extension $f^*: \bar{K}^n \to R$ of f over $\bar{K}^n = K_0 \cup K^n$. According to Eilenberg, [19, p. 239], f^* determines a cocycle $c^{n+1}(f^*)$ in $K^*=K\setminus K_0$ and hence an element $\gamma^{n+1}(f^*)$ of $H^{n+1}(K, K_0, \pi_n)$, called an obstruction element of f in $H^{n+1}(K, K_0, \pi_n)$. $\Omega^{n+1}(f, K)$ is defined to be the set of all obstruction elements of f in $H^{n+1}(K, K_0, \pi_n)$.

For a fixed partial order of the vertices of K, there is an injection $j: X \to P(X)$, (see § 8). It is obvious from the construction of j that j maps the complex K isomorphically onto a closed subcomplex $j(K)$ of $P(X)$. Further, it is clear that $j(K_0) \subset P(X_0)$. According to the invariance theorem, [23, pp. 418, 422], this simplicial map j induces an onto isomorphism:

$$j^*: H^{n+1}(X, X_0, \pi_n) \to H^{n+1}(K, K_0, \pi_n).$$

The following theorem proves the *topological invariance* of the obstruction set $\Omega^{n+1}(f, K)$.

(13. 5) $\Omega^{n+1}(f, K)=j^*\Omega^{n+1}(f)$.

Proof. First, let $\gamma^{n+1}(\phi)$ be an arbitrary element of $\Omega^{n+1}(f)$ where $\phi: \bar{P}^n \to R$ is an extension of $f\omega_0$. Since j maps $\bar{K}^n$ into $\bar{P}^n$, we may define a map $f^*: \bar{K}^n \to R$ by taking $f^*=\phi j|\bar{K}^n$. Since ωj is the identity map on X, we have $f^*|X_0=f$. Obviously, $\gamma^{n+1}(f^*)=j^*\gamma^{n+1}(\phi)$. Hence $j^*\Omega^{n+1}(f)$ is contained in $\Omega^{n+1}(f, K)$.

Next, let $\gamma^{n+1}(f^*)$ be an arbitrary element of $\Omega^{n+1}(f, K)$ where $f^*: \bar{K}^n \to R$ is an extension of f. It follows from a cellular approximation theorem of J. H. C. Whitehead, [73, p. 229], that there exists a homotopy $h_t: P \to X$ $(0 \leqq t \leqq 1)$ such that $h_0=\omega, h_1(P^n) \subset K^n$, and $h_t(y)=\omega(y)$ for each $y \in j(K)$ and each $0 \leqq t \leqq 1$. We may obviously assume that $h_t(P_0) \subset X_0$ for every $0 \leqq t \leqq 1$. Define a map $\psi: \bar{P}^n \to R$ by taking $\psi=f^*h_1|\bar{P}^n$. Since ψ has a partial homotopy $\psi_t: j(K^n) \cup P_0 \to R$ $(0 \leqq t \leqq 1)$ defined by

$$\psi_t=f^*h_t|j(K^n) \cup P_0, \qquad (0 \leqq t \leqq 1),$$

it follows from the homotopy extension property of $j(K^n) \cup P_0$ in $\bar{P}^n$ relative to R, [73, p. 228], that ψ_t has an extension $\psi_t^*: \bar{P}^n \to R$ $(0 \leqq t \leqq 1)$ such that $\psi_1^*=\psi$. Let $\phi=\psi_0^*$. Since $h_0=\omega$, we have $\phi|P_0=f\omega_0$ and $\phi j(x)=f^*(x)$ for each $x \in \bar{K}^n$. Hence $\gamma^{n+1}(\phi) \in \Omega^{n+1}(f)$ and $\gamma^{n+1}(f^*)=j^*\gamma^{n+1}(\phi)$. This completes the proof.

14. General extension theorem

The following theorem can be easily proved by the recurrent applications of (13. 2) and (13. 3).

(14.1) *If R is r-simple and $H^{r+1}(X, X_0, \pi_r(R))=0$ for each r such that $n \leqq r < m$, then the n-extensibility of a map $f: X_0 \to R$ with respect to X implies its m-extensibility with respect to X.*

For the remainder of the present section, let (X, X_0) be a pair of C_0 such that X_0 is closed in X and has the homotopy extension property in X relative to R, and let $m=\Delta(X, X_0)$. Combining (10.2) and (14.1), we obtain the following assertion.

(14.2) *If R is r-simple and $H^{r+1}(X, X_0, \pi_r(R))=0$ for each r such that $n \leqq r < m$, then the n-extensibility of a map $f: X_0 \to R$ with respect to X implies its extensibility over X.*

In particular, if we take $n=1$ or 2, we deduce the following two corollaries of (14.2) by means of (9.1) and (11.1) respectively.

(14.3) *If R is r-simple and $H^{r+1}(X, X_0, \pi_r(R))=0$ for each r such that $1 \leqq r < m$, then every map $f: X_0 \to R$ is extensible over X.*

(14.4) *Assume that X and X_0 be pathwise connected. If R is r-simple and $H^{r+1}(X, X_0, \pi_r(R))=0$ for each r such that $2 \leqq r < m$, then a necessary and sufficient condition for a map $f: X_0 \to R$ to be extensible over X is that its induced homomorphism $f^*: \pi_1(X_0, x_0) \to \pi_1(R, r_0)$ is extensible with respect to X, where $x_0 \in X_0$ and $r_0=f(x_0)$.*

Setting $R=X_0$, we obtain a sufficient condition for retraction as follows. Let (X, X_0) be a pathwise connected pair of C_0 such that X_0 is closed in X and has the homotopy extension property in X relative to X_0, and let $m=\Delta(X, X_0)$.

(14.5) *Assume that X_0 be r-simple and $H^{r+1}(X, X_0, \pi_r(X_0))=0$ for each r such that $2 \leqq r < m$, then X_0 is a retract of X if and only if there exists a homomorphism $h: \pi_1(X, x_0) \to \pi_1(X_0, x_0)$ such that $h\iota^*$ is the identity automorphism on $\pi_1(X_0, x_0)$, where $x_0 \in X_0$ and $\iota^*: \pi_1(X_0, x_0) \to \pi_1(X, x_0)$ denotes the homomorphism induced by the identity map $\iota: X_0 \to X$.*

IV. The General Theory of Homotopy Classification

Throughout the present chapter, we shall assume the same assumptions as given at the beginning of Chapter III.

15. n-Homotopy

Two maps $f, g: X \to R$ with $f|X_0=g|X_0$ are said to be *n-homotopic relative to X_0*, provided that the maps $f\omega, g\omega: P \to R$ are n-homotopic relative to P_0, i.e. there exists a homotopy $h_t: \bar{P}^n \to R$ $(0 \leqq t \leqq 1)$ such that $h_0=f\omega|\bar{P}^n$, $h_1=g\omega|\bar{P}^n$, and $f\omega|P_0=h_t|P_0=g\omega|P_0$ for every

$0 \leqq t \leqq 1$. Since R is pathwise connected, the following assertion is obvious.

(15. 1) *Every pair of maps* $f, g: X \to R$ *with* $f|X_0=g|X_0$ *are* 0*-homotopic with respect to* X_0.

For a given pair of maps $f, g: X \to R$ with $f|X_0=g|X_0$, the least upper bound of the set of integers n such that f and g are n-homotopic relative to X_0 is called the *homotopy index* of the pair (f, g) relative to X_0. Two pairs (f, g) and (f', g') are said to be *homotopic* relative to X_0, if $f \simeq f'$ and $g \simeq g'$ relative to X_0.

(15. 2) *Homotopic pairs have the same homotopy index.*

Proof. Suppose that (f, g) and (f', g') be two homotopic pairs of maps relative to X_0 and that f and g be n-homotopic relative to X_0. It needs only to show that f' and g' are also n-homotopic relative to X_0. Since $f \simeq f'$ and $g \simeq g'$ relative to X_0, we have $f\omega \simeq f'\omega$ and $g\omega \simeq g'\omega$ relative to P_0. Hence

$$f'\omega|\bar{P}^n \simeq f\omega|\bar{P}^n \simeq g\omega|\bar{P}^n \simeq g'\omega|\bar{P}^n$$

relative to P_0. This proves that f' and g' are n-homotopic relative to X_0. Q. E. D.

(15. 3) *Let* $\kappa: (M, M_0) \to (X, X_0)$ *be a map of a pair* (M, M_0) *into* (X, X_0). *If the maps* $f, g: X \to R$ *with* $f|X_0=g|X_0$ *are* n*-homotopic relative to* X_0, *then the maps* $f\kappa, g\kappa: M \to R$ *are* n*-homotopic relative to* M_0.

Proof. According to § 6, the map κ induces a map $\kappa_\#: P(M) \to P(X)$. Obviously $\kappa_\#$ maps $P(M_0)$ into $P(X_0)$ and $\bar{P}^n(M)$ into $\bar{P}^n(X)$. Since f and g are n-homotopic relative to X_0, the maps $f\omega_X|\bar{P}^n(X)$ and $g\omega_X|\bar{P}^n(X)$ are homotopic relative to $P(X_0)$. Therefore, the maps $f\omega_X\kappa_\#|\bar{P}^n(M)$ and $g\omega_X\kappa_\#|\bar{P}^n(M)$ are homotopic relative to $P(M_0)$. In accordance with (6. 3), we have $\omega_X\kappa_\#=\kappa\omega_M$. Hence the maps $f\kappa\omega_M|\bar{P}^n(M)$ and $g\kappa\omega_M|\bar{P}^n(M)$ are homotopic relative to $P(M_0)$. By definition, this implies that the maps $f\kappa$ and $g\kappa$ are n-homotopic relative to M_0. Q. E. D.

The following theorem proves the equivalence of our definition of n-homotopy with that of R. H. Fox, [26, p. 49].

(15. 4) *A necessary and sufficient condition for two maps* $f, g: X \to R$ *with* $f|X_0=g|X_0$ *to be* n*-homotopic relative to* X_0 *is that, for an arbitrary map* $\kappa:(M, M_0) \to (X, X_0)$ *of a simplicial polytope* M *and a closed subcomplex* $M_0 \subset M$, *the maps* $f\kappa|\bar{M}^n$ *and* $g\kappa|\bar{M}^n$ *are homotopic relative to* M_0, *where* $\bar{M}^n=M_0 \cup M^n$.

Proof. *Necessity.* Assume that f and g be n-homotopic relative to X_0. Then we have

$$f\omega_X|\overline{P}^n(X)\simeq g\omega_X|\overline{P}^n(X)$$

relative to $P(X_0)$. The map κ induces a map $\kappa\#: P(M)\to P(X)$ which clearly maps $P(M_0)$ into $P(X_0)$ and $\overline{P}^n(M)$ into $\overline{P}^n(X)$. Since M is a simplicial polytope and $M_0\subset M$ is a closed subpolytope, there is an injection $j: M\to P(M)$ which maps M_0 into $P(M_0)$ and $\overline{M}^n$ into $\overline{P}^n(M)$. By (6. 3) and (8. 1), we have

$$\omega_X\kappa\# j(y)=\kappa\omega_M j(y)=\kappa(y), \qquad (y\in M).$$

Hence we obtain

$$f\kappa|\overline{M}^n=f\omega_X\kappa\# j|\overline{M}^n\simeq g\omega_X\kappa\# j|\overline{M}^n=g\kappa|\overline{M}^n$$

relative to M_0. This proves the necessity.

Sufficiency. Assume that the condition holds. Let Q and $Q_0\subset Q$ denote the second barycentric subdivision of $P=P(X)$ and $P_0=P(X_0)$. According to (7. 1), Q is a simplicial polytope and Q_0 is a closed subpolytope of Q. As a topological space and a subspace, we have $P=Q$ and $P_0=Q_0$. Now, since the projection $\omega: P\to X$ maps (Q, Q_0) into (X, X_0), it follows from our condition that $f\omega|\overline{Q}^n\simeq g\omega|\overline{Q}^n$ relative to Q_0, where $\overline{Q}^n=Q_0\cup Q^n$. Since $\overline{P}^n\subset\overline{Q}^n$, this proves that f and g are n-homotopic relative to X_0. Q. E. D.

(15. 5) *If X is a simplicial polytope and X_0 a closed subpolytope of X, then a necessary and sufficient condition for two maps $f, g: X\to R$ with $f|X_0=g|X_0$ to be n-homotopic relative to X_0 is that $f|\overline{X}^n\simeq g|\overline{X}^n$ relative to X_0, where $\overline{X}^n=X_0\cup X^n$.*

Proof. *Necessity.* Assume that f and g be n-homotopic to X_0. Then, by definition, $f\omega|\overline{P}^n\simeq g\omega|\overline{P}^n$ relative to P_0. Since X is a simplicial polytope and X_0 a closed subpolytope of X, it follows from (8. 1) that there exists an injection $j: X\to P$ which clearly maps X_0 into P_0 and $\overline{X}^n$ into $\overline{P}^n$. According to (8. 1), we have $f\omega j=f$ and $g\omega j=g$. Hence $f|\overline{X}^n\simeq g|\overline{X}^n$ relative to X_0. This proves the necessity of the condition.

Sufficiency. Assume that $f|\overline{X}^n\simeq g|\overline{X}^n$ relative to X_0. Then, by definition, there is a homotopy $h_t: \overline{X}^n\to R$ $(0\leq t\leq 1)$ such that $h_0=f$, $h_1=g$, and $h_t(x)=f(x)$ for eacn $x\in X_0$ and each $0\leq t\leq 1$. Since both X and P are CW-complexes, [73, p. 223], it follows from a cellular approximation theorem of J. H. C. Whitehead, [73, p. 229], that there is a homotopy $\phi_t: P\to X$ $(0\leq t\leq 1)$ such that $\phi_0=\omega$ and $\phi_1(P^n)\subset X^n$ for each n. Clearly we may choose ϕ_t in such a way that $\phi_t(P_0)\subset X_0$

for every $0\leqq t\leqq 1$. Let I denote the closed interval of real numbers between 0 and 1. Define a map $F: \bar{P}^n\times I\to R$ by taking

$$F(p,t)=\begin{cases} f\phi_{3t}(p) & (p\in\bar{P}^n,\ 0\leqq t\leqq 1/3),\\ h_{3t-1}\phi_1(p) & (p\in\bar{P}^n,\ 1/3\leqq t\leqq 2/3),\\ g\phi_{3-3t}(p) & (p\in\bar{P}^n,\ 2/3\leqq t\leqq 1).\end{cases}$$

It is easily verified that, for each $p\in P_0$ and each $t\in I$, we have $F(p,t)=F(p,1-t)$. Consider the closed subset

$$T=(\bar{P}^n\times 0)\cup(P_0\times I)\cup(\bar{P}^n\times 1)$$

of the topological product $\bar{P}^n\times I$. Define a homotopy $F_\tau: T\to R$ $(0\leqq\tau\leqq 1)$ as follows:

$$F_\tau=F \quad\text{on}\quad (\bar{P}^n\times 0)\cup(\bar{P}^n\times 1);$$

$$F_\tau(p,t)=\begin{cases} F(p,(1-\tau)t), & (p\in P_0,\ 0\leqq t\leqq \tfrac{1}{2});\\ F_\tau(p,1-t), & (p\in P_0,\ \tfrac{1}{2}\leqq t\leqq 1).\end{cases}$$

Since $F_0=F|T$, it follows from a homotopy extension theorem of J. H. C. Whitehead, [73, p. 228], that the homotopy F_τ has an extension $H_\tau: \bar{P}^n\times I\to R$ $(0\leqq\tau\leqq 1)$ such that $H_0=F$. Define a homotopy $k_t: \bar{P}^n\to R$ $(0\leqq t\leqq 1)$ by taking

$$k_t(p)=H_1(p,t), \qquad (p\in\bar{P}^n,\ 0\leqq t\leqq 1).$$

Then $k_0=f\omega|\bar{P}^n$, $k_1=g\omega|\bar{P}^n$, and $k_t(p)=f\omega(p)$ for each $p\in P_0$ and every $0\leqq t\leqq 1$. Hence, f and g are n-homotopic relative to X_0. Q. E. D.

16. Homotopy

We recall the definition of homotopy relative to X_0 as follows: Two maps $f,g: X\to R$ with $f|X_0=g|X_0$ are said to be *homotopic relative to* X_0, if there exists a homotopy $h_t: X\to R$ $(0\leqq t\leqq 1)$ such that $h_0=f$, $h_1=g$, and $h_t(x)=f(x)$ for each $x\in X_0$ and each $0\leqq t\leqq 1$. The following assertion is clear.

(16.1) *If two maps $f,g: X\to R$ with $f|X_0=g|X_0$ are homotopic relative to X_0, then they are n-homotopic relative to X_0 for every n.*

In the remainder of the section, we shall return to the notations converse of (16.1). Let (X, X_0) be a pair of C_0 and $n=\Delta(X,X_0)$. In case that X_0 is non-empty, we shall further assume that X_0 is closed in X and that the closed subset

$$T=(X\times 0)\cup(X_0\times I)\cup(X\times 1)$$

of the topological product $X\times I$ has the homotopy extension property in $X\times I$ relative to R, where I denotes the closed unit interval of real numbers.

(16.2) *If two maps* $f, g: X\to R$ *with* $f|X_0=g|X_0$ *are n-homotopic relative to* X_0, *then they are homotopic relative to* X_0.

Proof. Let (M, M_0) be a simplicial pair with $\dim(M\setminus M_0)=n$ which dominates (X, X_0). By definition, there are maps

$$\xi: (X, X_0)\to(M, M_0), \qquad \eta: (M, M_0)\to(X, X_0),$$

and a homotopy $\lambda_t: (X, X_0)\to(X, X_0)$, $(0\leqq t\leqq 1)$, such that $\lambda_0=\eta\xi$ and λ_1 is the identity map on X. Since $\dim(M\setminus M_0)=n$, we have $\bar{M}^n=M$. By our hypothesis, f and g are n-homotopic relative to X_0; hence it follows from (15.4) that $f\eta$ and $g\eta$ are homotopic relative to M_0. Therefore, $f\eta\xi\simeq g\eta\xi$ relative to X_0. The homotopy $_t$ proves that $f\eta\xi\simeq f$ and $g\eta\xi\simeq g$. Hence $f\simeq g$. This proves the case that X_0 is empty. Now assume that X_0 is non-empty. Since $f\eta\simeq g\eta$ relative to M_0, there is a homotopy $\mu_t: M\to R$ $(0\leqq t\leqq 1)$ such that $\mu_0=f\eta$, $\mu_1=g\eta$, and $\mu_t|M_0=f\eta|M_0$ for every $0\leqq t\leqq 1$. Define a map $F: X\times I\to R$ by taking

$$F(x, t)=\begin{cases} f\lambda_{1-3t}(x), & (x\in X,\ 0\leqq t\leqq 1/3),\\ \mu_{3t-1}(x), & (x\in X,\ 1/3\leqq t\leqq 2/3),\\ g\lambda_{3t-2}(x), & (x\in X,\ 2/3\leqq t\leqq 1).\end{cases}$$

It is easily verified that, for each $x\in X_0$ and each $t\in I$, we have $F(x, t)=F(x, 1-t)$. Define a homotopy $F_\tau: T\to R$ $(0\leqq\tau\leqq 1)$ as follows:

$$F_\tau=F \quad\text{on}\quad (X\times 0)\cup(X\times 1),$$

$$F_\tau(x, t)=\begin{cases} F(x, (1-\tau)t) & (x\in X_0,\ 0\leqq t\leqq \tfrac{1}{2}),\\ F_\tau(x, 1-t) & (x\in X_0,\ \tfrac{1}{2}\leqq t\leqq 1).\end{cases}$$

Since $F_0=F|T$, it follows from the homotopy extension property of T in $X\times I$ relative to R that the homotopy F_τ has an extension H_τ: $X\times I\to R$ $(0\leqq\tau\leqq 1)$ such that $H_0=F$. Define a homotopy $h_t: X\to R$ $(0\leqq t\leqq 1)$ by taking

$$h_t(x)=H_1(x, t), \qquad (x\in X,\ 0\leqq t\leqq 1).$$

Then $h_0=f$, $h_1=g$, and $h_t|X_0=f|X_0$ for each $0\leqq t\leqq 1$. Hence, f and g are homotopic relative to X_0. Q.E.D.

17. Algebraic condition for 1-homotopy

Throughout the present section, let us assume that both X and X_0 are pathwise connected.

First, let us consider the case that X_0 is non-empty. Let $f, g: X \to R$ be two given maps with $f|X_0 = g|X_0$. Choose a fixed point $x_0 \in X_0$ and call $r_0 = f(x_0) = g(x_0)$. Consider the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(R, r_0)$ of the spaces X and R with basic points x_0 and r_0 respectively. The given maps f and g induce homomorphisms

$$f^*, g^*: \pi_1(X, x_0) \to \pi_1(R, r_0).$$

(17.1) *Two maps $f, g: X \to R$ with $f|X_0 = gX_0$ are 1-homotopic relative to X_0, if and only if their induced homomorphisms $f^* = g^*$.*

Proof. *Necessity.* Assume that f and g be 1-homotopic relative to X_0. Then the maps $f\omega|\bar{P}^1$ and $g\omega|\bar{P}^1$ are homotopic relative to P_0, i.e. there is a homotopy $h_t: \bar{P}^1 \to R$ $(0 \leqq t \leqq 1)$ such that $h_0 = f\omega|\bar{P}^1$, $h_1 = g\omega|\bar{P}^1$, and $h_t|P_0 = f\omega|P_0$ for each $0 \leqq t \leqq 1$. Let $e \in \pi_1(X, x_0)$ be an arbitrary element represented by a path $T: I \to X$ with $T(0) = x_0 = T(1)$. Since the closed unit interval I is an ordered geometric 1-simplex, the path T is a continuous 1-simplex in X and hence represents a singular 1-simplex $\xi = [T]$. Let s_ξ denote the ordered geometric 1-simplex associated with ξ whose interior is the open singular cell σ_ξ. Let $B_\xi: I \to s_\xi$ denote the barycentric map of I onto s_ξ which preserves the order of vertices. Using the characteristic map $\mu_\xi: s_\xi \to Cl\sigma_\xi$, we clearly have $T = \omega\mu_\xi B_\xi$. Define a homotopy $\phi_\tau: I \to R$ $(0 \leqq \tau \leqq 1)$ by taking $\phi_\tau = h_\tau \mu_\xi B_\xi$ for each $0 \leqq \tau \leqq 1$. Then we have:

$$\phi_0 = fT, \ \phi_1 = gT, \ \phi_\tau(0) = r_0 = \phi_\tau(1), \qquad (0 \leqq \tau \leqq 1).$$

Hence $f^*(e) = g^*(e)$. This proves the necessity of the condition.

Sufficiency. Assume that $f^* = g^*$. Consider the projection $\omega: (P, P_0) \to (X, X_0)$. Since both X aud X_0 are pathwise connected, one can easily show, by means of the homotopy extension property, that there exists a homotopy

$$\delta_t: (P, P_0) \to (X, X_0), \qquad (0 \leqq t \leqq 1),$$

such that $\delta_0 = \omega$ and δ_1 maps every vertex of P at x_0. We are going to construct a homotopy $\phi_t: \bar{P}^1 \to R$ $(0 \leqq t \leqq 1)$ as follows: Let σ_ξ be an arbitrary open singular 1-cell contained in $P \setminus P_0$ and s_ξ the ordered geometric 1-simplex associated with ξ. The path $\delta_1\mu_\xi B_\xi: I \to X$ represents an element $e_\xi \in \pi_1(X, x_0)$. Since $f^*(e_\xi) = g^*(e_\xi)$, there is a homotopy $\theta_t: I \to R$ $(0 \leqq t \leqq 1)$ such that $\theta_0 = f\delta_1\mu_\xi B_\xi$, $\theta_1 = g\delta_1\mu_\xi B_\xi$, and $\theta_t(0) = r_0 = \theta_t(1)$ for each $0 \leqq t \leqq 1$. The homotopy ϕ_t is given by

$$\phi_t(p)=\begin{cases} f\delta_1(p) & (p\in \overline{P}^0=P_0\cup P^0),\\ \theta_t B_\xi^{-1}(p) & (p\in\sigma_\xi\subset P^1\setminus P_0),\end{cases}$$

for every $0\leq t\leq 1$. The continuity of the homotopy ϕ_t is verified by the fact that δ_1 maps every vertex of P at x_0. ϕ_t has the following properties:

$$\phi_0=f\delta_1|\overline{P}^1,\ \phi_1=g\delta_1|\overline{P}^1,\ \phi_t|\overline{P}^0=f\delta_1|\overline{P}^0 \qquad (0\leq t\leq 1).$$

Define a map $F:\overline{P}^1\times I\to R$ by taking

$$F(p,t)=\begin{cases} f\delta_{3t}(p), & (p\in\overline{P}^1,\ 0\leq t\leq 1/3),\\ \phi_{3t-1}(p), & (p\in\overline{P}^1,\ 1/3\leq t\leq 2/3),\\ g\delta_{3-3t}(p), & (p\in\overline{P}^1,\ 2/3\leq t\leq 1).\end{cases}$$

It is easily verified that, for each $p\in P_0$ and each $t\in I$, we have $F(p,t)=F(p,1-t)$. By an argument used in the sufficiency proof of (15.5), one can prove that the maps $f\omega|\overline{P}^1$ and $g\omega|\overline{P}^1$ are homotopic relative to P_0. Hence, f and g are 1-homotopic relative to X_0. Q. E. D.

Throughout the remainder of the section, we shall assume that X_0 be the vacuous set. Let $f,g:X\to R$ be two given maps. Choose a point $x_0\in X$ and call $r_0=f(x_0)$ and $r_1=g(x_0)$. Let us use the following notations:

$$G=\pi_1(X,x_0),\ G_0=\pi_1(R,r_0),\ G_1=\pi_1(R,r_1).$$

The given maps $f,g:X\to R$ induce homomorphisms

$$f^*:G\to G_0,\qquad g^*:G\to G_1.$$

It is well-known that every path $\sigma:I\to R$ joining r_0 to r_1 induces an isomorphism $\sigma^*:G_0\approx G_1$ of G_0 onto G_1 which depends only on the homotopy class of σ leaving extremities fixed.

(17.2) *Two maps $f,g:X\to R$ are* 1-*homotopic if and only if there exists a path $\sigma:I\to R$, joining $r_0=f(x_0)$ to $r_1=g(x_0)$, such that $\sigma^*f^*=g^*$.*

Proof. *Necessity.* Assume that f and g be 1-homotopic. Then there is a homotopy $h_t:P^1\to R$ $(0\leq t\leq 1)$ such that $h_0=f\omega|P^1$ and $h_1=g\omega|P^1$. The point x_0 determines a vertex p_0 of P. Define a path $\sigma:I\to R$ by taking $\sigma(t)=h_t(p_0)$ for every $0\leq t\leq 1$. Let $e\in G$ be an arbitrary element represented by a path $T:I\to X$ with $T(0)=x_0=T(1)$. T is a continuous 1-simplex in X and represents a singular 1-simplex $\xi=[T]$. Define a homotopy $\phi_\tau:I\to R$ $(0\leq\tau\leq 1)$ by taking $\phi_\tau=h_\tau\mu_\xi B_\xi$ for each $0\leq\tau\leq 1$. Then we have

$$\phi_0=fT,\ \phi_1=gT,\ \phi_\tau(0)=\sigma(\tau)=\phi_\tau(1),\qquad (0\leqq\tau\leqq1).$$

Hence $\sigma^*f^*(e)=g^*(e)$. This proves the necessity.

Sufficiency. Assume that there exists a path $\sigma: I\to R$ joining r_0 to r_1 such that $\sigma^*f^*=g^*$. As in the sufficiency proof of (17.1), there is a homotopy $\delta_t: P\to X$ $(0\leqq t\leqq1)$ such that $\delta_0=\omega$ and δ_1 maps every vertex of P at x_0. Construct a homotopy $\phi_t: P^1\to R$ $(0\leqq t\leqq1)$ as follows: Let σ_ξ be an arbitrary open singular 1-cell of P. The path $\delta_1\mu_\xi B_\xi: I\to X$ represents an element $e_\xi\in G$. Since $\sigma^*f^*(e_\xi)=g^*(e_\xi)$, there is a homotopy $\theta_t: I\to R$ $(0\leqq t\leqq1)$ such that $\theta_0=f\delta_1\mu_\xi B_\xi$, $\theta_1=g\delta_1\mu_\xi B_\xi$, and $\theta_t(0)=\sigma(t)=\theta_t(1)$ for each $0\leqq t\leqq1$. The homotopy ϕ_t is given by

$$\phi_t(p)=\begin{cases} f\delta_1(p) & (p\in P^0),\\ \theta_t B_\xi^{-1}(p) & (p\in\sigma_\xi\subset P^1),\end{cases}$$

for every $0\leqq t\leqq1$. Then we have $\phi_0=f\delta_1|P^1$ and $\phi_1=g\delta_1|P^1$. Hence

$$f\omega|P^1\simeq f\delta_1|P^1\simeq g\delta_1|P^1\simeq g\omega|P^1.$$

This proves that f and g are 1-homotopic. Q.E.D.

Two homomorphisms $h, k: G\to H$ of a group G into a group H are said to be *equivalent*, provided that there exists an element $w\in H$ such that $k(e)=w^{-1}h(e)w$ for every element $e\in G$. The following assertion is an immediate corollary of (17.2).

(17.3) *Two maps $f, g: X\to R$ with $f(x_0)=r_0=g(x_0)$ are 1-homotopic, if and only if their induced homomorphisms $f^*, g^*: \pi_1(X, x_0)\to\pi_1(R, r_0)$ are equivalent.*

18. Deviation cocycles $d^n(\phi, \psi, \theta_t)$

As in § 12, with a purpose to simplify the arguments, we shall assume that R be n-simple in the sense of Eilenberg [17] where $n\geqq1$ is a given integer. Denote by $\pi_n=\pi_n(R)$ the n-th homotopy group of R.

Let $\phi, \psi: P\to R$ be two maps with $\phi|P_0=\psi|P_0$ and $\theta_t: \bar{P}^{n-1}\to R$ $(0\leqq t\leqq1)$ be a homotopy such that $\theta_0=\phi|\bar{P}^{n-1}$, $\theta_1=\psi|\bar{P}^{n-1}$, and $\theta_t|P_0=\phi|P_0$ for every $0\leqq t\leqq1$. The triple (ϕ, ψ, θ_t) defines an n-dimensional singular cochain $d^n(\phi, \psi, \theta_t)$ of X with coefficients in π_n as follows: Let $\xi\in S(X)$ be an arbitrary singular n-simplex. Let s_ξ be the ordered geometric n-simplex associated with ξ whose interior is the open singular cell σ_ξ. The characteristic map $\mu_\xi: s_\xi\to Cl\sigma_\xi$ maps the boundary sphere ∂s_ξ onto $\bar{P}^{n-1}$. We define a map D_ξ on the boundary sphere

$$\partial(s_\xi \times I) = (s_\xi \times 0) \cup (\partial s_\xi \times I) \cup (s_\xi \times 1)$$

of $s_\xi \times I$ with values in R by taking

$$D_\xi(y, t) = \begin{cases} \phi\mu_\xi(y) & (y \in s_\xi, \quad t=0), \\ \theta_t\mu_\xi(y) & (y \in \partial s_\xi, \quad t \in I), \\ \psi\mu_\xi(y) & (y \in s_\xi, \quad t=1). \end{cases}$$

The sphere $\partial(s_\xi \times I)$ may be oriented on such a way that $s_\xi \times 0$ lies negatively and $s_\xi \times 1$ positively on $\partial(s_\xi \times I)$. Then the map D_ξ determines an element $d_\xi \in \pi_n$ since R is n-simple. The association $\xi \to d_\xi$ defines an n-dimensional singular cochain $d^n(\phi, \psi, \theta_t)$.

(18.1) *$d^n(\phi, \psi, \theta_t)$ is a singular cocycle of X modulo X_0, called the deviation cocycle of the triple (ϕ, ψ, θ_t).*

Proof. Let $\eta \in S(X)$ be an arbitrary singular $(n+1)$-simplex. To show that $d^n(\phi, \psi, \theta_t)$ is a singular cocycle, it suffices to prove that $\delta d^n(\phi, \psi, \theta_t) \cdot \eta = 0$. Let s_η denote the ordered geometric $(n+1)$-simplex associated with η and $\mu_\eta: s_\eta \to Cl\sigma_\eta$ the characteristic map for η. Denote by s_η^{n-1} the $(n-1)$-dimensional skeleton of s_η and consider the maps $f = \phi\mu_\eta$, $g = \psi\mu_\eta$, and the homotopy $h_t = \theta_t\mu_\eta | s_\eta^{n-1}$. Since $h_0 = f | s_\eta^{n-1}$ and $h_1 = g | s_\eta^{n-1}$, it follows from the corresponding theorem for finite complex that the triple (f, g, h_t) determines a cocycle $d^n(f, g, h_t)$ of s_η. Let s be an arbitrary n-face of s_η, then μ_η maps the interior of s onto some open singular n-cell σ_ξ of P. It is easy to see that

$$d^n(\phi, \psi, \theta_t) \cdot \xi = d^n(f, g, h_t) \cdot s.$$

Hence it follows that

$$\delta d^n(\phi, \psi, \theta_t) \cdot \eta = \delta d^n(f, g, h_t) \cdot s_\eta = 0.$$

This proves that $d^n(\phi, \psi, \theta_t)$ is a singular cocycle.

Next, let ξ be an arbitrary singular n-simplex contained in $S(X_0)$. Since μ_ξ maps s_ξ onto $Cl\sigma_\xi \subset P_0$, the map D_ξ can be defined throughout $s_\xi \times I$. Therefore, $d^n(\phi, \psi, \theta_t) \cdot \xi = d_\xi = 0$. This proves that $d^n(\phi, \psi, \theta_t)$ is a singular cocycle of X modulo X_0. Q. E. D.

By means of the methods analogous to those used in § 12, one can prove the following assertions.

(18.2) *For a given triple (ϕ, ψ, θ_t) and any $d^n \sim d^n(\phi, \psi, \theta_t)$ modulo X_0, there exists a homotopy $\rho_t: \bar{P}^{n-1} \to R$ $(0 \leq t \leq 1)$ such that $\rho_0 = \phi | \bar{P}^{n-1}$, $\rho_1 = \psi | P^{n-1}$, $\rho_t | \bar{P}^{n-2} = \theta_t | \bar{P}^{n-2}$ for each $0 \leq t \leq 1$, and $d^n(\phi, \psi, \rho_t) = d^n$.*

(18.3) *For a given triple (ϕ, ψ, θ_t), we have*:

(i) *$d^n(\phi, \psi, \theta_t) = 0$ if and only if there is a homotopy $\theta_t^*: \bar{P}^n \to R$*

$(0\leqq t\leqq 1)$ *such that* $\theta_0^*=\phi|\bar{P}^n$, $\theta_1^*=\psi|\bar{P}^n$, *and* $\theta_t^*|\bar{P}^{n-1}=\theta_t$ *for each* $0\leqq t\leqq 1$.

(ii) $d^n(\phi, \psi, \theta_t)\sim 0$ *modulo* X_0 *if and only if there is a homotopy* $\theta_t^*: \bar{P}^n\to R$ $(0\leqq t\leqq 1)$ *such that* $\theta_0^*=\phi|\bar{P}^n$, $\theta_1^*=\psi|\bar{P}^n$, *and* $\theta_t^*|\bar{P}^{n-2}=\theta_t|\bar{P}^{n-2}$ *for each* $0\leqq t\leqq 1$.

19. The group $W^n(X, X_0, f)$

Let $f: X\to R$ be a given map. We are going to construct for each integer $n\geqq 1$ a group $W^n(X, X_0, f)$ as what follows.

Let us denote by V^n the totality of the maps $F: \bar{P}^{n-1}\times I\to R$ such that:

$$F(p, 0)=f\omega(p)=F(p, 1), \qquad (p\in\bar{P}^{n-1});$$
$$F(p, t)=f\omega(p), \qquad (p\in P_0,\ t\in I).$$

Two maps $F, G\in V^n$ are said to be *equivalent*, if there is a homotopy $H_\tau: \bar{P}^{n-1}\times I\to R$ $(0\leqq\tau\leqq 1)$ such that $H_0=F$, $H_1=G$, and $H_\tau\in V^n$ for each $0\leqq\tau\leqq 1$. This equivalence relation divides the maps of V^n into disjoint classes. We shall denote by the symbol $W^n(X, X_0, f)$ the totality of these classes and by $[F]$ the class which contains the map $F\in V^n$.

For any two given maps $F, G\in V^n$, we define a map $F\cdot G: \bar{P}^{n-1}\times I\to R$ by taking

$$(F\cdot G)(p, t)=\begin{cases} F(p, 2t) & (p\in\bar{P}^{n-1},\ 0\leqq t\leqq \tfrac{1}{2}),\\ G(p, 2t-1) & (p\in\bar{P}^{n-1},\ \tfrac{1}{2}\leqq t\leqq 1).\end{cases}$$

Obviously $F\cdot G$ is a map in V^n and the class $[F\cdot G]$ depends only on the classes $[F]$ and $[G]$. Therefore, we may define a multiplication in $W^n(X, X_0, f)$ by taking $[F]\cdot[G]=[F\cdot G]$. Let E denote the map in V^n defined by $E(p, t)=f\omega(p)$ for every $p\in\bar{P}^{n-1}$ and $t\in I$. For each $F\in V^n$, we defino a map $F^{-1}\in V^n$ by setting $(F^{-1})(p, t)=F(p, 1-t)$ for each $p\in\bar{P}^{n-1}$ and $t\in I$. The following theorem is obvious.

(19.1) *The elements of* $W^n(X, X_0, f)$ *form a group with the multiplication defined above as the group operation. The neutral element of* $W^n(X, X_0, f)$ *is the class* $[E]$ *and the inverse of the class* $[F]$ *is the class* $[F^{-1}]$.

Let $\kappa: (M, M_0)\to(X, X_0)$ be a map of a pair (M, M_0) into (X, X_0). According to §6, the map κ induces a map $\kappa\#: P(M)\to P(X)$ which maps $P(M_0)$ into $P(X_0)$ and $\bar{P}^n(M)$ into $\bar{P}^n(X)$ for each $n\geqq 0$. For every integer $n\geqq 1$ and any given map $f: X\to R$, $\kappa\#$ determines, in an obvious way, an induced homomorphism

(19. 2) $\kappa^*: W^n(X, X_0, f) \to W^n(M, M_0, f\kappa)$.

Now assume that X be a simplicial polytope with a fixed triangulation K and X_0 be a closed subpolytope of K, i. e. $X_0 \cap K$ is a closed subcomplex K_0 of K. For a given map $f: X \to R$, let us consider the maps $\phi: \bar{K}^{n-1} \times I \to R$ such that:

$$\phi(x, 0) = f(x) = \phi(x, 1), \qquad (x \in \bar{K}^{n-1})$$
$$\phi(x, t) = f(x), \qquad (x \in \bar{K}^{n-1},\ t \in I).$$

By the method used at the beginning of the section, one can define a group which will be denoted by $W^n(K, K_0, f)$.

For a fixed partial order of the vertices of K, there is an injection $j: X \to P(X)$, (see § 8). It is obvious from the construction of j that j maps the complex K isomorphically onto a closed subcomplex $j(K)$ of $P(X)$ and that ωj is the identity map on X. Further, it is also clear that $j(K_0) \subset P(X_0)$. The injection j determines an induced homomorphism

(19. 3) $j^*: W^n(X, X_0, f) \to W^n(K, K_0, f)$

for each integer $n \geq 1$ and any map $f: X \to R$ as follows: If an element $\alpha \in W^n(X, X_0, f)$ is represented by a map $F \in V^n$, then the element $j^*(\alpha) \in W^n(X, X_0, f)$ is represented by the map $\phi: \bar{K}^{n-1} \times I \to R$ defined by $\phi(x, t) = F(j(x), t)$ for each $x \in \bar{K}^{n-1}$ and $t \in I$.

(19. 4) *j^* maps $W^n(X, X_0, f)$ onto $W^n(K, K_0, f)$.*

Proof. Let $\beta \in W^n(K, K_0, f)$ be an arbitrary element represented by a map $\phi: \bar{K}^{n-1} \times I \to R$. It follows from a cellular approximation theorem of J. H. C. Whitehead, [73, p. 229], that there exists a homotopy $h_t: P \to X$ $(0 \leq t \leq 1)$ such that $h_0 = \omega$, $h_1(P^{n-1}) \subset K^{n-1}$, and $h_t(p) = \omega(p)$ for each $p \in j(K)$ and each $0 \leq t \leq 1$. Define a map $\Phi: \bar{P}^{n-1} \times I \to R$ by taking

$$\Phi(p, t) = \phi(h_1(p), t), \qquad (p \in \bar{P}^{n-1},\ t \in I).$$

Let $T = (\bar{P}^{n-1} \times 0) \cup (P_0 \times I) \cup (j(\bar{K}^{n-1}) \times I) \cup (\bar{P}^{n-1} \times 1)$ and define a homotopy $\Phi_\tau: T \to R$ $(0 \leq t \leq 1)$ by taking

$$\Phi_\tau(p, t) = \phi(h_\tau(p), t), \qquad (p, t) \in T.$$

Since $\Phi_1 = \Phi | T$, Φ_τ has an extension $\Phi_\tau^*: \bar{P}^{n-1} \times I \to R$ $(0 \leq t \leq 1)$ such that $\Phi_1^* = \Phi$. Call $F = \Phi_0^*$. Then we have $F \in V^n$ and $\phi(x, t) = F(j(x), t)$ for each $x \in \bar{K}^{n-1}$ and $t \in I$. Let $\alpha = [F] \in W^n(X, X_0, f)$, then $\beta = j^*(\alpha)$.

Hence j^* is onto. Q. E. D.

20. The homomorphisms k_n

Throughout the present section, we assume that R be n-simple, where $n \geq 1$ is a given integer. We shall construct a homomorphism k_n of $W^n(X, X_0, f)$ into the n-dimensional singular cohomology group $H^n(X, X_0, \pi_n)$ of X modulo X_0 with coefficients in $\pi_n = \pi_n(R)$.

Let $\alpha \in W^n(X, X_0, f)$ be an arbitrary element represented by a map $F: \bar{P}^{n-1} \times I \rightarrow R$ in V^n. Let $\phi = f\omega$ and define a homotopy $\theta_t: \bar{P}^{n-1} \rightarrow R$ $(0 \leq t \leq 1)$ by taking $\theta_t(p) = F(p, t)$ for each $p \in \bar{P}^{n-1}$ and $0 \leq t \leq 1$. Then we obtain a triple (ϕ, ϕ, θ_t). The deviation cocycle $d^n(\phi, \phi, \theta_t)$, which depends only on the class $\alpha = [F]$, represents an element $k_n(\alpha) \in H^n(X, X_0, \pi_n)$. The following theorem is obvious.

(20. 1) *The correspondence $\alpha \rightarrow k(\alpha)$ define a homomorphism*

$$k_n: W^n(X, X_0, f) \rightarrow H^n(X, X_0, \pi_n).$$

Let $J^n_f = J^n_f(X, X_0, \pi_n)$ denote the image of $W^n(X, X_0, f)$ under the homomorphism. J^n_f is a subgroup $H^n(X, X_0, \pi_n)$. We denote the quotient group by

$$Q^n_f = Q^n_f(X, X_0, \pi_n) = H^n(X, X_0, \pi_n)/J^n_f(X, X_0, \pi_n).$$

(20. 0) *The subgroup J^n_f (and hence the quotient group Q^n_f) depends only on the $(n-1)$-homotopy class of f relative to X_0.*

Proof. Let $g: X \rightarrow R$ be any map such that $f|X_0 = g|X_0$ and f, g are $(n-1)$-homotopic relative to X_0. Then there exists a homotopy $\chi_t: \bar{P}^{n-1} \rightarrow R$ $(0 \leq t \leq 1)$ such that $\chi_0 = \phi|\bar{P}^{n-1}$, $\chi_1 = \psi|\bar{P}^{n-1}$, and $\chi_t|P_0 = \phi|P_0$ for each $0 \leq t \leq 1$, where $\phi = f\omega$ and $\psi = g\omega$. Because of symmetry, it needs only to prove that $J^n_f \subset J^n_g$. As before, for an arbitrary element $\alpha = [F]$ of $W^n(X, X_0, f)$, the element $k_n(\alpha) \in H^n(X, X_0, \pi_n)$ is represented by the deviation cocycle $d^n(\phi, \phi, \theta_t)$. Define a homotopy $\rho_t: \bar{P}^{n-1} \rightarrow R$ $(0 \leq t \leq 1)$ by taking

$$\rho_t(p) = \begin{cases} \chi_{1-3t}(p) & (p \in \bar{P}^{n-1}, \quad 0 \leq t \leq 1/3), \\ \theta_{3t-1}(p) & (p \in \bar{P}^{n-1}, \ 1/3 \leq t \leq 2/3), \\ \chi_{3t-2}(p) & (p \in \bar{P}^{n-1}, \ 2/3 \leq t \leq 1). \end{cases}$$

Then $\rho_0 = |\bar{P}^{n-1} = \rho_1$ and $\rho_t|P_0 = \psi|P_0$ for every $0 \leq t \leq 1$. It is obvious that

$$d^n(\psi, \psi, \rho_t)=-d^n(\phi, \psi, \chi_t)+d^n(\phi, \phi, \theta_t)+d^n(\phi, \psi, \chi_t)=d^n(\phi, \phi, \theta_t).$$

Define a map $G: \bar{P}^{n-1}\times I\to R$ by setting $G(p, t)=\rho_t(p)$ for each $p\in \bar{P}^{n-1}$ and $t\in I$. Call $\beta=[G]\in W^n(X, X_0, g)$. Then the element $k_n(\beta)\in J_g^n$ is represented by $d^n(\psi, \psi, \rho_t)$. This proves $k_n(\alpha)=k_n(\beta)\in J_g^n$. Hence $J_f^n\subset J_g^n$. Q. E. D.

Let $\kappa: (M, M_0)\to(X, X_0)$ be a map of a pair (M, M_0) into the pair (X, X_0) and $f: X\to R$ be a map. In the following rectangle of homomorphisms

$$\begin{array}{ccc} W^n(X, X_0, f) & \xrightarrow[\kappa^*]{} & W^n(M, M_0, f) \\ \downarrow k_n & & \downarrow k_n \\ H^n(X, X_0, \pi_n) & \xrightarrow[\kappa^*]{} & H^n(M, M_0, \pi_n) \end{array}$$

we have obviously the following commutativity relation:

$$k_n\kappa^* = \kappa^* k_n. \tag{20.3}$$

For the remainder of the section, we assume that X be a locally finite simplicial polytope with a fixed triangulation K and X_0 be a closed subpolytope of K, i.e. $X_0\cap K$ is a closed subcomplex K_0 of K.

In an obviously analogous way, we may define the homomorphisms:

$$k_n: W^n(K, K_0, f)\to H^n(K, K_0, \pi_n), \tag{20.4}$$

where $H^n(K, K_0, \pi_n)$ denotes the n-dimensional cohomology group of the simplicial complex K modulo K_0 with coefficients in $\pi_n=\pi_n(R)$. The image of k_n is denoted by $J_f^n(K, K_0, \pi_n)$ and the quotient group by

$$Q^n(K, K_0, \pi_n)=H^n(K, K_0, \pi_n)/J(K, K_0, \pi_n).$$

For a fixed partial order of the vertices of K, there is an injection $j: X\to P(X)$. According to the invariance theorem, j induces an isomorphism j^* of $H^n(X, X_0, \pi_n)$ onto $H^n(K, K_0, \pi_n)$. In accordance with (19.3) and (19.4), j induces a homorphism j^* of $W^n(X, X_0, f)$ onto $W^n(K, K_0, f)$. In the following rectangle of homomorphisms

$$\begin{array}{ccc} W^n(X, X_0, f) & \xrightarrow{j^*} & W^n(K, K_0, f) \\ \downarrow k_n & & \downarrow k_n \\ H^n(X, X_0, \pi_n) & \xrightarrow{j^*} & H^n(K, K_0, \pi_n), \end{array}$$

one can easily see the following commutativity theorem.

(20.5) $$k_n j^* = j^* k_n.$$

The following theorem is an immediate consequence of (19.4), (20.4), and the fact that j^* maps $H^n(X, X_0, \pi_n)$ isomorphically onto $H^n(K, K_0, \pi_n)$.

(20.6) *j^* maps $J^n_f(X, X_0, \pi^n)$ isomorphyically onto $J^n_f(K, K_0, \pi)$.*

This indicates the topological invariance of $J^n_f(K, K_0, \pi^n)$ and hence of $Q^n_f(K, K_0, \pi^n)$.

21. Deviation sets

In the present section, we assume that R be n-simple, where $n \geqq 1$ is a given integer.

Let $f, g: X \to R$ be two given maps such that $f|X_0 = g|X_0$. We are going to define the *n-dimensional deviation set* $\Delta^n(f, g) \subset H^n(X, X_0, \pi_n)$ of the pair of maps (f, g) relative to X_0. If f and g are not $(n-1)$-homotopic relative to X_0, we define $\Delta^n(f, g)$ to be the vacuous set. Now suppose that f and g be $(n-1)$-homotopic relative to X_0. Then there is a homotopy $\theta_t: \bar{P}^{n-1} \to R\ (0 \leqq t \leqq 1)$ such that $\theta_0 = \phi|\bar{P}^{n-1}$, $\theta_1 = \psi|\bar{P}^{n-1}$, and $\theta_t|P_0 = \phi|P_0$ for every $0 \leqq t \leqq 1$, where $\theta = f\omega$ and $\psi = g\omega$. The deviation cocycle $d^n(\phi, \psi, \theta_t)$ represents an element $\delta^n(\phi, \psi, \theta_t)$ of the eingular cohomology group $H^n(X, X_0, \pi_n)$, called *an n-dimensional deviation element* of f and g. $\Delta^n(f, g)$ is defined to be the set of all n-dimensional deviation elements of f and g.

(21.1) *Homotopic pairs have the same n-dimensional deviation set.*

Proof. Suppose that (f, g) and (f', g') be two homotopic pairs of maps relative to X_0. It follows from (15.2) that our assertion is true if one and hence both of the pairs are not $(n-1)$-homotopic relative to X_0. On the other hand, let $\delta^n(\phi, \psi, \theta_t)$ be an arbitrary element of $\Delta^n(f, g)$, where $\phi = f\omega$, $\psi = g\omega$, and $\theta_t: \bar{P}^{n-1} \to R (0 \leqq t \leqq 1)$ is a homotopy such that $\theta_0 = \phi|\bar{P}^{n-1}$, $\theta_1 = \psi|\bar{P}^{n-1}$, and $\theta_t|P_0 = \phi|P_0$ for each $0 \leqq t \leqq 1$. Since $f \simeq f'$ and $g \simeq g'$ relative to X_0, there are homotopies $f_t, g_t: X \to R\ (0 \leqq t \leqq 1)$ such that $f_0 = f$, $f_1 = f'$, $g_0 = g$, $g_1 = g'$ and $f_t|X_0 = f|X_0 = g|X_0 = g_t|X_0$ for each $0 \leqq t \leqq 1$. Call $\phi' = f'\omega$ and $\psi' = g'\omega$. Define homotopies $\alpha_t, \beta_t: \bar{P}^{n-1} \to R\ (0 \leqq t \leqq 1)$ by taking

$$\alpha_t(p) = f_t\omega(p),\ \beta_t(p) = g_t\omega(p), \quad (p \in \bar{P}^{-1},\ 0 \leqq t \leqq 1).$$

According to (i) of (18.3), we have

$$d^n(\phi, \phi', \alpha_t) = 0, \quad d^n(\psi, \psi', \beta_t) = 0.$$

Define a homotopy $\theta'_t: \bar{P}^{n-1}\to R\ (0\leq t\leq 1)$ by taking

$$\theta'_t(p)=\begin{cases}\alpha_{1-3t}(p) & (p\in\bar{P}^{n-1},\quad 0\leq t\leq 1/3),\\ \theta_{3t-1}(p) & (p\in\bar{P}^{n-1},\ 1/3\leq t\leq 2/3),\\ \beta_{3t-2}(p) & (p\in\bar{P}^{n-1},\ 2/3\leq t\leq 1).\end{cases}$$

Obviously we have

$$d^n(\phi',\ \psi',\ \theta'_t)=-d^n(\phi,\ \phi',\ \alpha_t)+d^n(\phi,\ \psi,\ \theta_t)+d^n(\psi,\ \psi',\ \beta_t).$$

Therefore, $d^n(\phi',\ \psi',\ \theta'_t)=d^n(\phi,\ \psi,\ \theta_t)$ and the element $\delta^n(\phi,\psi,\theta_t)$ is in $\Delta^n(f',\ g')$. This proves that $\Delta^n(f,\ g)\subset\Delta^n(f',g')$. Similarly, one can show that $\Delta^n(f',\ g')\subset\Delta^n(f,\ g)$. Hence $\Delta^n(f,\ g)=\Delta^n(f',\ g')$. Q. E. D.

The following assertion is obvious.

(21. 2) *Two maps $f,\ g: X\to R$ with $f|X_0=g|X_0$ are $(n-1)$-homotopic relative to X_0 if and only if $\Delta^n(f,\ g)$ is non-empty.*

The above statement is strengthened as follows:

(21. 3) *Two maps $f,\ g: X\to R$ with $f|X_0=g|X_0$ are $(n-1)$-homotopic relative to X_0 if and only if $\Delta^n(f,g)$ is a coset of $J^n_f(X,\ X_0,\ \pi_n)$ in $H^n(X,\ X_0,\ \pi_n)$.*

Proof. Because of (21. 2), it needs only to prove the necessity. Suppose f and g to be $(n-1)$-homotopic relative to X_0. Call $\phi=f\omega$ and $\psi=g\omega$. Let $\delta^n(\phi,\ \psi,\ \alpha_t)$ and $\delta^n(\phi,\ \psi,\ \beta_t)$ be any two deviation elements of f and g. Define a map $F: \bar{P}^{n-1}\times I\to R$ by taking

$$F(p,\ t)=\begin{cases}\alpha_{2t}(p) & (p\in\bar{P}^{n-1},\quad 0\leq t\leq \tfrac{1}{2}),\\ \beta_{2-2t}(p) & (p\in\bar{P}^{n-1},\ \tfrac{1}{2}\leq t\leq 1).\end{cases}$$

Evidently $F\in V^n$. F represents an element w of the group $W^n(X,\ X_0,\ f)$. If follows from the definition of F that

$$k_n(w)=\delta^n(\phi,\ \psi,\ \alpha_t)-\delta^n(\phi,\ \psi,\ \beta_t).$$

Since $k_n(w)\in J^n_f$, this proves that $\Delta^n(f,\ g)$ is contained in the coset $\delta^n(\phi,\ \psi,\ \alpha_t)+J^n_f$.

Conversely, let $w=[F]$ be an arbitrary element of $W^n(X,\ X_0,\ f)$. Define a homotopy $\theta_t: \bar{P}^{n-1}\to R\ (0\leq t\leq 1)$ by setting $\theta_t(p)=F(p,\ t)$ for every $p\in\bar{P}^{n-1}$ and $0\leq t\leq 1$. Then $k_n(w)=\delta^n(\phi,\ \phi,\ \theta_t)$. Define a homotopy $\beta_t: \bar{P}^{n-1}\to R\ (0\leq t\leq 1)$ by taking

$$\beta_t(p)=\begin{cases}\theta_{2t}(p) & (p\in\bar{P}^{n-1},\quad 0\leq t\leq \tfrac{1}{2}),\\ \alpha_{2t-1}(p) & (p\in\bar{P}^{n-1},\ \tfrac{1}{2}\leq t\leq 1).\end{cases}$$

Then we have $\delta^n(\phi, \psi, \beta_t)=k_n(w)+\delta^n(\phi, \psi, \alpha_t)$. Hence every element of the coset $\delta^n(\phi, \psi, \alpha_t)+J^n_f$ is in $\Delta^n(f, g)$. This completes the proof that $\Delta^n(f, g)$ is a coset of J^n_f in $H^n(X, X_0, \pi_n)$. Q. E. D.

(21.4) FUNDAMENTAL HOMOTOPY LEMMA. *Two maps $f, g: X\to R$ with $f|X_0=g|X_0$ are n-homotopic relative to X_0, if and only if $\Delta^n(f, g)=J^n_f(X, X_0, \pi_n)$.*

Proof. *Necessity.* Suppose that f and g be n-homotopic relative to X_0. Call $\phi=f\omega$ and $\psi=g\omega$. Then there exists a homotopy $\alpha^*_t: \bar{P}\to R\ (0\leq t\leq 1)$ such that $\alpha^*_0=\phi|\bar{P}^n$, $\alpha^*_1=\psi|\bar{P}^n$, and $\alpha^*_t|P_0=\phi|P_0$ for each $0\leq t\leq 1$. Call $\alpha_t=\alpha^*_t|\bar{P}^{n-1}$. According to (i) of (18.3), we have $d^n(\phi, \psi, \alpha_t)=0$ and hence $\delta^n(\phi, \psi, \alpha_t)=0$. This implies that $\Delta^n(f, g)$ contains the zero element of $H^n(X, X_0, \pi_n)$. By (21.3), we have $\Delta^n(f, g)=J^n_f(X, X_0, \pi_n)$.

Sufficiency. Suppose that $\Delta^n(f, g)=J^n_f(X, X_0, \pi_n)$. Then $\Delta^n(f, g)$ contains the zero element of $H^n(X, X_0, \pi_n)$. Hence there is a homotopy $\alpha_t: \bar{P}^{n-1}\to R\ (0\leq t\leq 1)$ such that $\alpha_0=\phi|\bar{P}^{n-1}$, $\alpha_1=\psi|\bar{P}^{n-1}$, $\alpha_t|P_0=\phi|P_0$ for each $0\leq t\leq 1$, and $\delta^n(\phi, \psi, \alpha_t)=0$. This implies that $d^n(\phi, \psi, \alpha_t)\sim 0$ modulo X_0. It follows from (ii) of (18.3) that f and g are n-homotopic relative to X_0. Q. E. D.

(21.5) *Let $\kappa: (M, M_0)\to(X, X_0)$ be a map of a pair (M, M_0) into (X, X_0), and $f, g: X\to R$ be any two given maps with $f|X_0=g|X_0$. Then $\Delta^n(f\kappa, g\kappa)$ contains the image $\kappa^*(\Delta^n(f, g))$ under the induced homomorphism*

$$\kappa^*: H^n(X, X_0, \pi_n)\to H^n(M, M_0, \pi_n).$$

Proof. The assertion is obviously true if f and g are not $(n-1)$-homotopic relative to X_0. Assume that f and g be $(n-1)$-homotopic relative to X_0 and $\delta^n(\phi, \psi, \alpha_t)$ be an arbitrary element of $\Delta^n(f, g)$ where $\phi=f\omega_X$, $\psi=g\omega_Y$, and $\alpha_t: \bar{P}^{n-1}(X)\to R\ (0\leq t\leq 1)$ is a homotopy such that $\alpha_0=\phi|\bar{P}^{n-1}(X)$, $\alpha_1=\psi|\bar{P}^{n-1}(X)$, and $\alpha_t P(X_0)=\phi|P(X_0)$ for every $0\leq t\leq 1$. According to (6.1) and (6.4), the map κ induces a map

$$\kappa^{\#}: (P(M), P(M_0))\to(P(X), P(X_0)),$$

which maps $\bar{P}^{n-1}(M)$ into $\bar{P}^{n-1}(X)$. Call $\phi'=\phi\kappa^{\#}$, $\psi'=\psi\kappa^{\#}$, and $\alpha'_t=\alpha_t\kappa^{\#}|\bar{P}^{n-1}(M)$, $(0\leq t\leq 1)$. By means of (6.3), we have

$$\phi'=\phi\kappa^{\#}=f\omega_X\kappa^{\#}=f\kappa\omega_M,\quad \psi'=\psi\kappa^{\#}=g\omega_X\kappa^{\#}=g\kappa\omega_M.$$

The triple $(\phi', \psi', \alpha'_t)$ determines an element $\delta^n(\phi', \psi', \alpha'_t)$ of $\Delta^n(f\kappa, g\kappa)$.

Since we have obviously

$$\delta^n(\phi', \psi', \alpha_t') = \kappa^*(\delta^n(\phi, \psi, \alpha_t)),$$

it follows that $\Delta^n(f\kappa, g\kappa) \supset \kappa^*(\Delta^n(f, g))$. Q. E. D.

Taking $f=g$, we obtain the following corollary of (21. 5).

(21. 6) *For any given maps* $\kappa:(M, M_0)\to(X, X_0)$ *and* $f: X\to R$, *we always have*

$$\kappa^*(J_f^n(X, X_0, \pi_n)) \subset J_{f\kappa}^n(M, M_0, \pi_n).$$

Hence the map κ *induces a homomorphism*

$$\kappa^\square: Q_f^n(X, X_0, \pi_n) \to Q_{f\kappa}^n(M, M_0, \pi_n).$$

Once more, let X be a locally finite simplicial polytope and X_0 be a closed subpolytope of X with a given triangulation K of X such that $X_0 \wedge K = K_0$ is a closed subcomplex of K. For any two maps $f, g: X\to R$ with $f|X_0 = g|X_0$, we can define their *deviation set* $\Delta^n(f, g, K)$ in $H^n(K, K_0, \pi_n)$ as follows. If f and g are not $(n-1)$-homotopic relative X_0, we define $\Delta^n(f, g, K)$ to be the vacuous set of $H^n(K, K_0, \pi_n)$. Now suppose f and g to be $(n-1)$-homotopic relative to X_0. It follows from (15. 5) that there exists a homotopy $h_t: \overline{K}^{n-1}\to R$ $0(\leqq t\leqq 1)$ such that $h_0 = f|\overline{K}^{n-1}$, $h_1 = g|\overline{K}^{n-1}$, and $h_t|K_0 = f|K_0$ for each $0\leqq t\leqq 1$. The triple (f, g, h_t) determines, in an obvious way, a cocycle $d^n(f, g, h_t)$ of K modulo K_0 with coefficients in π_n. $d^n(f, g, h_t)$ represents an element $\delta^n(f, g, h_t)$ of $H^n(K, K_0, \pi_n)$, called a *deviation element* of (f, g) in $H^n(K, K_0, \pi_n)$. $\Delta^n(f, g, K)$ is defined to be the set of all deviation elements of (f, g) in $H^n(K, K_0, \pi_n)$.

Analogous to (21. 3) and (21. 4), one can easily prove the following assertion.

(21. 7) *If two maps* $f, g: X\to R$ *with* $f|X_0 = g|X_0$ *are* $(n-1)$*-homotopic relative to* X_0, *then* $\Delta^n(f, g, K)$ *is a coset of* $J_f^n(K, K_0, \pi_n)$ *in* $H^n(K, K_0, \pi_n)$. f *and* g *are* n*-homotopic relative to* X_0 *if and only if* $\Delta^n(f, g, K) = J_f^n(K, K_0, \pi_n)$.

For a fixed partial order of the vertices of K, there in an injection $j: X\to P(X)$. j induces an isomorphism j^* of $H^n(X, X_0, \pi_n)$ onto $H^n(K, K_0, \pi_n)$. According to (20. 6), j^* maps $J_f^n(X, X_0, \pi_n)$ onto $J_f^n(K, K_0, \pi_n)$. One can also easily prove the following assertion.

(21. 8.) $$\Delta^n(f, g, K) = j^*\Delta^n(f, g).$$

22. General homotopy theorems

The following main homotopy theorem is an immediate consequence of (21.3) and (21.4).

(22.1) *If R is n-simple, then any two maps $f, g: X \to R$ with $f|X_0 = g|X_0$ which are $(n-1)$-homotopic relative to X_0 determine a unique element $\chi^n(f, g)$ of $Q_f^n(X, X_0, \pi_n)$, called the n-dimensional characteristic element of (f, g). $\chi^n(f, g)=0$ if and only if f and g are n-homotopic relative to X_0.*

By the recurrent application of (22.1), we obtain the following theorem.

(22.2) *Let $f: X \to R$ be a given map. If R is r-simple and $Q_f^n(X, X_0, \pi_r)=0$ for each r such that $n < r \leq m$, then the n-homotopy relative to X_0 of two maps $f, g: X \to R$ with $f|X_0 = g|X_0$ implies that they are m-homotopic relative to X_0.*

For the remainder of the present section, let (X, X_0) be a pair of C_0 and $m=\Delta(X, X_0)$. In case that X_0 is non-empty, we assume that X_0 is closed in X and that $(X\times 0)\cup(X_0\times I)\cup(X\times 1)$ has the homotopy extension property in $X\times I$ relative to R. Combining (16.2) and (22.2), we obtain the following assertion.

(22.3) *Let $f: X \to R$ be a given map. If R is r-simple and $Q_f^r(X, X_0, \pi_r)=0$ for each r such that $n < r \leq m$, then the n-homotopy relative to X_0 of two maps $f, g: X \to R$ with $f|X_0 = g|X_0$ implies that they are homotopic relative to X_0.*

In particular, if we take $n=0$ or 1, we deduce the following corollaries of (22.3) by means of (15.1), (17.1) and (17.2).

(22.4) *Let $f: X \to R$ be a given map. If R is r-simple and $Q_f^r(X, X_0, \pi_r)=0$ for each r such that $1 \leq r \leq m$, then every map $g: X \to R$ with $f|X_0 = g|X_0$ is homotopic with f relative to X_0.*

(22.5) *Assume that X and X_0 be pathwise connected, X_0 be non-empty, and $f: X \to R$ be a given map. If R is r-simple and $Q_f^r(X, X_0, \pi_r)=0$ for each r such that $2 \leq r \leq m$, then a necessary and sufficient condition for a map $g: X \to R$ with $f|X_0 = g|X_0$ to be homotopic with f relative to X_0 is that $f^* = g^*$, where $f^*, g^*: \pi_1(X, x_0) \to \pi_1(R, r_0)$ are the homomorphisms induced by f, g respectively, $x_0 \in X_0$, $r_0 = f(x_0)$.*

(22.6) *Assume that X be pathwise connected, X_0 be empty, $x_0 \in X$, and $f: X \to R$ be a given map. If R is r-simple and $Q_f^r(X, \pi_r)=0$ for each r such that $2 \leq r \leq m$, then a necessary and sufficient condition for a map $g: X \to R$ to be homotopic with f is the existence of a path $\sigma: I \to R$ joining $f(x_0)$ to $g(x_0)$ such that $\sigma^* f^* = g^*$, where*

$$f^*: \pi_1(X, x_0) \to \pi_1(R, f(x_0)), \qquad g^*: \pi_1(X, x_0) \to \pi_1(R, g(x_0))$$

are the induced homomorphisms and σ^* denotes the isomorphism of $\pi_1(R, f(x_0))$ onto $\pi_1(R, g(x_0))$ determined by the path σ.

If we assume X to be pathwise connected, X_0 to be vacuous, and $R=X$, (22. 4) implies the following assertion.

(22. 7) *The following statements are equivalent:*

(i) *X is contractible to a point.*

(ii) *$\pi_r(X)=0$ for each $1 \leq r \leq m$.*

(iii) *X is r-simple and $H^r(X, \pi_r(X))=0$ for each $1 \leq r \leq m$.*

(iv) *X is r-simple and $Q_i^r(X, \pi_r(X))=0$ for each $1 \leq r \leq m$, where $i: X \to X$ denotes the identity map.*

23. Classification theorems

Throughout the present section, let $f: X \to R$ be a given map. Let us denote by $\underline{\underline{M}}=\underline{\underline{M}}(X, X_0, R, f)$ the totality of the maps $g: X \to R$ such that $f|X_0=g|X_0$. The maps of $\underline{\underline{M}}$ are divided into disjoint homotopy classes relative to X_0. The classification problem is to enumerate these classes by means of some convenient invariants.

The relation of n-homotopy relative to X_0 among the maps $\underline{\underline{M}}$ divides $\underline{\underline{M}}$ into disjoint *n-homotopy classes* relative to X_0. For each $n \geq 1$, every $(n-1)$-homotopy class relative to X_0 of $\underline{\underline{M}}$ contains a certain collection of n-homotopy classes relative to X_0. Theoretically, the classification problem could be considered as solved, if there is a definite way to count the n-homotopy classes contained in a given $(n-1)$-homotopy class by means of the elements of some cohomology invariant.

In the sequel, let θ be a given $(n-1)$-homotopy class relative to X_0 of the maps $\underline{\underline{M}}$. We are going to give a method to enumerate the n-homotopy classes relative to X_0 of the maps of $\underline{\underline{M}}$, which are contained in θ. We assume that n be a given positive integer and R be n-simple.

According to (20. 2), the $(n-1)$-homotopy class θ relative to X_0 determines a subgroup $J_\theta^n(X, X_0, \pi_n)$ of the singular cohomology group $H^n(X, X_0, \pi_n)$ and hence the quotient group:

$$Q_\theta^n(X, X_0, \pi_n)=H^n(X, X_0, \pi_n)/J_\theta^n(X, X_0, \pi_n).$$

Now let us cooose a map $g: X \to R$ from the class θ as our reference map. According to (22.1), every map $h \in \theta$ determines a characteristic element $\chi^n(g, h)$ of the group $Q^n_\theta(X, X_0, \pi_n)$. An element $\alpha \in Q^n(X, X_0, \pi_n)$ is said to be *g-admissble* if there is a map $h \in \theta$ such that $\chi^n(g, h)=\alpha$. The g-admissible elements of $Q^n_\theta(X, X_0, \pi_n)$ form a set A^n_g, called *the g-admissible set.* The following lemma is clear.

(23.1) *For any two maps $g, h: X \to R$ of the $(n-1)$-homotopy class θ relative to X_0, A^n_g is the image of A^n_h under the translation determined by their characteristic element $\chi^n(g, h)$,* i.e.

$$A^n_g = \chi^n(g, h) + A^n_h.$$

(22.2) CLASSIFICATION THEOREM. *Given an $(n-1)$-homotopy class θ relative to X_0 of the maps $\underline{\underline{M}}$, the n-homotopy classes relative to X_0 of those maps which are contained in θ are in a (1-1)-correspondence with the elements of the g-admissible set A^n_g in the quotient group $Q^n_0(X, X_0, \pi_n)$, where g is an arbitrarily given map of θ.*

Proof. According to (22.1), every map $h \in \theta$ determines a unique element $\chi^n(g, h) \in A^n_g$. We assert that $\chi^n(g, h)$ depends only on the n-homotopy class relative to X_0 which contains h. For, if $h, k \in \theta$ are n-homotopic relative to X_0, (22.1) gives

$$\chi^n(g, h) - \chi^n(g, k) = \chi^n(k, h) = 0,$$

i.e. $\chi^n(g, h) = \chi^n(g, k)$. Hence the correspondence $h \to \chi^n(g, h)$ defines a transformation τ of the n-homotopy classes relative to X_0 contained in θ into the elements of A^n_g. It remains to show that τ is one-to-one. That τ is onto follows from the definition of A^n_g. To prove that τ is univalent, suppose $\chi^n(g, h) = \chi^n(g, k)$. Then we have

$$\chi^n(h, k) = \chi^n(g, k) - \chi^n(g, h) = 0.$$

By (22.1), h and k are n-homotopic relative to X_0. This completes the proof.

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NOTES

1) A major part of the results contained in the present paper were carried out under sponsorship of the Office of Naval Research in the Summer Session of 1949. The homotopy and classification theorems in §§21–23 were first obtained for a finite polyhedron when the author was working at

Academia Sinica in 1948. The essentials of the paper were delivered in an adress before the Conference on Topology held at the University of Chicago in May, 1950.

2) If X,R are topological spaces, a *map* $f:X\to R$ is a continuous transformation f of X into R. If $X_0\subset X$ and $R_0\subset R$, a map $f:(X, X_0)\to(R, R_0)$ is a map $f:X\to R$ such that $f(X_0)\subset R_0$.

3) Numbers in brackets refer to the bibliography at the end of the paper.

4) The singular polytope of a topological space was independently introduced by J. B. Giever [31] in proving the equivalence of the two singular homology theories.

5) The circumflex over v_i indicates that v_i is omitted.

6) A subspace X_0 of a topological space X is a subset (not necessarily closed) of X with the relative topology.

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